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(p, q)-Analysis of Montgomery identity and estimates of (p, q)-bounds with applications

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Abstract

The main objective of this article is to establish a new post quantum version of Montgomery identity. Some estimates of associated post quantum bounds are also obtained. In order to obtain the main results of the article, we use the preinvexity property of the functions. Some special cases are also discussed in detail. Finally, we present some applications of the obtained results, which shows the significance of the discussed results.

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1 Introduction and preliminaries

Quantum calculus, which is often known as q-calculus or calculus without limits, is based on finite difference. In quantum calculus we obtain q-analogues of mathematical objects which can be recaptured by taking $q \to 1^-$. The history of quantum calculus can be traced back to Euler for investigating the q-binomial formula and Euler's identities for q-exponential functions. In the last century extensive research was done on the study of quantum calculus. Consequently, this particular area of mathematics has expanded in different directions. With the research study of Jackson [6] the in-depth study of quantum calculus started. He is considered a pioneer in developing the first systematic definitions of q-derivatives and q-integrals. Geometrical interpretation of the quantum calculus has been recognized through study on quantum groups. Another reason behind its rapid development is that it can be viewed as a bridge between mathematics and physics. It has numerous applications in various branches of mathematics and physics such as ordinary fractional calculus, orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, the theory of relativity, optimal control problems, q-difference and q-integral equations, q-transform analysis, etc. For some recent studies on quantum calculus and its applications, see [1, 5, 7]. Tariboon et al. [12] introduced the notion of q-derivatives and q-integrals on finite intervals and developed several new q-analogues of classical inequalities. Since then many new q-analogues of classical inequalities have been obtained using the concept of q-integral on finite interval; for instance, Sudsutad et al. [11] obtained first q-analogues of certain classical inequalities. Alp et al. [2] gave

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a corrected q-analogue of Hermite–Hadamard's inequality. Noor et al. [10] obtained q-analogues of Ostrowski type inequalities using the convexity property of the functions. Zhang et al. [15] obtained a generalized q-integral identity and obtained many significant generalized q-analogues of classical integral identities. Another significant development in the field of quantum calculus is the introduction of post quantum calculus. In quantum calculus we deal with q-number with one base q; however, post quantum calculus includes p and q-numbers with two independent variables p and q. This was first investigated by Chakarabarti and Jagannathan [3]. It is worth to mention here that quantum calculus cannot be obtained directly by substituting q by $\frac{\mathrm{q}}{\mathrm{p}}$ in q-calculus. But q-calculus can be recaptured by taking $\mathrm{p}=1$ in (p,q)-calculus. Recently Tunc and Gov [13] gave the notion of (p,q)-derivatives and (p,q)-integrals on finite intervals as follows.

Definition 1.1 ([13]) Let $\Psi: I \to \mathbb{R}$ be a continuous function, and let $x \in I$ and $0 < \mathrm{q} < \mathrm{p} \leq 1$. Then the (p,q)-derivative on I of function Ψ at x is defined as

$$D^{R}_{\mathrm{p},\mathrm{q}}\Psi(x) = \frac{\Psi(px + (1-\mathrm{p})e)) - \Psi(qx + (1-\mathrm{q})e)}{(\mathrm{p}-\mathrm{q})(x-e)}, \quad x \neq e.$$

Definition 1.2 ([13]) Let $\Psi: I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the (p,q)-integral on I is defined as follows:

$$\int_{e}^{x} \Psi(\lambda)\, \mathrm{d}^{R}_{\mathrm{p},\mathrm{q}}\lambda = (\mathrm{p}-\mathrm{q})(x-e)\sum_{n=0}^{\infty} \frac{\mathrm{q}^{n}}{\mathrm{p}^{n+1}}\Psi\left(\frac{\mathrm{q}^{n}}{\mathrm{p}^{n+1}}x + \left(1-\frac{\mathrm{q}^{n}}{\mathrm{p}^{n+1}}\right)e\right)$$

for $x \in I$.

Convexity is one of the most important and significant notions in mathematical analysis. Although it is very simple in nature, it is very powerful. It has many applications in various areas of pure and applied sciences, such as in economics, management sciences, optimization theory, in engineering sciences, etc. Extensive study on the theory of convexity leads to many new extensions and generalizations of classical concepts of convex functions. Note that convex functions depend on convex sets. In the literature we can see several new diversified forms of the convex sets. This naturally leads us to several new generalizations of convex functions. Mititelu [9] introduced an important generalization of convex sets called invex sets.

Definition 1.3 ([9]) A set $\mathcal{X} \in \mathbb{R}$ is said to be invex with respect to ζ if

$$x + t\zeta(y,x) \in \mathcal{X}, \quad \forall x,y \in \mathcal{X}, t \in [0,1].$$

Note that if we take $\zeta(y,x) = y - x$, then invexity reduces to convexity. Using invex sets as domain, Weir and Mond [14] introduced the class of preinvex functions. This class is defined as follows.

Definition 1.4 ([14]) A function $\mathcal{F}: \mathcal{X} \to \mathbb{R}$ is said to be preinvex with respect to ζ if

$$\mathcal{F}\big(x + t\zeta(y,x)\big) \leq (1-t)\mathcal{F}(x) + t\mathcal{F}(y), \quad \forall x,y \in \mathcal{X}, t \in [0,1].$$

For $\zeta(y,x) = y - x$, the class of preinvex functions reduces to the class of convex functions.

