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Dunkl-type generalization of the second kind beta operators via (p,q)-calculus

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Abstract

The main purpose of this research article is to construct a Dunkl extension of (p,q)-variant of Szász–Beta operators of the second kind by applying a new parameter. We obtain Korovkin-type approximation theorems, local approximations, and weighted approximations. Further, we study the rate of convergence by using the modulus of continuity, Lipschitz class and Peetre's K-functionals.

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1 Introduction and preliminaries

The q-analogues of Bernstein operators were independently given by Lupaş [25] and Phillips [42]. Consequently, Mursaleen et al. [33] applied the (p,q)-integers and studied the approximation properties of Bernstein operators. Recently, a Dunkl-type generalization of Szász operators [47] via post-quantum calculus was studied by Alotaibi et al. [10]. For more details and research motivation in Dunkl-type generalizations, we mention here some research articles [13, 27, 34, 35, 37–39, 45, 46]. Let $[s]_{p,q}$ be the (p,q)-integer defined as

$$[s]_{p,q} = p^{s-1} + qp^{s-3} + \cdots + q^{s-1} = \begin{cases} \frac{p^s-q^s}{p-q} & (p \neq q \neq 1), \\ \frac{1-q^s}{1-q} & (p = 1), \\ s & (p = q = 1), \end{cases} \tag{1.1}$$

$$(au+bv)^s_{p,q} := \sum_{\ell=0}^{s} p^{\frac{(s-\ell)(s-\ell-1)}{2}} q^{\frac{\ell(\ell-1)}{2}} \begin{bmatrix} s \\ \ell \end{bmatrix}_{p,q} a^{s-\ell} b^{\ell} u^{s-\ell} v^{\ell},$$

$$(1-u)^s_{p,q} = (1-u)(p-qu)\bigl(p^2-q^2u\bigr)\cdots\bigl(p^{s-1}-q^{s-1}u\bigr),$$

$$(u-y)^s_{p,q} = \begin{cases} \prod_{j=0}^{s-1}(p^j u - q^j y) & \text{if } s \in \mathbb{N}, \\ 1 & \text{if } s = 0. \end{cases}$$

The (p,q)-power basis is explained as

$$(u \oplus v)_{p,q}^{s} = (u+v)(pu+qv)\left(p^{2}u+q^{2}v\right)\cdots\left(p^{s-1}u+q^{s-1}v\right).$$

Furthermore, the (p,q)-analogues of the exponential function are defined by

$$e_{p,q}(u) = \sum_{\ell=0}^{\infty} p^{\frac{\ell(\ell-1)}{2}} \frac{u^{\ell}}{[\ell]_{p,q}!}, \qquad E_{p,q}(u) = \sum_{\ell=0}^{\infty} q^{\frac{\ell(\ell-1)}{2}} \frac{u^{\ell}}{[\ell]_{p,q}!}.$$

Moreover, the (p,q)-Dunkl analogue of the exponential function is defined by

$$e_{\tau,p,q}(u) = \sum_{\ell=0}^{\infty} p^{\frac{\ell(\ell-1)}{2}} \frac{u^{\ell}}{\gamma_{\tau,p,q}(\ell)}, \tag{1.2}$$

$$\gamma_{\tau,p,q}(\ell) = \frac{\prod_{i=0}^{[\frac{\ell+1}{2}]-1} p^{2\tau(-1)^{i+1}+1}((p^{2})^{i}p^{2\tau+1}-(q^{2})^{i}q^{2\tau+1}) \prod_{j=0}^{[\frac{\ell}{2}]-1} p^{2\tau(-1)^{j}+1}((p^{2})^{j}p^{2}-(q^{2})^{j}q^{2})}{(p-q)^{\ell}}. \tag{1.3}$$

And a recursion identity is defined as

$$\gamma_{\tau,p,q}(\ell+1) = \frac{p^{2\tau(-1)^{\ell+1}+1}(p^{2\tau\theta_{\ell+1}+\ell+1}-q^{2\tau\theta_{\ell+1}+\ell+1})}{(p-q)} \gamma_{\tau,p,q}(\ell), \tag{1.4}$$

where

$$\theta_{\ell} = \begin{cases} 0 & \text{for } \ell = 2m, m = 0,1,2,\ldots, \\ 1 & \text{for } \ell = 2m+1, m = 0,1,2,\ldots \end{cases} \tag{1.5}$$

For $m = 0,1,2,\ldots s$, the number $[\frac{m}{2}]$ denotes the greatest integer function evaluated at $m/2$.

In our demonstration, we let $u \geq 0$ and $C[0,\infty)$ be the class of all continuous functions on $[0,\infty)$. Recent investigation in [10, 38] defined the (p,q)-Dunkl analogue of Szász operators by

$$D_{s,p,q}(f;u) = \frac{1}{e_{\tau,p,q}([s]_{p,q}u)} \sum_{\ell=0}^{\infty} \frac{([s]_{p,q}u)^{\ell}}{\gamma_{\tau,p,q}(\ell)} p^{\frac{\ell(\ell-1)}{2}} f\left(\frac{p^{\ell+2\tau\theta_{\ell}}-q^{\ell+2\tau\theta_{\ell}}}{p^{\ell-1}(p^{s}-q^{s})}\right). \tag{1.6}$$

2 Operators and basic estimates

In this section we construct a class of (p,q)-variant of Szász–Beta operators of the second kind generated by an exponential function via Dunkl generalization in Definition 2.1. Such operators are a generalized version of the operators studied in [7, 22, 28, 29, 31, 36, 45].

