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Identities of symmetry for Bernoulli polynomials and power sums

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Abstract

Identities of symmetry in two variables for Bernoulli polynomials and power sums had been investigated by considering suitable symmetric identities. T. Kim used a completely different tool, namely the p -adic Volkenborn integrals, to find the same identities of symmetry in two variables. Not much later, it was observed that this p -adic approach can be generalized to the case of three variables and shown that it gives some new identities of symmetry even in the case of two variables upon specializing one of the three variables. In this paper, we generalize the results in three variables to those in an arbitrary number of variables in a suitable setting and illustrate our results with some examples.

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1 Introduction and preliminaries

Tuenter [17] obtained the following identity of symmetry involving the Bernoulli numbers and the power sums. This was done by showing that the exponential generating function of the sum on the left-hand side of (1) is invariant under the interchange of w_1 and w_2 .

$$\sum_{i=0}^n \binom{n}{i} B_i S_{n-i}(w_1 - 1) w_1^{i-1} w_2^{n-i} = \sum_{i=0}^n \binom{n}{i} B_i S_{n-i}(w_2 - 1) w_2^{i-1} w_1^{n-i}, \quad (1)$$

where w_1, w_2 are any positive integers, n is a nonnegative integer, B_n are Bernoulli numbers in (8), $S_k(n)$ are the power sums in (10).

When $w_2 = 1$, equation (1) reduces to the following recurrence relation for the Bernoulli numbers:

$$B_n = \frac{1}{w_1(1 - w_1^n)} \sum_{i=0}^{n-1} \binom{n}{i} B_i S_{n-i}(w_1 - 1) w_1^i, \quad (2)$$

which was proved by Deeba and Rodriguez [2] and Gessel [3]. Actually, it was a conjecture posed by Namias who found identity (2) for $w_1 = 2, 3$ by using the multiplication formula for the gamma function (see [2]).

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A completely different approach to identities of symmetry was given in [12], where the p -adic Volkenborn integrals for uniformly differential functions (see (5)) were used. In particular, the following two identities were obtained:

$$\sum_{i=0}^n \binom{n}{i} B_i(w_2x) S_{n-i}(w_1 - 1) w_1^{i-1} w_2^{n-i} = \sum_{i=0}^n \binom{n}{i} B_i(w_1x) S_{n-i}(w_2 - 1) w_2^{i-1} w_1^{n-i}, \tag{3}$$

$$\sum_{i=0}^{w_1-1} B_n\left(w_2x + \frac{w_2}{w_1}i\right) w_1^{n-1} = \sum_{i=0}^{w_2-1} B_n\left(w_1x + \frac{w_1}{w_2}i\right) w_2^{n-1}. \tag{4}$$

We note here that (3) becomes (1) when $x = 0$, and that (3) and (4) can be generalized to higher-order Bernoulli polynomials (see [18]). It turns out that this p -adic approach to identities of symmetry has the merit of being easily generalized. In addition, the identities of symmetry can be found also for Euler polynomials [9], q -Bernoulli polynomials [5, 13, 14], and q -Euler polynomials [6], respectively by using p -adic fermionic integrals, p -adic q -Volkenborn integrals, and p -adic fermionic q -integrals [11]. For the q -Bernoulli and q -Euler polynomials, we let the reader refer to the papers [1, 4, 16]. Indeed, in [10] many identities of symmetry for three variables were obtained for the first time by adopting the p -adic Volkenborn integral approach initiated in [12] (see Example 3.1). As was mentioned in Sect. 1 of [10], by specializing the variable w_3 as 1 in (c-1), (c-2), and (c-3) of Example 3.1, it was shown that (3) and (4) are all equal and that they are further equal to the following identities:

$$\begin{aligned} & \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) S_\ell(w_1 - 1) S_m(w_2 - 1) w_1^{k+m-1} w_2^{k+\ell-1} \\ &= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k\left(y_1 + \frac{i}{w_1}\right) S_{n-k}(w_2 - 1) w_2^{k-1} \\ &= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k\left(y_1 + \frac{i}{w_2}\right) S_{n-k}(w_1 - 1) w_1^{k-1} \\ &= (w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} B_n\left(y_1 + \frac{i}{w_1} + \frac{j}{w_2}\right). \end{aligned}$$

Thus we may say that the abundance of symmetries in (c-1), (c-2), and (c-3) shed new light even on the existing identities in two variables. These would not be unearthed if more symmetries had not been available. Moreover, the identities of symmetry for higher-order Bernoulli polynomials in two variables in [18], which was done by manipulations of identities, can be done by using the p -adic Volkenborn integrals so as to give abundant symmetries in three variables (see [7, 8]).