Another charming aspect of the theory of convexity is its close relationship with the theory of inequalities. Many inequalities are direct consequences of the applications of the convexity property of functions. One of the most interesting results relating to convexity is Hermite–Hadamard's inequality which is just estimates for the integral average of a continuous convex function on a compact interval. It reads as follows:

Let $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then for $a, b \in I$ we have

$$\Psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \Psi(x)\,\mathrm{d}x \leq \frac{\Psi(a)+\Psi(b)}{2}.$$

This result has a lot of applications in numerical analysis and in theory of means. In recent years intensive study has been done on the generalizations and applications of Hermite–Hadamard's inequality. For example, Dragomir and Pearce [4] wrote a very informative monograph on the significance of Hermite–Hadamard's inequality, its recent development, and its applications. Interested readers can find very useful information in that book. Note that the left Hermite–Hadamard inequality can be estimated by the inequality of Ostrowski which reads as follows:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a)$$

with the best possible constant $\frac{1}{4}$ if $f : [a,b] \mapsto \mathbb{R}$ is differentiable, where $\|f'\|_\infty = \max\{|f(x)| : x \in [a,b]\}$.

Kunt et al. [8] used the concept of post quantum integrals and gave new generalizations of Hermite–Hadamard's inequality.

The motivation behind the study of this paper is to derive a new generalization of the classical Montgomery identity using (p,q)-integrals. We also give estimation of associated bounds essentially using the class of preinvex functions. Finally, we present some applications to means of the obtained results. We hope that ideas and techniques of this paper will attract interested readers.

2 Results and discussions

We now discuss our main results of the paper.

2.1 Post quantum Montgomery identity

We now derive a significant result of this paper. Next results of this paper depend on this lemma.

Lemma 2.1 *Let* $\Psi : [e, e+\xi(f,e)] \to \mathbb{R}$ *be a* (p,q)*-differentiable function such that* ${}_eD_{\mathrm{p},\mathrm{q}}\Psi$ *is* (p,q)*-integrable on* $[e, e+\xi(f,e)]$, *then*

$$\Psi(x) - \frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)} \Psi(\lambda)\,{}_e\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \xi(f,e)\int_0^1 K_{\mathrm{q}}(\lambda)\,{}_eD_{\mathrm{p},\mathrm{q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda,$$

where

$$K_{\mathrm{q}}(\lambda) = \begin{cases} \mathrm{q}\lambda & \textit{for } \lambda \in [0, \frac{x-e}{\xi(f,e)}], \\ \mathrm{q}\lambda - 1 & \textit{for } \lambda \in (\frac{x-e}{\xi(f,e)}, 1]. \end{cases}$$