Definition 2.1 Let $f \in C_{\zeta}[0,\infty) = \{f(t) : f(t) = O(t^{\zeta}), t \to \infty, f \in C[0,\infty)\}$ and consider $u \geq 0$, $\zeta > s$, and $s \in \mathbb{N}$. Then for all $0 < q < p \leq 1$, $\tau > -\frac{1}{2}$, and θ_{ℓ} given by (1.5), we define

$$\mathcal{P}_{s,p,q}^{\tau}(f;u) = \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{1}{\mathcal{B}_{p,q}(\ell+2\tau\theta_{\ell}+1,s)} \int_{0}^{\infty} \frac{t^{\ell+2\tau\theta_{\ell}}}{(1\oplus pt)_{p,q}^{\ell+2\tau\theta_{\ell}+s+1}} f(t)\, \mathrm{d}_{p,q}t, \tag{2.1}$$

where

$$\mathcal{Q}_{s,p,q}(u) = \frac{1}{e_{\tau,p,q}([s]_{p,q}u)} \frac{([s]_{p,q}u)^\ell}{\gamma_{\tau,p,q}(\ell)} p^{\frac{\ell(\ell-1)}{2}},$$

and $\mathcal{B}_{p,q}(\ell + 2\tau\theta_\ell + 1, s)$ is the Beta function of the second kind in post-quantum calculus defined by

$$\mathcal{B}_{p,q}(\alpha,\beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1 \oplus pt)_{p,q}^{\alpha+\beta}} \,\mathrm{d}_{p,q}t, \quad \alpha,\beta \in \mathbb{N}, \tag{2.2}$$

where a formula for the (p,q)-Beta function is given by

$$\mathcal{B}_{p,q}(\alpha,\beta) = \frac{[\alpha-1]_{p,q}}{p^{\alpha-1}[\beta]_{p,q}} \mathcal{B}_{p,q}(\alpha-1,\beta+1), \quad \alpha,\beta \in \mathbb{N}. \tag{2.3}$$

Moreover, to obtain the basic estimates here, we use the following relations:

$$[\ell + 1 + 2\tau\theta_\ell]_{p,q} = q[\ell + 2\tau\theta_\ell]_{p,q} + p^{\ell+2\tau\theta_\ell}, \tag{2.4}$$

$$[\ell + 2 + 2\tau\theta_\ell]_{p,q} = q^2[\ell + 2\tau\theta_\ell]_{p,q} + (p+q)p^{\ell+2\tau\theta_\ell}. \tag{2.5}$$

For more related results on (p,q)-analogues, we refer to [1–6, 8, 9, 11, 14–21, 26, 30, 43, 44, 48] and also see [12, 32, 40], for example, if $p = 1$, the operators $\mathcal{P}^\tau_{s,p,q}$ reduce to those considered recently (see [45]). We have the following inequalities.

Lemma 2.2 *Let $f(t) = 1, t, t^2$. Then the operators $\mathcal{P}^\tau_{s,p,q}(\cdot;\cdot)$ defined by (2.1) satisfy $\mathcal{P}^\tau_{s,p,q}(1;u) = 1$, and the following inequalities hold*:

$$\mathcal{P}^\tau_{s,p,q}(f;u) \leq \begin{cases} \frac{[s]_{p,q}}{[s-1]_{p,q}}u + \frac{1}{[s-1]_{p,q}} & \text{for } f(t) = t, \\ \frac{[s]^2_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}u^2 + \frac{[s]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}(1 + [2]_{p,q} + [1+2\tau]_{p,q})u \\ \quad + \frac{[2]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}} & \text{for } f(t) = t^2, \end{cases} \tag{2.6}$$

and

$$\mathcal{P}^\tau_{s,p,q}(f;u) \geq \begin{cases} \frac{q[s]_{p,q}}{[s-1]_{p,q}}u + \frac{1}{[s-1]_{p,q}} & \text{for } f(t) = t, \\ \frac{q^3[s]^2_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}u^2 \\ \quad + \frac{q[s]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}(q + [2]_{p,q} + q^{2+2\tau}[1-2\tau]_{p,q}\frac{e_{\tau,p,q}(\frac{q}{p}[s]_{p,q}u)}{e_{\tau,p,q}([s]_{p,q}u)})u \\ \quad + \frac{[2]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}} & \text{for } f(t) = t^2. \end{cases}$$

Proof To prove the results of this lemma, we use (2.2)–(2.5). Take $f(t) = 1$. Then

$$\begin{aligned} \mathcal{P}^\tau_{s,p,q}(1;u) &= \sum_{\ell=0}^\infty \mathcal{Q}_{s,p,q}(u) \frac{1}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} \int_0^\infty \frac{t^{\ell+2\tau\theta_\ell}}{(1\oplus pt)_{p,q}^{\ell+2\tau\theta_\ell+s+1}} \,\mathrm{d}_{p,q}t \\ &= \sum_{\ell=0}^\infty \mathcal{Q}_{s,p,q}(u) \frac{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} = 1. \end{aligned}$$