The aim of this paper is to generalize the results in three variables [10] to those in an arbitrary number of variables in a suitable setting. Theorem 2.2 is the main result of this paper. Further, we illustrate our results with examples in Sect. 3. In the rest of this section, we recall the facts that are needed throughout this paper.

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p -adic integers, the field of p -adic rational numbers, and the completion of the

algebraic closure of \mathbb{Q}_p . For a uniformly differentiable function $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$, the p -adic Volkenborn integral of f is defined by

$$\int_{\mathbb{Z}_p} f(z) d\mu(z) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j). \tag{5}$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z + 1) d\mu(z) = \int_{\mathbb{Z}_p} f(z) d\mu(z) + f'(z). \tag{6}$$

Let $|\cdot|_p$ be the normalized non-Archimedean absolute value of \mathbb{C}_p such that $|p|_p = \frac{1}{p}$, and let

$$E = \{t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p-1}}\}. \tag{7}$$

Then, for each fixed $t \in E$, the function $f(z) = e^{zt}$ is analytic on \mathbb{Z}_p , and by applying (6) to this f , we get the p -adic integral expression of the generating function for Bernoulli numbers B_n given by

$$\int_{\mathbb{Z}_p} e^{zt} d\mu(z) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (t \in E). \tag{8}$$

So we have the following p -adic integral expression of the generating function for the Bernoulli polynomials $B_n(x)$:

$$\int_{\mathbb{Z}_p} e^{(x+z)t} d\mu(z) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (t \in E, x \in \mathbb{Z}_p). \tag{9}$$

Here and throughout this paper, we will have many instances to be able to interchange integral and infinite sum. That is justified by Proposition 55.4 in [15]. Let $S_k(n)$ denote the k th power sum of the first $n + 1$ nonnegative integers, namely

$$S_k(n) = \sum_{i=0}^n i^k = 0^k + 1^k + \dots + n^k. \tag{10}$$

In particular, we have

$$S_0(n) = n + 1, \quad S_k(0) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases} \tag{11}$$

From (8) and (10), one easily derives the following identities: for $w \in \mathbb{Z}_{>0}$,

$$\frac{w \int_{\mathbb{Z}_p} e^{xt} d\mu(x)}{\int_{\mathbb{Z}_p} e^{wyt} d\mu(y)} = \sum_{i=0}^{w-1} e^{it} \tag{12}$$

$$= \sum_{k=0}^{\infty} S_k(w-1) \frac{t^k}{k!} \quad (t \in E). \tag{13}$$

Let $n \geq 2$, and let $I_n = \{1, 2, \dots, n\}$. Then the symmetric group S_n acts on I_n in a natural way as $(\sigma, j) \mapsto \sigma(j)$. For each integer j with $1 \leq j \leq n$, let Ω_j be the subset of 2^{I_n} consisting of all j -element subsets of I_n .

Proposition 1.1 *Let Ω be a nonempty subset of 2^{I_n} not containing the empty set ϕ . Then there is an action of S_n on Ω induced by the natural action of S_n on I_n if and only if $\Omega = \bigcup_{j \in J} \Omega_j$ for some nonempty subset J of I_n . Moreover, such an action of S_n on Ω is transitive if and only if $\Omega = \Omega_j$ for some j ($1 \leq j \leq n$).*

Proof (\Leftarrow) For each j ($1 \leq j \leq n$), the natural action of S_n on I_n induces an action of S_n on Ω_j , which is obviously transitive. In turn, this induces an action of S_n on $\Omega = \bigcup_{j \in J} \Omega_j$ for some nonempty subset J of I_n .