Proof It suffices to show that

$$
\begin{aligned}
&\xi(f,e)\int_0^1 K_{\mathrm{q}}(\lambda)\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&= \xi(f,e)\bigg[\int_0^{\frac{x-e}{\xi(f,e)}} \mathrm{q}\lambda\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad + \int_{\frac{x-e}{\xi(f,e)}}^1 (\mathrm{q}\lambda-1)\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg] \\
&= \xi(f,e)\bigg[\int_0^{\frac{x-e}{\xi(f,e)}} \mathrm{q}\lambda\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad + \int_0^1 (\mathrm{q}\lambda-1)\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad - \int_0^{\frac{x-e}{\xi(f,e)}} (\mathrm{q}\lambda-1)\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg] \\
&= \xi(f,e)\bigg[\int_0^1 (\mathrm{q}\lambda-1)\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad + \int_0^{\frac{x-e}{\xi(f,e)}} {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg] \\
&= \xi(f,e)\bigg[\int_0^1 \mathrm{q}\lambda\, {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad - \int_0^1 {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad + \int_0^{\frac{x-e}{\xi(f,e)}} {}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg] \\
&= \frac{1}{\mathrm{p}-\mathrm{q}}\bigg[\mathrm{q}\bigg[\int_0^1 \Psi\big(e+\mathrm{q}\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda - \int_0^1 \Psi\big(e+\mathrm{q}\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg] \\
&\quad - \bigg[\int_0^1 \frac{\Psi(e+\mathrm{q}\lambda\xi(f,e))}{\lambda}\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda - \int_0^1 \frac{\Psi(e+\mathrm{q}\lambda\xi(f,e))}{\lambda}\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg] \\
&\quad + \bigg[\int_0^{\frac{x-e}{\xi(f,e)}} \frac{\Psi(e+\mathrm{q}\lambda\xi(f,e))}{\lambda}\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda - \int_0^{\frac{x-e}{\xi(f,e)}} \frac{\Psi(e+\mathrm{q}\lambda\xi(f,e))}{\lambda}\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\bigg]\bigg] \\
&= \frac{1}{\mathrm{p}-\mathrm{q}}\bigg[\mathrm{q}(\mathrm{p}-\mathrm{q})\bigg[\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\Psi\left(e+\frac{\mathrm{q}^n}{\mathrm{p}^n}\xi(f,e)\right) \\
&\quad - \sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\Psi\left(e+\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\xi(f,e)\right)\bigg] \\
&\quad - (\mathrm{p}-\mathrm{q})\bigg[\sum_{n=0}^{\infty}\Psi\left(e+\frac{\mathrm{q}^n}{\mathrm{p}^n}\xi(f,e)\right) \\
&\quad - \sum_{n=0}^{\infty}\Psi\left(e+\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\xi(f,e)\right)\bigg] \\
&\quad + (\mathrm{p}-\mathrm{q})\left(\frac{x-e}{\xi(f,e)}\right)\bigg[\sum_{n=0}^{\infty}\frac{\Psi(e+\frac{\mathrm{q}^n}{\mathrm{p}^n}(\frac{x-e}{\xi(f,e)})\xi(f,e))}{\frac{x-e}{\xi(f,e)}}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{n=0}^{\infty} \frac{\Psi(e + \frac{q^{n+1}}{p^{n+1}} (\frac{x-e}{\xi(f,e)}) \xi(f,e))}{\frac{x-e}{\xi(f,e)}} \Bigg] \Bigg] \\
&= \Bigg[q \Bigg[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + \frac{q^n}{p^n} \xi(f,e) \right) \\
&\quad - \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + \frac{q^{n+1}}{p^{n+1}} \xi(f,e) \right) \Bigg] \\
&\quad - \Bigg[\sum_{n=0}^{\infty} \Psi\left(e + \frac{q^n}{p^n} \xi(f,e) \right) \\
&\quad - \sum_{n=0}^{\infty} \Psi\left(e + \frac{q^{n+1}}{p^{n+1}} \xi(f,e) \right) \Bigg] \\
&\quad + \Bigg[\sum_{n=0}^{\infty} \Psi\left(e + \frac{q^n}{p^n} \left(\frac{x-e}{\xi(f,e)} \right) \xi(f,e) \right) \\
&\quad - \sum_{n=0}^{\infty} \Psi\left(e + \frac{q^{n+1}}{p^{n+1}} \left(\frac{x-e}{\xi(f,e)} \right) \xi(f,e) \right) \Bigg] \Bigg] \\
&= \Bigg[q \Bigg[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + p \frac{q^n}{p^{n+1}} \xi(f,e) \right) \\
&\quad - \frac{p}{q} \sum_{n=0}^{\infty} \frac{q^{n+1}}{p^{n+2}} \Psi\left(e + p \frac{q^{n+1}}{p^{n+2}} \xi(f,e) \right) \Bigg] \\
&\quad - \left[\Psi\left(e + \xi(f,e)\right) - \Psi(e) \right] \\
&\quad + \left[\Psi\left(e + \left(\frac{x-e}{\xi(f,e)} \right) \xi(f,e) \right) - \Psi(e) \right] \Bigg] \\
&= \Bigg[q \Bigg[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + p \frac{q^n}{p^{n+1}} \xi(f,e) \right) \\
&\quad - \frac{p}{q} \sum_{n=1}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + p \frac{q^n}{p^{n+1}} \xi(f,e) \right) \Bigg] \\
&\quad - \Psi\left(e + \xi(f,e)\right) + \Psi\left(e + \left(\frac{x-e}{\xi(f,e)} \right) \xi(f,e) \right) \Bigg] \\
&= \Bigg[q \Bigg[\frac{\Psi(e + \xi(f,e))}{q} - \left(\frac{p}{q} - 1 \right) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + p \frac{q^n}{p^{n+1}} \xi(f,e) \right) \Bigg] \\
&\quad - \Psi\left(e + \xi(f,e)\right) + \Psi\left(e + \left(\frac{x-e}{\xi(f,e)} \right) \xi(f,e) \right) \Bigg] \\
&= \Bigg[q \Bigg[\frac{\Psi(e + \xi(f,e))}{q} - \left(\frac{p-q}{q} \right) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(e + p \frac{q^n}{p^{n+1}} \xi(f,e) \right) \Bigg] \\
&\quad - \Psi\left(e + \xi(f,e)\right) + \Psi\left(e + \left(\frac{x-e}{\xi(f,e)} \right) \xi(f,e) \right) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \Psi(x) - (\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\Psi\left(e + \mathrm{p}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\xi(f,e)\right)\\
&= \Psi(x) - \frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)}\Psi(\lambda)\,{}_e\mathrm{d}_{\mathrm{p,q}}\lambda.
\end{aligned}
$$

This completes the proof. □

Remark 2.2 If we take $x = \frac{(\mathrm{p}+\mathrm{q})e+\mathrm{p}\xi(f,e)}{\mathrm{p}+\mathrm{q}}$ in Lemma 2.1, then we have the following new equality:

$$
\begin{aligned}
&\Psi\left(\frac{(\mathrm{p}+\mathrm{q})e+\mathrm{p}\xi(f,e)}{\mathrm{p}+\mathrm{q}}\right) - \frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)}\Psi(\lambda)\,{}_e\mathrm{d}_{\mathrm{p,q}}\lambda\\
&= \xi(f,e)\left[\int_0^{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}\mathrm{q}\lambda\,{}_eD_{\mathrm{p,q}}\Psi\big(+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\right.\\
&\quad\left. + \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^1(\mathrm{q}\lambda-1)\,{}_eD_{\mathrm{p,q}}\Psi\big(e+\lambda\xi(f,e)\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\right].
\end{aligned}
$$

2.2 Estimation of bounds

We now discuss some results depending upon Lemma 2.1.