If $f(t) = t$, then

$$\begin{aligned}
\mathcal{P}^{\tau}_{s,p,q}(t;u) &= \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{1}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} \int_0^{\infty} \frac{t^{\ell+2\tau\theta_\ell+1}}{(1\oplus pt)_{p,q}^{\ell+2\tau\theta_\ell+s+1}}\,\mathrm{d}_{p,q}t \\
&= \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+2,s-1)}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} \\
&= \frac{q}{[s-1]_{p,q}} \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{1}{p^{\ell+2\tau\theta_\ell+1}} [\ell+2\tau\theta_\ell]_{p,q} + \frac{1}{p[s-1]_{p,q}} \\
&= \frac{1}{p[s-1]_{p,q}} + \frac{q[s]_{p,q}}{p^2[s-1]_{p,q}} \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \left(\frac{p^{2\ell+2\tau\theta_{2\ell}} - q^{2\ell+2\tau\theta_{2\ell}}}{p^{2\ell-1}(p^s-q^s)} \right) \\
&\quad + \frac{q[s]_{p,q}}{p^{2+2\tau}[s-1]_{p,q}} \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \left(\frac{p^{2\ell+1+2\tau\theta_{2\ell+1}} - q^{2\ell+1+2\tau\theta_{2\ell+1}}}{p^{2\ell}(p^s-q^s)} \right).
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
\mathcal{P}^{\tau}_{s,p,q}(t;u) &\geq \frac{1}{[s-1]_{p,q}} + \frac{q[s]_{p,q}}{[s-1]_{p,q}} \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \left(\frac{p^{\ell+2\tau\theta_\ell} - q^{\ell+2\tau\theta_\ell}}{p^{\ell-1}(p^s-q^s)} \right) \\
&= \frac{1}{[s-1]_{p,q}} + \frac{q[s]_{p,q}}{[s-1]_{p,q}} D_{s,p,q}(t;u) \\
&= \frac{1}{[s-1]_{p,q}} + \frac{q[s]_{p,q}}{[s-1]_{p,q}} u
\end{aligned}$$

and

$$\mathcal{P}^{\ell,\tau}_{s,p,q}(t;u) \leq \frac{1}{[s-1]_{p,q}} + \frac{[s]_{p,q}}{[s-1]_{p,q}} u.$$

Similarly, for $f(t) = t^2$, we have

$$\begin{aligned}
\mathcal{P}^{\tau}_{s,p,q}(t^2;u) &= \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{1}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} \int_0^{\infty} \frac{t^{\ell+2\tau\theta_\ell+2}}{(1\oplus pt)_{p,q}^{\ell+2\tau\theta_\ell+s+1}}\,\mathrm{d}_{p,q}t \\
&= \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+3,s-2)}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} \\
&= \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+3,s-2)}{\mathcal{B}_{p,q}(\ell+2\tau\theta_\ell+1,s)} \\
&= \frac{1}{[s-1]_{p,q}[s-2]_{p,q}} \\
&\quad \times \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{1}{p^{3+2\ell+4\tau\theta_\ell+1}} [\ell+2\tau\theta_\ell+1]_{p,q}[\ell+2\tau\theta_\ell+2]_{p,q} \\
&= \frac{q^3[s]_{p,q}^2}{[s-1]_{p,q}[s-2]_{p,q}} \sum_{\ell=0}^{\infty} \mathcal{Q}_{s,p,q}(u) \frac{1}{p^{5+4\tau\theta_\ell}} \left(\frac{p^{\ell+2\tau\theta_\ell} - q^{\ell+2\tau\theta_\ell}}{p^{\ell-1}(p^s-q^s)} \right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{q(p+2q)[s]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}\sum_{\ell=0}^{\infty}\mathcal{Q}_{s,p,q}(u)\frac{1}{p^{4+2\tau\theta_\ell}}\left(\frac{p^{\ell+2\tau\theta_\ell}-q^{\ell+2\tau\theta_\ell}}{p^{\ell-1}(p^s-q^s)}\right)\\
&+\frac{(p+q)}{p^3[s-1]_{p,q}[s-2]_{p,q}}\sum_{\ell=0}^{\infty}\mathcal{Q}_{s,p,q}(u).
\end{aligned}
$$

Now by separating the even and odd terms and applying θ_ℓ from (1.5), i.e., taking $\ell=2m$ and $\ell=2m+1$ for all $m=0,1,2,\dots$, we have

$$
\begin{aligned}
\mathcal{P}^{\tau}_{s,p,q}(t^2;u) &\geq \frac{q^3[s]^2_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}\sum_{\ell=0}^{\infty}\mathcal{Q}_{s,p,q}(u)\left(\frac{p^{\ell+2\tau\theta_\ell}-q^{\ell+2\tau\theta_\ell}}{p^{\ell-1}(p^s-q^s)}\right)^2\\
&\quad+\frac{q(q+[2]_{p,q})[s]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}\sum_{\ell=0}^{\infty}\mathcal{Q}_{s,p,q}(u)\left(\frac{p^{\ell+2\tau\theta_\ell}-q^{\ell+2\tau\theta_\ell}}{p^{\ell-1}(p^s-q^s)}\right)\\
&\quad+\frac{[2]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}\\
&=\frac{q^3[s]^2_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}D_{s,p,q}(t^2;u)+\frac{q(q+[2]_{p,q})[s]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}D_{s,p,q}(t;u)\\
&\quad+\frac{[2]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{P}^{\tau}_{s,p,q}(t^2;u) &\leq \frac{[s]^2_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}D_{s,p,q}(t^2;u)+\frac{(1+[2]_{p,q})[s]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}D_{s,p,q}(t;u)\\
&\quad+\frac{[2]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}.
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