(\Rightarrow) Assume that there is an action of S_n on Ω induced by the natural action of S_n on I_n . If $A \in \Omega$, with $|A| = j$ ($1 \leq j \leq n$), then $\sigma A \in \Omega$ for all $\sigma \in S_n$, and hence $\Omega_j \subset \Omega$. This shows that $\Omega = \bigcup_{j \in J} \Omega_j$ for some nonempty subset J of I_n . For the second statement, note that the action cannot be transitive if $\Omega = \bigcup_{j \in J} \Omega_j$ with $|J| \geq 2$. □

2 Main results

Assume that n is any fixed integer ≥ 2 . Then we will introduce notations that will be used throughout this paper.

- $I = I_n = \{1, 2, \dots, n\}$.
- $\Omega_j = \Omega_j^{(n)}$ is the set consisting of all j -element subsets of I for any $j = 1, 2, \dots, n - 1$ ($|\Omega_j| = \binom{n}{j}$).

We give the reversed lexicographic ordering on each Ω_j and also on any nonempty subset Ω of Ω_j . For example, $\Omega_3 = \{\bar{1} = \{2, 3, 4\} < \bar{2} = \{1, 3, 4\} < \bar{3} = \{1, 2, 4\} < \bar{4} = \{1, 2, 3\}\}$, when $n = 4, j = 3$; $\Omega = \{\{2, 3, 4\} < \{1, 2, 4\}\}$ is a subset of Ω_j . We write $\Omega < \Omega'$ for disjoint nonempty subsets Ω, Ω' of Ω_j if every member of Ω is smaller than that of Ω' . Also, we agree that $\phi < \Omega, \Omega < \phi$ for every (including the empty set) subset Ω of Ω_j .

Every permutation σ in S_n gives rise to a natural bijection of Ω_j onto itself given by $A \mapsto \sigma A$, where σA is the set obtained from A by applying σ to each member of A .

- w_1, w_2, \dots, w_n typical n positive integers.
- $w_A = \prod_{i \in A} w_i$ for any subset A of I ($w_\phi = 1$). For example, $w_{\{1,2,3\}} = w_1 w_2 w_3$. Also, we note that $\prod_{A \in \Omega_j} w_A = w_I^{\binom{n-1}{j-1}}$.
- $\hat{w}_i = \frac{w_i}{w_I}$.
- \bar{A} = the complement of A in I for any subset A of I , so that $w_A w_{\bar{A}} = w_I$ for any subset A of I .
- $\hat{w}_A = \prod_{i \in A} \hat{w}_i = (w_I)^{|A|-1} w_{\bar{A}}$ ($\hat{w}_I = (w_I)^{|A|-1}$).
- $x_A = x_{j_1 j_2 \dots j_r}$ for any nonempty subset $A = \{j_1, j_2, \dots, j_r\}$ of I with $j_1 < j_2 < \dots < j_r$. For example, $x_{\{1,2,3\}} = x_{123}$. These are typical variables of integration.
- $\phi_j : \Omega_j \rightarrow \Omega_{n-j}$ ($A \mapsto \bar{A}$) is a bijection.
- $p_j(w; x) = p_j^{(n)}(w; x) = \sum_{A \in \Omega_j} w_A x_{\bar{A}}$ for $j = 1, 2, \dots, n - 1$. For example, with $n = 4$, we have

$$p_1(w; x) = w_1 x_{234} + w_2 x_{134} + w_3 x_{124} + w_4 x_{123},$$

$$p_2(w; x) = w_1 w_2 x_{34} + w_1 w_3 x_{24} + w_1 w_4 x_{23} + w_2 w_3 x_{14} + w_2 w_4 x_{13} + w_3 w_4 x_{12},$$

$$p_3(w; x) = w_1 w_2 w_3 x_4 + w_1 w_2 w_4 x_3 + w_1 w_3 w_4 x_2 + w_2 w_3 w_4 x_1.$$

• $d\mu(\Omega) = \prod_{A \in \Omega} d\mu(x_{\bar{A}})$ for any nonempty subset Ω of Ω_j .