Theorem 2.3 *Let* $\Psi : [e, e+\xi(f,e)]$ *be a function such that* ${}_eD_{\mathrm{p,q}}\Psi$ *is* (p,q)*-integrable on* $[e, e+\xi(f,e)]$. *If* $|{}_eD_{\mathrm{p,q}}\Psi|^r$, $r > 1$ *is preinvex on* $[e, e+\xi(f,e)]$, *then*

$$
\begin{aligned}
&\left|\Psi(x) - \frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)}\Psi(\lambda)\,{}_e\mathrm{d}_{\mathrm{p,q}}\lambda\right|\\
&\le \xi(f,e)\big[L_1^{1-\frac{1}{r}}\big[\big|{}_eD_{\mathrm{p,q}}\Psi(e)\big|^r L_2 + \big|{}_eD_{\mathrm{p,q}}\Psi(f)\big|^r L_3\big]^{\frac{1}{r}}\\
&\quad + L_4^{1-\frac{1}{r}}\big[\big|{}_eD_{\mathrm{p,q}}\Psi(e)\big|^r L_5 + \big|{}_eD_{\mathrm{p,q}}\Psi(f)\big|^r L_6\big]^{\frac{1}{r}}\big],
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= \int_0^{\frac{x-e}{\xi(f,e)}}\mathrm{q}\lambda\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda = \frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}}\left(\frac{x-e}{\xi(f,e)}\right)^2,\\
L_2 &= \int_0^{\frac{x-e}{\xi(f,e)}}\big(\mathrm{q}\lambda-\mathrm{q}\lambda^2\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda = L_1 - L_3,\\
L_3 &= \int_0^{\frac{x-e}{\xi(f,e)}}\mathrm{q}\lambda^2\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda = \frac{\mathrm{q}}{\mathrm{p}+pq+\mathrm{q}^2}\left(\frac{x-e}{\xi(f,e)}\right)^3,\\
L_4 &= \int_{\frac{x-e}{\xi(f,e)}}^1(1-\mathrm{q}\lambda)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda = \frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}} - \frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}}\left(\frac{x-e}{\xi(f,e)}\right)\left(1-\frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}}\frac{x-e}{\xi(f,e)}\right),\\
L_5 &= \int_{\frac{x-e}{\xi(f,e)}}^1\big(1-\mathrm{q}\lambda-\lambda+\mathrm{q}\lambda^2\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda = L_4 - L_6,\\
L_6 &= \int_{\frac{x-e}{\xi(f,e)}}^1\big(\lambda-\mathrm{q}\lambda^2\big)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\\
&= \frac{\mathrm{p}^2}{(\mathrm{p}+\mathrm{q})(\mathrm{p}^2+pq+\mathrm{q}^2)} - \frac{1}{\mathrm{p}+\mathrm{q}}\left(\frac{x-e}{\xi(f,e)}\right)^2 + \frac{\mathrm{q}}{\mathrm{p}^2+pq+\mathrm{q}^2}\left(\frac{x-e}{\xi(f,e)}\right)^3.
\end{aligned}
$$

Proof Using Lemma 2.1, the power mean integral inequality, and the preinvexity of $|{}_eD_{p,q}\Psi|^r$, we obtain