Lemma 2.3 *Let $\Phi_j=(t-u)^j$ for $j=1,2$, then we have following inequalities*:

1. $\mathcal{P}^{\tau}_{s,p,q}(\Phi_1;u)\leq\left(\frac{[s]_{p,q}}{[s-1]_{p,q}}-1\right)u+\frac{1}{[s-1]_{p,q}}$, *for $s>1, s\in\mathbb{N}$*,
2. $$
\begin{aligned}
\mathcal{P}^{\tau}_{s,p,q}(\Phi_2;u) &\leq \left(\frac{[s]^2_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}}-\frac{2[s]_{p,q}}{[s-1]_{p,q}}+1\right)u^2\\
&\quad+\frac{1}{[s-1]_{p,q}}\left(\frac{[s]_{p,q}}{[s-2]_{p,q}}(1+[2]_{p,q}+[1+2\tau]_{p,q})-2\right)u\\
&\quad+\frac{[2]_{p,q}}{[s-1]_{p,q}[s-2]_{p,q}},\quad \textit{for } s>2, s\in\mathbb{N}.
\end{aligned}
$$

3 Approximation results

Let us denote by $C_B[0,\infty)$ the set of all bounded and continuous functions defined on $[0,\infty)$, equipped with the norm $\|f\|_{C_B}=\sup_{u\geq 0}|f(u)|$. We write

$$
\mathcal{L}:=\left\{f:\lim_{u\to\infty}\frac{f(u)}{1+u^2}\text{ exits}\right\},
$$

$$B_\sigma[0,\infty) := \{f : |f(u)| \leq \mathcal{M}_f \sigma(u)\},$$

where $\mathcal{M}_f$ is a constant depending on f, and σ is the weight function with $\sigma(u) = 1 + u^2$. Moreover,

$$C_\sigma[0,\infty) := B_\sigma[0,\infty) \cap C[0,\infty),$$
$$C_\sigma^k[0,\infty) := \left\{f : f \in C_\sigma[0,\infty) \text{ and } \lim_{u\to\infty} \frac{f(u)}{\sigma(u)} = k < \infty\right\}.$$

Note that $C_\sigma[0,\infty)$ is a normed space with the norm given by $\|f\|_\sigma = \sup_{u\geq 0} \frac{|f(u)|}{\sigma(u)}$.

Theorem 3.1 *Take the sequences of positive numbers $q = q_s$, $p = p_s$ satisfying $q_s \in (0,1)$, $p_s \in (q_s, 1]$ such that $\lim_{s\to\infty} q_s = 1$, $\lim_{s\to\infty} p_s = 1$. Then, $\mathcal{P}^\tau_{s,p_s,q_s}$ is uniformly convergent on each compact subset of $[0,\infty)$ and such that*

$$\lim_{s\to\infty} \mathcal{P}^\tau_{s,p_s,q_s}(f;u) = f(u),$$

where $f \in C[0,\infty) \cap \mathcal{L}$.

Proof To prove the uniform convergence on each compact subset of $[0,\infty)$, it is obvious from the well-known Korovkin's theorem [23] that $\lim_{s\to\infty} \mathcal{P}^{\ell,\tau}_{s,p_s,q_s}(t^\eta;u) = u^\eta$ for $\eta = 0,1,2$. Whenever, $q_s = 1$, $p_s = 1$ as $s \to \infty$, then clearly for all $i = 1,2$ we have $\frac{1}{[s-i]_{p_s,q_s}} \to 0$, $\frac{[s]_{p_s,q_s}}{[s-i]_{p_s,q_s}} \to 1$, which imply that

$$\lim_{s\to\infty} \mathcal{P}^\tau_{s,p_s,q_s}(1;u) = 1, \qquad \lim_{s\to\infty} \mathcal{P}^\tau_{s,p_s,q_s}(t;u) = u, \qquad \lim_{s\to\infty} \mathcal{P}^\tau_{s,p_s,q_s}(t^2;u) = u^2. \qquad \square$$

Theorem 3.2 *For each $f \in C_\sigma^k[0,\infty)$, consider the sequences of positive numbers $0 < q_s < p_s \leq 1$ such that $\lim_{s\to\infty} q_s = 1$, $\lim_{s\to\infty} p_s = 1$. Then the operators $\mathcal{P}^\tau_{s,p_s,q_s}$ satisfy*

$$\lim_{s\to\infty} \left\|\mathcal{P}^\tau_{s,p_s,q_s}(f) - f\right\|_\sigma = 0. \tag{3.1}$$

Proof We take $f(t) = t^\eta$ with $\eta = 0,1,2$. From Theorem 3.1, since $\mathcal{P}^\tau_{s,p_s,q_s}(t^\eta;u)$ is uniformly convergent to u^η for all $\eta = 0,1,2$, and applying Lemma 2.2, we conclude that

$$\lim_{s\to\infty} \left\|\mathcal{P}^\tau_{s,p_s,q_s}(1) - 1\right\|_\sigma = 0. \tag{3.2}$$