As before, assume that n is any fixed integer ≥ 2 and that j is an integer with $1 \leq j \leq n - 1$. Then, in view of Proposition 1.1, for any subset Ω for Ω_j , we consider the following quotients of integrals given by

$$I_j(\Omega) = I_j^{(n)}(\Omega) = \frac{\int_{\mathbb{Z}_p^{|\Omega_j|}} e^{(p_j(w;x) + w_I(\sum_{A \in \Omega_j - \Omega} \gamma_{\bar{A}}))t} d\mu(\Omega_j)}{(\int_{\mathbb{Z}_p} e^{w_I z t} d\mu(z))^{|\Omega|}} \tag{14}$$

$$= \frac{w_I^{\binom{n-1}{j-1} - |\Omega|} t^{|\Omega_j| - |\Omega|} e^{w_I(\sum_{A \in \Omega_j - \Omega} \gamma_{\bar{A}})t} (e^{w_I t} - 1)^{|\Omega|}}{\prod_{A \in \Omega_j} (e^{w_A t} - 1)}. \tag{15}$$

Here, we have to observe that

$$\begin{aligned} \int_{\mathbb{Z}_p^{|\Omega_j|}} e^{p_j(w;x)t} d\mu(\Omega_j) &= \prod_{A \in \Omega_j} \int_{\mathbb{Z}_p} e^{w_A x_{\bar{A}} t} d\mu(x_{\bar{A}}) \\ &= \prod_{A \in \Omega_j} \frac{w_A t}{e^{w_A t} - 1} = \frac{w_I^{\binom{n-1}{j-1}} t^{|\Omega_j|}}{\prod_{A \in \Omega_j} (e^{w_A t} - 1)}. \end{aligned} \tag{16}$$

It is important to observe here, either from (14) or from (15), that the integrals $I_j(\Omega)$ are invariant under any permutation of w_1, w_2, \dots, w_n .

Now, we decompose Ω into a disjoint union $\Omega = \Omega^{(e)} \cup \Omega^{(s)}$ with $\Omega^{(e)} < \Omega^{(s)}$. As we allow either $\Omega^{(e)}$ or $\Omega^{(s)}$ to be the empty set, there are $|\Omega| + 1$ ways of doing this. Then, by invoking (12) and (13), we write the integral in (14) as follows:

$$\begin{aligned} I_j(\Omega) &= \frac{1}{\prod_{A \in \Omega} w_{\bar{A}}} \times \prod_{A \in \Omega_j - \Omega} \int_{\mathbb{Z}_p} e^{w_A(x_{\bar{A}} + w_{\bar{A}} \gamma_{\bar{A}})t} d\mu(x_{\bar{A}}) \times \prod_{A \in \Omega} \frac{w_{\bar{A}} \int_{\mathbb{Z}_p} e^{w_A x_{\bar{A}} t} d\mu(x_{\bar{A}})}{\int_{\mathbb{Z}_p} e^{w_{\bar{A}} w_A z t} d\mu(z)} \\ &= \frac{1}{\prod_{A \in \Omega} w_{\bar{A}}} \times \prod_{A \in \Omega_j - \Omega} \int_{\mathbb{Z}_p} e^{w_A(x_{\bar{A}} + w_{\bar{A}} \gamma_{\bar{A}})t} d\mu(x_{\bar{A}}) \\ &\quad \times \prod_{A \in \Omega^{(e)}} \left(\sum_{i_A=0}^{w_{\bar{A}}-1} e^{i_A w_A t} \right) \times \prod_{A \in \Omega^{(s)}} \left(\sum_{j_A=0}^{\infty} S_{j_A}(w_{\bar{A}} - 1) \frac{(w_A t)^{j_A}}{j_A!} \right). \end{aligned} \tag{17}$$

Note here that we used the identity in (12) for all $A \in \Omega^{(e)}$, and that in (13) for all $A \in \Omega^{(s)}$.