$$\begin{aligned}
&\left|\Psi(x)-\frac{1}{p\xi(f,e)}\int_e^{e+p\xi(f,e)}\Psi(\lambda)\,{}_e d_{p,q}\lambda\right| \\
&\quad\le \xi(f,e)\biggl[\int_0^{\frac{x-e}{\xi(f,e)}} q\lambda\bigl|{}_eD_{p,q}\Psi\bigl(e+\lambda\xi(f,e)\bigr)\bigr|{}_0 d_{p,q}\lambda \\
&\qquad+\int_{\frac{x-e}{\xi(f,e)}}^1 (1-q\lambda)\bigl|{}_eD_{p,q}\Psi\bigl(e+\lambda\xi(f,e)\bigr)\bigr|{}_0 d_{p,q}\lambda\biggr] \\
&\quad\le \xi(f,e)\biggl[\biggl(\int_0^{\frac{x-e}{\xi(f,e)}} q\lambda\,{}_0 d_{p,q}\lambda\biggr)^{1-\frac{1}{r}}\biggl(\int_0^{\frac{x-e}{\xi(f,e)}} q\lambda\bigl|{}_eD_{p,q}\Psi\bigl(e+\lambda\xi(f,e)\bigr)\bigr|^r\,{}_0 d_{p,q}\lambda\biggr)^{\frac{1}{r}} \\
&\qquad+\biggl(\int_{\frac{x-e}{\xi(f,e)}}^1 (1-q\lambda)\,{}_0 d_{p,q}\lambda\biggr)^{1-\frac{1}{r}}\biggl(\int_{\frac{x-e}{\xi(f,e)}}^1 (1-q\lambda)\bigl|{}_eD_{p,q}\Psi\bigl(e+\lambda\xi(f,e)\bigr)\bigr|^r\,{}_0 d_{p,q}\lambda\biggr)^{\frac{1}{r}}\biggr] \\
&\quad\le \xi(f,e)\biggl[\biggl(\int_0^{\frac{x-e}{\xi(f,e)}} q\lambda\,{}_0 d_{p,q}\lambda\biggr)^{1-\frac{1}{r}} \\
&\qquad\times\biggl(\bigl|{}_eD_{p,q}\Psi(e)\bigr|^r\int_0^{\frac{x-e}{\xi(f,e)}}\bigl(q\lambda-q\lambda^2\bigr)\,{}_0 d_{p,q}\lambda+\bigl|{}_eD_{p,q}\Psi(f)\bigr|^r\int_0^{\frac{x-e}{\xi(f,e)}} q\lambda^2\,{}_0 d_{p,q}\lambda\biggr)^{\frac{1}{r}} \\
&\qquad+\biggl(\int_{\frac{x-e}{\xi(f,e)}}^1 (1-q\lambda)\,{}_0 d_{p,q}\lambda\biggr)^{1-\frac{1}{r}} \\
&\qquad\times\biggl(\bigl|{}_eD_{p,q}\Psi(e)\bigr|^r\int_{\frac{x-e}{\xi(f,e)}}^1\bigl(1-q\lambda-\lambda+q\lambda^2\bigr)\,{}_0 d_{p,q}\lambda \\
&\qquad+\bigl|{}_eD_{p,q}\Psi(f)\bigr|^r\int_{\frac{x-e}{\xi(f,e)}}^1\bigl(\lambda-q\lambda^2\bigr)\,{}_0 d_{p,q}\lambda\biggr)^{\frac{1}{r}}\biggr],
\end{aligned}$$

The proof is accomplished. □

Corollary 2.4 *In Theorem 2.3, the following quantum estimates hold under the following conditions:*

I. $r=1$

$$\begin{aligned}
&\left|\Psi(x)-\frac{1}{\xi(f,e)}\int_e^{e+\xi(f,e)}\Psi(\lambda)\,{}_e d_{p,q}\lambda\right| \\
&\quad\le \xi(f,e)\bigl[\bigl|{}_eD_{p,q}\Psi(e)\bigr|[L_2+L_5]+\bigl|{}_eD_{p,q}\Psi(f)\bigr|[L_3+L_6]\bigr].
\end{aligned}$$

II. $x=\frac{(p+q)e+p\xi(f,e)}{p+q}$, *then we have a new inequality*:

$$\begin{aligned}
&\left|\Psi(x)-\frac{1}{p\xi(f,e)}\int_e^{e+p\xi(f,e)}\Psi(\lambda)\,{}_e d_{p,q}\lambda\right| \\
&\quad\le \xi(f,e)\Bigl[L_7^{1-\frac{1}{r}}\bigl[\bigl|{}_eD_{p,q}\Psi(e)\bigr|^r L_8+\bigl|{}_eD_{p,q}\Psi(f)\bigr|^r L_9\bigr]^{\frac{1}{r}} \\
&\qquad+L_{10}^{1-\frac{1}{r}}\bigl[\bigl|{}_eD_{p,q}\Psi(e)\bigr|^r L_{11}+\bigl|{}_eD_{p,q}\Psi(f)\bigr|^r L_{12}\bigr]^{\frac{1}{r}}\Bigr],
\end{aligned}$$

where

$$L_7 = \int_0^{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}} \mathrm{q}\lambda \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}}\left(\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}\right)^2,$$

$$L_8 = \int_0^{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}} \left(\mathrm{q}\lambda - \mathrm{q}\lambda^2\right) {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = L_7 - L_9,$$

$$L_9 = \int_0^{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}} \mathrm{q}\lambda^2 \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}}{\mathrm{p}+pq+\mathrm{q}^2}\left(\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}\right)^3,$$

$$L_{10} = \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^1 (1-\mathrm{q}\lambda) \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}}\left(1-\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}\right)\left(1-\frac{\mathrm{q}}{\mathrm{p}+\mathrm{q}}\frac{x-e}{\xi(f,e)}\right),$$

$$L_{11} = \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^1 \left(1-\mathrm{q}\lambda - \lambda + \mathrm{q}\lambda^2\right) {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = L_{10} - L_{12},$$

$$L_{12} = \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^1 \left(\lambda - \mathrm{q}\lambda^2\right) {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{1}{\mathrm{p}+\mathrm{q}} - \frac{\mathrm{p}^2}{(\mathrm{p}+\mathrm{q})^3} + \frac{1}{\mathrm{p}^2+\mathrm{q}^2+pq}\left(1-\left(\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}\right)^3\right).$$