For $\eta = 1$,

$$\begin{aligned}\left\|\mathcal{P}^\tau_{s,p_s,q_s}(t) - u\right\|_\sigma &= \sup_{u\geq 0} \frac{|\mathcal{P}^\tau_{s,p_s,q_s}(t;u) - u|}{1+u^2} \\ &\leq \left(\frac{[s]_{p_s,q_s}}{[s-1]_{p_s,q_s}} - 1\right) \sup_{u\geq 0} \frac{u}{1+u} + \frac{1}{[s-1]_{p_s,q_s}} \sup_{u\geq 0} \frac{1}{1+u}.\end{aligned}$$

Then

$$\lim_{s\to\infty} \left\|\mathcal{P}^\tau_{s,p_s,q_s}(t) - u\right\|_\sigma = 0. \tag{3.3}$$

Similarly, if we take $\eta = 2$, then

$$\begin{aligned}
\left\|\mathcal{P}^{\tau}_{s,p_s,q_s}\left(t^2\right) - u^2\right\|_{\sigma} &= \sup_{u\geq 0} \frac{|\mathcal{P}^{\tau}_{s,p_s,q_s}(t^2;u) - u^2|}{1+u^2} \\
&\leq \left(\frac{[s]^2_{p_s,q_s}}{[s-1]_{p_s,q_s}[s-2]_{p,q}} - 1\right) \sup_{u\geq 0} \frac{u^2}{1+u^2} \\
&\quad + \frac{[s]_{p_s,q_s}}{[s-1]_{p_s,q_s}[s-2]_{p_s,q_s}}\left(1 + [2]_{p_s,q_s} + [1+2\tau]_{p_s,q_s}\right) \sup_{u\geq 0} \frac{u}{1+u^2} \\
&\quad + \frac{[2]_{p_s,q_s}}{[s-1]_{p_s,q_s}[s-2]_{p,q}} \sup_{u\geq 0} \frac{1}{1+u^2},
\end{aligned}$$

$$\lim_{s\to\infty}\left\|\mathcal{P}^{\tau}_{s,p_s,q_s}\left(t^2\right) - u^2\right\|_{\sigma} = 0. \tag{3.4}$$

This completes the proof. □

Let

$$\omega_{\mu}(f;\delta) = \sup_{|t-u|\leq\delta} \sup_{u,t\in[0,\mu]} \left|f(t) - f(u)\right|. \tag{3.5}$$

It is obvious that $\lim_{\delta\to 0+}\omega_{\mu}(f;\delta) = 0$ and for $f \in C[0,\infty)$,

$$\left|f(t) - f(u)\right| \leq \left(\frac{|t-u|}{\delta} + 1\right)\omega_{\mu}(f;\delta). \tag{3.6}$$

Theorem 3.3 *Let $f \in C_{\sigma}[0,\infty)$, and $0 < q_s < p_s \leq 1$ be such that $\lim_{s\to\infty} q_s = 1$, $\lim_{s\to\infty} p_s = 1$. Moreover, suppose $\omega_{\mu}(f;\delta)$ is defined by (3.5) on the interval $[0,\mu+1] \subset [0,\infty)$, for $\mu > 0$. Then for every $s > 2$, we get*

$$\left|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u) - f(u)\right| \leq 2\omega_{\mu+1}\left(f;\delta_s(u)\right) + 6\mathcal{C}_f\left(1+\mu^2\right)\left(\delta_s(u)\right)^2,$$

where $\mathcal{C}_f$ is a constant depending only on f and $\delta_s(u) = \sqrt{\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)}$.

Proof For $u \in [0,\mu]$ and $t \leq \mu + 1$, with $\mu > 0$, we have

$$\left|f(t) - f(u)\right| \leq \mathcal{C}_f\left(2 + u^2 + t^2\right) \leq 6\mathcal{C}_f\left(1+\mu^2\right)(t-u)^2. \tag{3.7}$$

Furthermore, for any $\delta > 0$, $u \in [0,\mu]$, and $t > \mu + 1$, with $\mu > 0$,

$$\left|f(t) - f(u)\right| \leq \omega_{\mu+1}\left(f;|t-u|\right) \leq \left(1 + \frac{|t-u|}{\delta}\right)\omega_{\mu+1}(f;\delta). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\left|f(t) - f(u)\right| \leq 6\mathcal{C}_f\left(1+\mu^2\right)(t-u)^2 + \left(1 + \frac{|t-u|}{\delta}\right)\omega_{\mu+1}(f;\delta). \tag{3.9}$$

Applying operators $\mathcal{P}^{\tau}_{s,p_s,q_s}$ and the well-known Cauchy–Schwartz inequality, we have

$$\begin{aligned}\mathcal{P}^{\tau}_{s,p_s,q_s}\big(\big|f(t)-f(u)\big|;u\big) &\leq 6\mathcal{C}_f\big(1+\mu^2\big)\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)\\ &\quad+\mathcal{P}^{\tau}_{s,p_s,q_s}\left(1+\frac{|t-u|}{\delta};u\right)\omega_{\mu+1}(f;\delta).\end{aligned}$$