Remark 2.1 We observe that (15) depends essentially only on the size $|\Omega|$ of Ω . Namely, if $\Omega^{(1)}$ and $\Omega^{(2)}$ are two subsets of Ω_j , with $|\Omega^{(1)}| = |\Omega^{(2)}|$, and $f : \Omega_j \rightarrow \Omega_j$ is any bijective map sending $\Omega^{(1)}$ onto $\Omega^{(2)}$, then we have

$$\begin{aligned} &\frac{w_I^{\binom{n-1}{j-1} - |\Omega^{(2)}|} t^{|\Omega_j| - |\Omega^{(2)}|} e^{w_I(\sum_{A \in \Omega_j - \Omega^{(2)}} \gamma_{\bar{A}})t} (e^{w_I t} - 1)^{|\Omega^{(2)}|}}{\prod_{A \in \Omega_j} (e^{w_A t} - 1)} \\ &= \frac{w_I^{\binom{n-1}{j-1} - |\Omega^{(1)}|} t^{|\Omega_j| - |\Omega^{(1)}|} e^{w_I(\sum_{A \in \Omega_j - \Omega^{(1)}} \gamma_{\bar{A}})t} (e^{w_I t} - 1)^{|\Omega^{(1)}|}}{\prod_{A \in \Omega_j} (e^{w_A t} - 1)}. \end{aligned} \tag{18}$$

Thus (14) with $\Omega^{(2)}$ is the same as that with $\Omega^{(1)}$ with the ‘y variables renamed.’ Hence we only need to consider (17) for only one subset Ω of Ω_j with the given size. So, for each positive integer k with $k \leq |\Omega_j|$, we only consider the subset Ω of Ω_j consisting of the first k (smaller) elements of Ω_j . We denote this subset by Ω_{jk} , and the empty subset of Ω_j by Ω_{j0} . From now on, we assume that $\Omega = \Omega_{jk}$ for some integer k ($0 \leq k \leq |\Omega_j| = \binom{n}{j}$). For example, when $n = 3$, we see that

$$\Omega_{20} = \phi, \quad \Omega_{21} = \{2, 3\}, \quad \Omega_{22} = \{2, 3, 1\}, \quad \Omega_{23} = \{2, 3, 1, 2\}.$$

Further, we assume that we have a decomposition of $\Omega^{(e)}$ as the disjoint union

$$\Omega^{(e)} = \bigcup_{A \in \Omega_j - \Omega} \Omega_A^{(e)} \tag{19}$$

satisfying the following conditions:

- (i) $|\Omega_A^{(e)}| \leq |\Omega_{A'}^{(e)}|$ for all $A, A' \in \Omega_j - \Omega$ with $A < A'$,
- (ii) $\Omega_A^{(e)} < \Omega_{A'}^{(e)}$ for all $A, A' \in \Omega_j - \Omega$ with $A < A'$.

(Note: in view of (19), this requires in particular that we should choose $\Omega^{(e)} = \phi$ for $\Omega_j - \Omega = \phi$.)

We assume that we are given such a decomposition as in (19) satisfying (i) and (ii). Then we write (17) as

$$\begin{aligned} I_j(\Omega) &= \frac{1}{\prod_{A \in \Omega} w_{\bar{A}}} \prod_{A \in \Omega_j - \Omega} \prod_{E \in \Omega_A^{(e)}} \sum_{i_E=0}^{w_{\bar{E}}-1} \int_{\mathbb{Z}_p} e^{w_A(x_{\bar{A}} + w_{\bar{A}}y_{\bar{A}} + \sum_{E \in \Omega_A^{(e)}} i_E \frac{w_E}{w_A})t} d\mu(x_{\bar{A}}) \tag{20} \\ &\times \prod_{A \in \Omega^{(s)}} \sum_{j_A=0}^{\infty} S_{j_A}(w_{\bar{A}} - 1) \frac{(w_A t)^{j_A}}{j_A!} \\ &= \frac{1}{\prod_{A \in \Omega} w_{\bar{A}}} \prod_{A \in \Omega_j - \Omega} \sum_{l_A=0}^{\infty} \prod_{E \in \Omega_A^{(e)}} \sum_{i_E=0}^{w_{\bar{E}}-1} B_{l_A} \left(w_{\bar{A}} y_{\bar{A}} + \sum_{E \in \Omega_A^{(e)}} i_E \frac{w_E}{w_A} \right) \frac{(w_A t)^{l_A}}{l_A!} \\ &\times \prod_{A \in \Omega^{(s)}} \sum_{j_A=0}^{\infty} S_{j_A}(w_{\bar{A}} - 1) \frac{(w_A t)^{j_A}}{j_A!}. \end{aligned}$$