Theorem 2.5 *Let $\Psi : [e, e+\xi(f,e)]$ be a function such that ${}_eD_{\mathrm{p},\mathrm{q}}\Psi$ is (p,q)-integrable on $[e, e+\xi(f,e)]$. If $|{}_eD_{\mathrm{p},\mathrm{q}}\Psi|^r, r > 1, s^{-1}+r^{-1} = 1$ is preinvex on $[e, e+\xi(f,e)]$, then*

$$\begin{aligned}
&\left|\Psi(x) - \frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)} \Psi(\lambda) \, {}_e\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda\right| \\
&\quad \le \mathrm{q}\xi(f,e)\left[K_1^{\frac{1}{s}}\left[\left|{}_eD_{\mathrm{p},\mathrm{q}}\Psi(e)\right|^r K_2 + \left|{}_eD_{\mathrm{p},\mathrm{q}}\Psi(f)\right|^r K_3\right]^{\frac{1}{r}}\right. \\
&\qquad \left. + K_4^{\frac{1}{s}}\left[\left|{}_eD_{\mathrm{p},\mathrm{q}}\Psi(e)\right|^r K_5 + \left|{}_eD_{\mathrm{p},\mathrm{q}}\Psi(f)\right|^r K_6\right]^{\frac{1}{r}}\right],
\end{aligned}$$

where

$$K_1 = \int_0^{\frac{x-e}{\xi(f,e)}} \lambda^s \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \left(\frac{x-e}{\xi(f,e)}\right)^{s+1}\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}},$$

$$K_2 = \int_0^{\frac{x-e}{\xi(f,e)}} (1-\lambda) \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{x-e}{\xi(f,e)} - \frac{1}{\mathrm{p}+\mathrm{q}}\left(\frac{x-e}{\xi(f,e)}\right)^2,$$

$$K_3 = \int_0^{\frac{x-e}{\xi(f,e)}} \lambda \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{1}{\mathrm{p}+\mathrm{q}}\left(\frac{x-e}{\xi(f,e)}\right)^2,$$

$$\begin{aligned}
K_4 &= \int_{\frac{x-e}{\xi(f,e)}}^1 \left(\frac{1}{\mathrm{q}} - \lambda\right)^s {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda \\
&= (\mathrm{p}-\mathrm{q})\left[\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\left(\frac{1}{\mathrm{q}} - \frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)^s\right. \\
&\quad \left. - \frac{x-e}{\xi(f,e)}\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\left(\frac{1}{\mathrm{q}} - \frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\left(\frac{x-e}{\xi(f,e)}\right)\right)^s\right],
\end{aligned}$$

$$K_5 = \int_{\frac{x-e}{\xi(f,e)}}^1 (1-\lambda) \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{p}+\mathrm{q}-1}{\mathrm{p}+\mathrm{q}} - \frac{x-e}{\xi(f,e)} + \frac{1}{\mathrm{p}+\mathrm{q}}\left(\frac{x-e}{\xi(f,e)}\right)^2,$$

$$K_6 = \int_{\frac{x-e}{\xi(f,e)}}^1 \lambda \, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{1}{\mathrm{p}+\mathrm{q}}\left(1-\left(\frac{x-e}{\xi(f,e)}\right)^2\right).$$

Proof Using Lemma 2.1, Holder's inequality, and the preinvexity of $|{}_eD_{\mathrm{p,q}}\Psi|^r$, we obtain

$$\begin{aligned}
&\left|\Psi(x)-\frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)}\Psi(\lambda)\,{}_e\mathrm{d}_{\mathrm{p,q}}\lambda\right| \\
&\le \xi(f,e)\biggl[\int_0^{\frac{x-e}{\xi(f,e)}}\mathrm{q}\lambda\bigl|{}_eD_{\mathrm{p,q}}\Psi\bigl(e+\xi(f,e)\bigr)\bigr|{}_0\mathrm{d}_{\mathrm{p,q}}\lambda \\
&\quad+\int_{\frac{x-e}{\xi(f,e)}}^1(1-\mathrm{q}\lambda)\bigl|{}_eD_{\mathrm{p,q}}\Psi\bigl(e+\xi(f,e)\bigr)\bigr|{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr] \\
&\le \xi(f,e)\biggl[\biggl(\int_0^{\frac{x-e}{\xi(f,e)}}(\mathrm{q}\lambda)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{s}}\biggl(\int_0^{\frac{x-e}{\xi(f,e)}}\bigl|{}_eD_{\mathrm{p,q}}\Psi\bigl(e+\xi(f,e)\bigr)\bigr|^r\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{r}} \\
&\quad+\biggl(\int_{\frac{x-e}{\xi(f,e)}}^1(1-\mathrm{q}\lambda)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{s}}\biggl(\int_{\frac{x-e}{\xi(f,e)}}^1\bigl|{}_eD_{\mathrm{p,q}}\Psi\bigl(e+\xi(f,e)\bigr)\bigr|^r\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{r}}\biggr] \\
&\le \mathrm{q}\xi(f,e)\biggl[\biggl(\int_0^{\frac{x-e}{\xi(f,e)}}\lambda^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{s}} \\
&\quad\times\biggl(\bigl|{}_eD_{\mathrm{p,q}}\Psi(e)\bigr|^r\int_0^{\frac{x-e}{\xi(f,e)}}(1-\lambda)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda+\bigl|{}_eD_{\mathrm{p,q}}\Psi(f)\bigr|^r\int_0^{\frac{x-e}{\xi(f,e)}}\lambda\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{r}} \\
&\quad+\biggl(\int_{\frac{x-e}{\xi(f,e)}}^1\biggl(\frac{1}{\mathrm{q}}-\lambda\biggr)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{s}} \\
&\quad\times\biggl(\bigl|{}_eD_{\mathrm{p,q}}\Psi(e)\bigr|^r\int_{\frac{x-e}{\xi(f,e)}}^1(1-\lambda)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda+\bigl|{}_eD_{\mathrm{p,q}}\Psi(f)\bigr|^r\int_{\frac{x-e}{\xi(f,e)}}^1\lambda\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda\biggr)^{\frac{1}{r}}\biggr].
\end{aligned}$$