Moreover, for any $g\in C_\sigma[0,\infty)$, we know

$$\begin{aligned}\mathcal{P}^{\tau}_{s,p_s,q_s}(g;u)-g(u) &= \mathcal{P}^{\tau}_{s,p_s,q_s}(g;u)-g(u)\mathcal{P}^{\tau}_{s,p_s,q_s}(1;u)\\ &= \mathcal{P}^{\tau}_{s,p_s,q_s}\big(g(t)-g(u);u\big)\\ &\leq \mathcal{P}^{\tau}_{s,p_s,q_s}\big(\big|g(t)-g(u)\big|;u\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\big|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u)-f(u)\big| &\leq 6\mathcal{C}_f\big(1+\mu^2\big)\mathcal{P}^{\tau}_{s,p_s,q_s}(t-u)^2\\ &\quad+\left(1+\frac{1}{\delta}\mathcal{P}^{\tau}_{s,p_s,q_s}\big(|t-u|;u\big)\right)\omega_{\mu+1}(f;\delta)\\ &\leq 6\mathcal{C}_f\big(1+\mu^2\big)\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)\\ &\quad+\left(1+\frac{1}{\delta}\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)^{\frac{1}{2}}\right)\omega_{\mu+1}(f;\delta),\end{aligned}$$

where

$$\mathcal{P}^{\tau}_{s,p_s,q_s}\big(|t-u|;u\big)\leq\mathcal{P}^{\tau}_{s,p_s,q_s}(1;u)^{\frac{1}{2}}\mathcal{P}^{\tau}_{s,p_s,q_s}\big((t-u)^2;u\big)^{\frac{1}{2}}=\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)^{\frac{1}{2}}.$$

Finally, if we choose $\delta=\delta_s(u)=\sqrt{\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)}$, then we get the desired result. □

4 Rate of convergence

In 1963, to measure the smoothness, a mathematical formula of a certain functional was given by Peetre [41]. For all $\delta>0$ and $f\in C[0,\infty)$, Peetre defined the K-functional, which we write as $K_2(f;\delta)$. The formulas below give its definition, as well as a bound for some constant $\mathcal{C}>0$ and the second-order modulus of continuity $\omega_2(f;\delta)$ defined as follows:

$$K_2(f;\delta)=\inf_{u\geq0}\big\{\big(\|f-\psi\|_{C_B[0,\infty)}+\delta\big\|\psi''\big\|_{C_B[0,\infty)}\big):\psi\in C_B^2[0,\infty)\big\},\tag{4.1}$$

$$K_2(f;\delta)\leq\mathcal{C}\big\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\|f\|_{C_B[0,\infty)}\big\},$$

$$\omega_2(f;\delta)=\sup_{0<v<\delta}\sup_{u\geq0}\big|f(u+2v)-2f(u+v)+f(u)\big|.\tag{4.2}$$

Theorem 4.1 *Let $q=q_s$, $p=p_s$ with $q_s\in(0,1)$, $p_s\in(q_s,1]$ and $\mathcal{R}^{\tau}_{s,p,q}(f;u)=\mathcal{P}^{\tau}_{s,p,q}(f;u)+f(u)-f(\frac{[s]_{p,q}u+1}{[s-1]_{p,q}})$. Then, for every $\psi\in C_B^2[0,\infty)$ and $s>2$, we have*

$$\big|\mathcal{R}^{\tau}_{s,p_s,q_s}(\psi;u)-\psi(u)\big|\leq\chi_n(u)\big\|\psi''\big\|,$$

where $\chi_n(u)=\delta_s^2(u)+(\mathcal{P}_{s,p,q}(\Phi_1;u))^2$, in which $\delta_s(u)$ is defined in Theorem 3.3 and $\mathcal{P}_{s,p,q}(\Phi_1;u)$ is defined by Lemma 2.3.

Proof Let $\psi \in C_B^2[0,\infty)$. We easily get $\mathcal{R}^{\tau}_{s,p_s,q_s}(1;u) = 1$ and

$$\mathcal{R}^{\tau}_{s,p_s,q_s}(t;u) = \mathcal{P}^{\tau}_{s,p_s,q_s}(t;u) + u - \left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}}\right) = u.$$

Also

$$\begin{aligned} &\left\|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u)\right\| \leq \|f\|, \\ &\left|\mathcal{R}^{\tau}_{s,p_s,q_s}(f;u)\right| \leq \left|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u)\right| + \left|f(u)\right| - \left|f\left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}}\right)\right| \leq 3\|f\|. \end{aligned} \tag{4.3}$$

From the Taylor series expansion, we have

$$\psi(t) = \psi(u) + (t-u)\psi'(u) + \int_u^t (t-\alpha)\psi''(\alpha)\,\mathrm{d}\alpha.$$