Further, by rearranging sums (20) can be written as

$$\begin{aligned} I_j(\Omega) &= \frac{1}{\prod_{A \in \Omega} w_{\bar{A}}} \left\{ \sum_{n=0}^{\infty} \sum_{\sum_{A \in \Omega_j - \Omega} l_A + \sum_{A \in \Omega^{(s)}} j_A = n} \binom{n}{\dots, l_A, \dots, j_A, \dots} \tag{21} \right. \\ &\times \prod_{A \in \Omega_j - \Omega} \prod_{E \in \Omega_A^{(e)}} \sum_{i_E=0}^{w_{\bar{E}}-1} B_{l_A} \left(w_{\bar{A}} y_{\bar{A}} + \sum_{E \in \Omega_A^{(e)}} i_E \frac{w_E}{w_A} \right) \\ &\times \left. \prod_{A \in \Omega^{(s)}} S_{j_A}(w_{\bar{A}} - 1) \prod_{A \in \Omega_j - \Omega} w_A^{l_A} \prod_{A \in \Omega^{(s)}} w_A^{j_A} \frac{t^n}{n!} \right\}. \end{aligned}$$

Here, $\binom{n}{\dots, l_A, \dots, j_A, \dots}$ denotes the multinomial coefficient, where l_A and j_A are nonnegative integers varying respectively over the index sets $A \in \Omega_j - \Omega$ and $A \in \Omega^{(s)}$.

For the next theorem, we assume that $\Omega = \Omega_{jk}$ for some k , Ω has a decomposition into a disjoint union $\Omega = \Omega^{(e)} \cup \Omega^{(s)}$ with $\Omega^{(e)} < \Omega^{(s)}$, and that $\Omega^{(e)}$ has a decomposition $\Omega^{(e)} = \bigcup_{A \in \Omega_j - \Omega} \Omega_A^{(e)}$ satisfying conditions (i) and (ii) in (19). As we noted earlier, the p -adic integrals in (14) are invariant under every permutation of w_1, w_2, \dots, w_n , so that it gives the identities of symmetry with respect to w_1, w_2, \dots, w_n , involving Bernoulli polynomials and power sums. Now, we have our main result from (21).

Theorem 2.2 *The following expression is invariant under every permutation of w_1, w_2, \dots, w_n , so that it gives the identities of symmetry with respect to w_1, w_2, \dots, w_n :*

$$\begin{aligned} & \frac{1}{\prod_{A \in \Omega} w_{\bar{A}}} \sum_{\sum_{A \in \Omega_j - \Omega} l_A + \sum_{A \in \Omega^{(s)}} j_A = n} \binom{n}{\dots, l_A, \dots, j_A, \dots} \\ & \times \prod_{A \in \Omega_j - \Omega} \prod_{E \in \Omega_A^{(e)}} \sum_{i_E = 0}^{w_{\bar{E}} - 1} B_{l_A} \left(w_{\bar{A}} y_{\bar{A}} + \sum_{E \in \Omega_A^{(e)}} i_E \frac{w_{\bar{E}}}{w_{\bar{A}}} \right) \\ & \times \prod_{A \in \Omega^{(s)}} S_{j_A}(w_{\bar{A}} - 1) \prod_{A \in \Omega_j - \Omega} w_{\bar{A}}^{l_A} \prod_{A \in \Omega^{(s)}} w_{\bar{A}}^{j_A}. \end{aligned} \tag{22}$$