The proof is accomplished. □

Remark 2.6 If we take $x=\frac{e+\mathrm{q}(e+\xi(f,e))}{1+\mathrm{q}}$ in Theorem 2.5, then we have a new inequality:

$$\begin{aligned}
&\left|\Psi(x)-\frac{1}{\mathrm{p}\xi(f,e)}\int_e^{e+\mathrm{p}\xi(f,e)}\Psi(\lambda)\,{}_e\mathrm{d}_{\mathrm{p,q}}\lambda\right| \\
&\le \mathrm{q}\xi(f,e)\bigl[K_7^{\frac{1}{s}}\bigl[\bigl|{}_eD_{\mathrm{p,q}}\Psi(e)\bigr|^rK_8+\bigl|{}_eD_{\mathrm{p,q}}\Psi(f)\bigr|^rK_9\bigr]^{\frac{1}{r}} \\
&\quad+K_{10}^{\frac{1}{s}}\bigl[\bigl|{}_eD_{\mathrm{p,q}}\Psi(e)\bigr|^rK_{11}+\bigl|{}_eD_{\mathrm{p,q}}\Psi(f)\bigr|^rK_{12}\bigr]^{\frac{1}{r}}\bigr],
\end{aligned}$$

where

$$\begin{aligned}
K_7&=\int_0^{\frac{\mathrm{p}}{\mathrm{p+q}}}\lambda^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda=\biggl(\frac{\mathrm{p}}{\mathrm{p+q}}\biggr)^{s+1}\frac{\mathrm{p-q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}}, \\
K_8&=\int_0^{\frac{\mathrm{p}}{\mathrm{p+q}}}(1-\lambda)\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda=\frac{\mathrm{p}}{\mathrm{p+q}}-\frac{\mathrm{p}^2}{(\mathrm{p+q})^3}, \\
K_9&=\int_0^{\frac{\mathrm{p}}{\mathrm{p+q}}}\lambda\,{}_0\mathrm{d}_{\mathrm{p,q}}\lambda=\frac{\mathrm{p}^2}{(\mathrm{p+q})^3},
\end{aligned}$$

$$\begin{aligned}
K_{10} &= \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^{1} \left(\frac{1}{\mathrm{q}} - \lambda\right)^{s} {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda, \\
&= (\mathrm{p}-\mathrm{q})\Bigg[\sum_{n=0}^{\infty} \frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\left(\frac{1}{\mathrm{q}} - \frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)^{s} \\
&\quad - \frac{1}{\mathrm{p}+\mathrm{q}} \sum_{n=0}^{\infty} \frac{\mathrm{q}^n}{\mathrm{p}^n}\left(\frac{1}{\mathrm{q}} - \frac{\mathrm{q}^n}{\mathrm{p}^n}\left(\frac{1}{\mathrm{p}+\mathrm{q}}\right)\right)^{s}\Bigg], \\
K_{11} &= \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^{1} (1-\lambda)\, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}-1}{\mathrm{p}+\mathrm{q}} - \frac{\mathrm{p}^2}{(\mathrm{p}+\mathrm{q})^3}, \\
K_{12} &= \int_{\frac{\mathrm{p}}{\mathrm{p}+\mathrm{q}}}^{1} \lambda\, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{1}{\mathrm{p}+\mathrm{q}} - \frac{\mathrm{p}^2}{(\mathrm{p}+\mathrm{q})^3}.
\end{aligned}$$

2.3 Applications

We now discuss some applications of the results obtained in the previous section. First of all we recall some previously known concepts. For arbitrary real numbers, consider the following means:

$$\text{Arithmetic mean}: \quad A(e,f) = \frac{e+f}{2},$$

$$\text{Generalized logarithmic mean}: \quad L_{\mathrm{q}}(e,f) = \left[\frac{f^{\mathrm{q}+1} - e^{\mathrm{q}+1}}{(\mathrm{q}+1)(f-e)}\right]^{\frac{1}{\mathrm{q}}},$$

where $\mathrm{q} \in \mathbb{R} \setminus \{-1, 0\}$, $e,f \in \mathbb{R}$ with $e \neq f$.