Applying the operator $\mathcal{R}^{\tau}_{s,p_s,q_s}$, we conclude that

$$\begin{aligned} \mathcal{R}^{\tau}_{s,p_s,q_s}(\psi;u) - \psi(u) &= \psi'(u)\mathcal{R}^{\tau}_{s,p_s,q_s}(t-u;u) + \mathcal{R}^{\tau}_{s,p_s,q_s}\left(\int_u^t (t-\alpha)\psi''(\alpha)\,\mathrm{d}\alpha;u\right) \\ &= \mathcal{R}^{\tau}_{s,p_s,q_s}\left(\int_u^t (t-\alpha)\psi''(\alpha)\,\mathrm{d}\alpha;u\right) \\ &= \mathcal{P}^{\tau}_{s,p_s,q_s}\left(\int_u^t (t-\alpha)\psi''(\alpha)\,\mathrm{d}\alpha;u\right) \\ &\quad - \int_u^{\frac{[s]_{p_s,q_s}u+1}{[s-1]_{p_s,q_s}}} \left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}} - \alpha\right)\psi''(\alpha)\,\mathrm{d}\alpha \\ \left|\mathcal{R}^{\tau}_{s,p_s,q_s}(\psi;u) - \psi(u)\right| &\leq \left|\mathcal{P}^{\tau}_{s,p_s,q_s}\left(\int_u^t (t-\alpha)\psi''(\alpha)\,\mathrm{d}\alpha;u\right)\right| \\ &\quad + \left|\int_u^{\frac{[s]_{p_s,q_s}u+1}{[s-1]_{p_s,q_s}}} \left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}} - \alpha\right)\psi''(\alpha)\,\mathrm{d}\alpha\right|. \end{aligned}$$

Since

$$\left|\int_u^t (t-\alpha)\psi''(\alpha)\,\mathrm{d}\alpha\right| \leq (t-u)^2\left\|\psi''\right\|,$$

we conclude that

$$\left|\int_u^{\frac{[s]_{p_s,q_s}u+1}{[s-1]_{p_s,q_s}}} \left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}} - \alpha\right)\psi''(\alpha)\,\mathrm{d}\alpha\right| \leq \left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}} - u\right)^2\left\|\psi''\right\|.$$

Hence,

$$\left|\mathcal{R}^{\tau}_{s,p_s,q_s}(\psi;u) - \psi(u)\right| \leq \left\{\mathcal{P}^{\tau}_{s,p_s,q_s}\left((t-u)^2;u\right) + \left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}} - u\right)^2\right\}\left\|\psi''\right\|.$$

Thus we complete the proof. □

Theorem 4.2 *Let $q = q_s$, $p = p_s$ with $q_s \in (0,1)$, $p_s \in (q_s, 1]$ and $f \in C_B[0,\infty)$. Then, for every $\psi \in C_B^2[0,\infty)$ and $s > 2$ there exits a positive constant $\mathcal{C} >$ satisfying the inequality*

$$\begin{aligned}\left|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u) - f(u)\right| \leq \mathcal{A}&\left\{\omega_2\left(f;\frac{\sqrt{\chi_s(u)}}{2}\right) + \min\left(1,\frac{\chi_s(u)}{4}\right)\|f\|\right\}\\ &+ \omega\left(f;\left|\mathcal{P}_{s,p_s,q_s}(\Phi_1;u)\right|\right).\end{aligned}$$

Proof For all $f \in C_B[0,\infty)$ and $\psi \in C_B^2[0,\infty)$, it is very easy to see the result from Theorem 4.1. Indeed,

$$\begin{aligned}\left|\mathcal{R}^{\tau}_{s,p_s,q_s}(f;u) - f(u)\right| &= \left|\mathcal{R}^{\tau}_{s,p_s,q_s}(f;u) - f(u) + f\left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p,q}}\right) - f(u)\right|\\ &\leq \left|\mathcal{R}^{\tau}_{s,p_s,q_s}(f-\psi;u)\right| + \left|\mathcal{R}^{\tau}_{s,p_s,q_s}(\psi;u) - \psi(u)\right|\\ &\quad + \left|\psi(u) - f(u)\right| + \left|f\left(\frac{[s]_{p_s,q_s}u + 1}{[s-1]_{p_s,q_s}}\right) - f(u)\right|\\ &\leq 4\|f-\psi\| + \chi_s(u)\left\|\psi''\right\|\\ &\quad + \omega\left(f;\left|\left(\frac{[s]_{p_s,q_s}}{[s-1]_{p_s,q_s}} - 1\right)u + \frac{1}{[s-1]_{p_s,q_s}}\right|\right).\end{aligned}$$

By taking the infimum over all $\psi \in C_B^2[0,\infty)$ and using (4.1), we get

$$\begin{aligned}\left|\mathcal{R}^{\tau}_{s,p_s,q_s}(f;u) - f(u)\right| &\leq 4K_2\left(f;\frac{\chi_s(u)}{4}\right) + \omega\left(f;\left|\left(\frac{[s]_{p_s,q_s}}{[s-1]_{p_s,q_s}} - 1\right)u + \frac{1}{[s-1]_{p_s,q_s}}\right|\right)\\ &\leq \mathcal{A}\left\{\omega_2\left(f;\frac{\sqrt{\chi_s(u)}}{2}\right) + \min\left(1;\frac{\chi_s(u)}{4}\right)\|f\|\right\}\\ &\quad + \omega\left(f;\left|\left(\frac{[s]_{p_s,q_s}}{[s-1]_{p_s,q_s}} - 1\right)u + \frac{1}{[s-1]_{p_s,q_s}}\right|\right).\end{aligned}$$

□

We consider the following Lipschitz-type maximal function [24] and obtain the local approximation. For $f \in C[0,\infty]$, $0 < \kappa \leq 1$ and $t,u \geq 0$, we recall that

$$\mathrm{Lip}_M(\kappa) = \left\{f : \left|f(t) - f(u)\right| \leq M|t-u|^{\kappa}\right\}. \tag{4.4}$$

Theorem 4.3 *For all $\kappa \in (0,1]$, $s > 2$, and $f \in C_B[0,\infty)$, we have*

$$\left|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u) - f(u)\right| \leq M\left(\delta_s(u)\right)^{\kappa},$$

where $\delta_s(u)$ is given in Theorem 3.3.