In other words, for all permutations $\sigma \in S_n$, the following expressions are all the same:

$$\begin{aligned} & \frac{1}{\prod_{A \in \Omega} w_{\sigma \bar{A}}} \sum_{\sum_{A \in \Omega_j - \Omega} l_A + \sum_{A \in \Omega^{(s)}} j_A = n} \binom{n}{\dots, l_A, \dots, j_A, \dots} \\ & \times \prod_{A \in \Omega_j - \Omega} \prod_{E \in \Omega_A^{(e)}} \sum_{i_E = 0}^{w_{\sigma \bar{E}} - 1} B_{l_A} \left(w_{\sigma \bar{A}} y_{\bar{A}} + \sum_{E \in \Omega_A^{(e)}} i_E \frac{w_{\sigma \bar{E}}}{w_{\sigma \bar{A}}} \right) \\ & \times \prod_{A \in \Omega^{(s)}} S_{j_A}(w_{\sigma \bar{A}} - 1) \prod_{A \in \Omega_j - \Omega} w_{\sigma \bar{A}}^{l_A} \prod_{A \in \Omega^{(s)}} w_{\sigma \bar{A}}^{j_A}. \end{aligned}$$

Here, $\binom{n}{\dots, l_A, \dots, j_A, \dots}$ denotes the multinomial coefficient, where l_A and j_A are nonnegative integers varying respectively over the index sets $A \in \Omega_j - \Omega$ and $A \in \Omega^{(s)}$.

3 Examples

Here, we would like to illustrate our Theorem 2.2.

Example 3.1 Assume that $n = 3, j = 2$. Here, $\Omega_2 = \Omega_2^{(3)} = \{\bar{1} = \{2, 3\} < \bar{2} = \{1, 3\} < \bar{3} = \{1, 2\}\}$. In view of our discussion leading up to Theorem 2.2, we may consider only the subsets $\Omega = \Omega_{2i}$ ($i = 0, 1, 2, 3$) of Ω_2 .

- (a) $\Omega_{20} = \phi$ ($\Omega_2 - \Omega_{20} = \Omega_2$)
 - (a-1) $\Omega_{20}^{(e)} = \phi$ ($\Omega_{20,A}^{(e)} = \phi$ for each $A \in \Omega_2$), $\Omega_{20}^{(s)} = \phi$.
- (b) $\Omega_{21} = \{\bar{1}\}$ ($\Omega_2 - \Omega_{21} = \{\bar{2}, \bar{3}\}$)
 - (b-1) $\Omega_{21}^{(e)} = \phi$ ($\Omega_{21,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{21}$), $\Omega_{21}^{(s)} = \{\bar{1}\}$,
 - (b-2) $\Omega_{21}^{(e)} = \{\bar{1}\}$ ($\Omega_{21,2}^{(e)} = \phi, \Omega_{21,3}^{(e)} = \{\bar{1}\}$), $\Omega_{21}^{(s)} = \phi$.
- (c) $\Omega_{22} = \{\bar{1}, \bar{2}\}$ ($\Omega_2 - \Omega_{22} = \{\bar{3}\}$)
 - (c-1) $\Omega_{22}^{(e)} = \phi$ ($\Omega_{22,3}^{(e)} = \phi$), $\Omega_{22}^{(s)} = \{\bar{1}, \bar{2}\}$,
 - (c-2) $\Omega_{22}^{(e)} = \{\bar{1}\}$ ($\Omega_{22,3}^{(e)} = \{\bar{1}\}$), $\Omega_{22}^{(s)} = \{\bar{2}\}$,
 - (c-3) $\Omega_{22}^{(e)} = \{\bar{1}, \bar{2}\}$ ($\Omega_{22,3}^{(e)} = \{\bar{1}, \bar{2}\}$), $\Omega_{22}^{(s)} = \phi$.

(d) $\Omega_{23} = \{\bar{1}, \bar{2}, \bar{3}\}$ ($\Omega_2 - \Omega_{23} = \phi$)

(d-1) $\Omega_{23}^{(e)} = \phi, \Omega_{23}^{(s)} = \Omega_{23}$.