Proposition 2.7 *Let* $0 < e < f$, $n \in \mathbb{N}$, $0 < \mathrm{q} < \mathrm{p} < 1$, *then*

$$\begin{aligned}
&\left|A^n(e,f) - \frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}(\mathrm{p}^{n+1} - \mathrm{q}^{n+1})} L_n^n\big(e, (1-\mathrm{p})e + pf\big)\right| \\
&\quad \le (f-e)\Bigg[H_1^{1-\frac{1}{r}}\left[\left|ne^{n-1}\right|^r H_2 + \left|\frac{(pf + (1-\mathrm{p})e)^n - (qf + (1-\mathrm{q})e)^n}{(f-e)(1-\mathrm{q})}\right|^r H_3\right]^{\frac{1}{r}} \\
&\quad\quad + H_4^{1-\frac{1}{r}}\left[\left|ne^{n-1}\right|^r H_5 + \left|\frac{(pf + (1-\mathrm{p})e)^n - (qf + (1-\mathrm{q})e)^n}{(f-e)(1-\mathrm{q})}\right|^r H_6\right]^{\frac{1}{r}}\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
H_1 &= \int_0^{\frac{1}{2}} \mathrm{q}\lambda\, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}}{4(\mathrm{p}+\mathrm{q})}, \\
H_2 &= \int_0^{\frac{1}{2}} \left(\mathrm{q}\lambda - \mathrm{q}\lambda^2\right) {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}(\mathrm{p} + 2pq + 2\mathrm{q}^2 - \mathrm{q})}{8(\mathrm{p}+\mathrm{q})(\mathrm{p}^2 + pq + \mathrm{q}^2)}, \\
H_3 &= \int_0^{\frac{1}{2}} \mathrm{q}\lambda^2\, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{\mathrm{q}}{8(\mathrm{p} + pq + \mathrm{q}^2)}, \\
H_4 &= \int_{\frac{1}{2}}^{1} (1-\mathrm{q}\lambda)\, {}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\lambda = \frac{2\mathrm{p}-\mathrm{q}}{4(\mathrm{p}+\mathrm{q})},
\end{aligned}$$

$$H_5 = \int_{\frac{1}{2}}^{1} \left(1 - q\lambda - \lambda + q\lambda^2\right) {}_0d_{p,q}\lambda = \frac{4p^3 + 2pq^2 + 2p^2q - 2q^3 - 6p^2 + pq + q^2}{8(p+q)(p^2+pq+q^2)},$$

$$H_6 = \int_{\frac{1}{2}}^{1} \left(\lambda - q\lambda^2\right) {}_0d_{p,q}\lambda = \frac{6p^2 - pq - q^2}{8(p+q)(p^2+pq+q^2)}.$$

Proof The proof directly follows from Theorem 2.3 applied for $\Psi(x) = x^n$, $\xi(f,e) = f - e$ and considering $x = \frac{e+f}{2}$. □

Proposition 2.8 *Let* $0 < e < f$, $n \in \mathbb{N}$, $0 < q < 1$, *then*

$$\begin{aligned}
&\left| A^n(e,f) - \frac{(n+1)(p-q)}{p(p^{n+1} - q^{n+1})} L_n^n\bigl(e,(1-p)e + pf\bigr) \right| \\
&\quad \le q(f-e)\left[M_1^{\frac{1}{s}} \left[\left| ne^{n-1} \right|^r M_2 + \left| \frac{(pf + (1-p)e)^n - (qf + (1-q)e)^n}{(f-e)(1-q)} \right|^r M_3 \right]^{\frac{1}{r}} \right. \\
&\qquad \left. + M_4^{\frac{1}{s}} \left[\left| ne^{n-1} \right|^r M_5 + \left| \frac{(pf + (1-p)e)^n - (qf + (1-q)e)^n}{(f-e)(1-q)} \right|^r M_6 \right]^{\frac{1}{r}} \right],
\end{aligned}$$

where

$$M_1 = \int_0^{\frac{1}{2}} \lambda^s {}_0d_{p,q}\lambda = \frac{1}{2^{s+1}} \frac{p-q}{p^{s+1} - q^{s+1}},$$

$$M_2 = \int_0^{\frac{1}{2}} (1-\lambda) {}_0d_{p,q}\lambda = \frac{2p + 2q - 1}{4(p+q)},$$

$$M_3 = \int_0^{\frac{1}{2}} \lambda {}_0d_{p,q}\lambda = \frac{1}{4(p+q)},$$

$$\begin{aligned}
M_4 &= \int_{\frac{1}{2}}^{1} \left(\frac{1}{q} - \lambda\right)^s {}_0d_{p,q}\lambda \\
&= (p-q)\left[\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(\frac{1}{q} - \frac{q^n}{p^{n+1}}\right)^s \right. \\
&\qquad \left. - \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(\frac{1}{q} - \frac{q^n}{2p^{n+1}}\right)^s \right],
\end{aligned}$$

$$M_5 = \int_{\frac{1}{2}}^{1} (1-\lambda) {}_0d_{p,q}\lambda = \frac{2p + 2q - 3}{4(p+q)},$$

$$M_6 = \int_{\frac{1}{2}}^{1} \lambda {}_0d_{p,q}\lambda = \frac{3}{4(p+q)}.$$

Proof The proof directly follows from Theorem 2.5 applied for $\Psi(x) = x^n$, $\xi(f,e) = f - e$ and considering $x = \frac{e+f}{2}$. □

3 Conclusion

In the article, we have found a new post quantum version of Montgomery identity and presented several inequalities involving the post quantum bounds via certain properties of the preinvex functions. Our obtained results are the generalizations and improvements of some previously known results. Moreover, we also provided some applications to support our obtained results.

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