Proof We prove the claim by applying (4.4) and the well-known Hölder's inequality:

$$\begin{aligned}\left|\mathcal{P}^{\tau}_{s,p_s,q_s}(f;u) - f(u)\right| &\leq \mathcal{P}^{\tau}_{s,p_s,q_s}\left(\left|f(t) - f(u)\right|;u\right)\\ &\leq M|\mathcal{P}^{\tau}_{s,p_s,q_s}\left(|t-u|^{\kappa};u\right)\end{aligned}$$

$$\begin{aligned}&\leq M\big(\mathcal{P}^{\tau}_{s,p_s,q_s}(1;u)\big)^{\frac{2-\kappa}{2}}\big(\mathcal{P}^{\tau}_{s,p_s,q_s}\big(|t-u|^2;u\big)\big)^{\frac{\kappa}{2}}\\&= M\big(\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)\big)^{\frac{\kappa}{2}}.\end{aligned}$$

This gives the desired result. □

We denote

$$C^2_B[0,\infty)=\big\{\psi:\psi\in C_B[0,\infty)\text{ and }\psi',\psi''\in C_B[0,\infty)\big\},\tag{4.5}$$

$$\|\psi\|_{C^2_B[0,\infty)}=\|\psi\|_{C_B[0,\infty)}+\big\|\psi'\big\|_{C_B[0,\infty)}+\big\|\psi''\big\|_{C_B[0,\infty)},\tag{4.6}$$

$$\|\psi\|_{C_B[0,\infty)}=\sup_{u\geq 0}\big|\psi(u)\big|.\tag{4.7}$$

Theorem 4.4 *Let the positive sequences of numbers* $0<q_s<p_s\leq 1$ *satisfy* $\lim_{s\to\infty}q_s=1$, $\lim_{s\to\infty}p_s=1$. *Then for all* $\psi\in C^2_B[0,\infty)$, *the operators* $\mathcal{P}^{\tau}_{s,p_s,q_s}$ *have the property*

$$\big|\mathcal{P}^{\tau}_{s,p_s,q_s}(\psi;u)-\psi(u)\big|\leq\Theta_s(u)\|\psi\|_{C^2_B[0,\infty)},\tag{4.8}$$

where $\Theta_s(u)=\sqrt{\delta_s(u)}+\frac{(\delta_s(u))^2}{2}$.

Proof Let $\psi\in C^2_B[0,\infty)$. Then

$$\psi(t)=\psi(u)+\psi'(u)(t-u)+\psi''(\varphi)\frac{(t-u)^2}{2}\quad\text{for }\varphi\in(u,t),$$

where if we take

$$\begin{aligned}\mathcal{S}&=\sup_{u\geq 0}\big|\psi'(u)\big|=\big\|\psi'\big\|_{C_B[0,\infty)}\leq\|\psi\|_{C^2_B[0,\infty)},\\\mathcal{T}&=\sup_{u\geq 0}\big|\psi''(u)\big|=\big\|\psi''\big\|_{C_B[0,\infty)}\leq\|\psi\|_{C^2_B[0,\infty)},\end{aligned}$$

then we have

$$\begin{aligned}\big|\psi(t)-\psi(u)\big|&\leq\mathcal{S}|t-u|+\frac{1}{2}\mathcal{T}(t-u)^2\\&\leq\left(|t-u|+\frac{1}{2}(t-u)^2\right)\|\psi\|_{C^2_B[0,\infty)}.\end{aligned}$$

Therefore

$$\begin{aligned}\big|\mathcal{P}^{\tau}_{s,p_s,q_s}(\psi;u)-\psi(u)\big|&\leq\left(\mathcal{P}^{\tau}_{s,p_s,q_s}\big(|t-u|;u\big)+\frac{1}{2}\mathcal{P}^{\tau}_{s,p_s,q_s}\big((t-u)^2;u\big)\right)\|\psi\|_{C^2_B[0,\infty)}\\&\leq\left(\big(\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)\big)^{\frac{1}{2}}+\frac{1}{2}\mathcal{P}^{\tau}_{s,p_s,q_s}(\Phi_2;u)\right)\|\psi\|_{C^2_B[0,\infty)}.\end{aligned}$$

This completes the proof of Theorem 4.4. □

5 Conclusion

We constructed a (p,q)-variant of Szász operators by using the Beta functions of the second kind by introducing the Dunkl generalization. We obtained the approximation results involving local and global approximations in Korovkin's and weighted Korovkin's spaces. We applied some techniques of earlier investigation and discussed the convergence of operators by employing the modulus of continuity, Lipschitz class and Peetre's K-functionals.

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