One checks now that the invariance of (14) under any permutation of w_1, w_2, w_3 , applied to each of the cases (a-1), (b-1), (b-2), (c-1), (c-2), and (c-3), yields the results in Theorems 1, 2, 5, 8, 11, 14 in [10]. As we noted in [10], not all of these give the full six identities of symmetry corresponding to the symmetric group S_3 . The possible numbers of distinct identities of symmetry are 1, 2, 3, and 6 corresponding to the quotient $|S_3|/|H|$, where H is a subgroup of S_3 , with the respective orders 6, 3, 2, and 1. In our case, (a-1), (b-1), (b-2), and (c-2) give the full six identities of symmetry, and (c-1) and (c-3) yield three identities of symmetry, while (d-1) gives no identities of symmetry. For convenience of the reader, we reproduce those results here with appropriate change of notations in (22).

$$\begin{aligned}
 & \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_2 y_2) B_m(w_3 y_3) w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_1) B_\ell(w_3 y_2) B_m(w_2 y_3) w_1^{\ell+m} w_3^{k+m} w_2^{k+\ell} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(w_1 y_2) B_m(w_3 y_3) w_2^{\ell+m} w_1^{k+m} w_3^{k+\ell} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_1) B_\ell(w_3 y_2) B_m(w_1 y_3) w_2^{\ell+m} w_3^{k+m} w_1^{k+\ell} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) B_\ell(w_1 y_2) B_m(w_2 y_3) w_3^{\ell+m} w_1^{k+m} w_2^{k+\ell} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_1) B_\ell(w_2 y_2) B_m(w_1 y_3) w_3^{\ell+m} w_2^{k+m} w_1^{k+\ell},
 \end{aligned} \tag{a-1}$$

$$\begin{aligned}
 & \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_2) B_\ell(w_3 y_3) S_m(w_1 - 1) w_2^{\ell+m} w_3^{k+m} w_1^{k+\ell-1} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_2) B_\ell(w_1 y_3) S_m(w_3 - 1) w_2^{\ell+m} w_1^{k+m} w_3^{k+\ell-1} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_2) B_\ell(w_3 y_3) S_m(w_2 - 1) w_1^{\ell+m} w_3^{k+m} w_2^{k+\ell-1} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_2) B_\ell(w_2 y_3) S_m(w_3 - 1) w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell-1} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_2) B_\ell(w_2 y_3) S_m(w_1 - 1) w_3^{\ell+m} w_2^{k+m} w_1^{k+\ell-1} \\
 &= \sum_{k+l+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_2) B_\ell(w_1 y_3) S_m(w_2 - 1) w_3^{\ell+m} w_1^{k+m} w_2^{k+\ell-1},
 \end{aligned} \tag{b-1}$$

$$\begin{aligned}
 & w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_2) \sum_{i=0}^{w_1-1} B_{n-k} \left(w_3 y_3 + \frac{w_3}{w_1} i \right) w_2^{n-k} w_3^k \\
 &= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_3 y_2) \sum_{i=0}^{w_1-1} B_{n-k} \left(w_2 y_3 + \frac{w_2}{w_1} i \right) w_3^{n-k} w_2^k
 \end{aligned}$$

$$= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_2) \sum_{i=0}^{w_2-1} B_{n-k} \left(w_3 y_3 + \frac{w_3}{w_2} i \right) w_1^{n-k} w_3^k \tag{b-2}$$

$$= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_3 y_2) \sum_{i=0}^{w_2-1} B_{n-k} \left(w_1 y_3 + \frac{w_1}{w_2} i \right) w_3^{n-k} w_1^k$$

$$= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_1 y_2) \sum_{i=0}^{w_3-1} B_{n-k} \left(w_2 y_3 + \frac{w_2}{w_3} i \right) w_1^{n-k} w_2^k$$

$$= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} B_k(w_2 y_2) \sum_{i=0}^{w_3-1} B_{n-k} \left(w_1 y_3 + \frac{w_1}{w_3} i \right) w_2^{n-k} w_1^k,$$

$$\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_3 y_3) S_\ell(w_1 - 1) S_m(w_2 - 1) w_3^{\ell+m} w_1^{k+m-1} w_2^{k+\ell-1}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_1 y_3) S_\ell(w_2 - 1) S_m(w_3 - 1) w_1^{\ell+m} w_2^{k+m-1} w_3^{k+\ell-1} \tag{c-1}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(w_2 y_3) S_\ell(w_1 - 1) S_m(w_3 - 1) w_2^{\ell+m} w_1^{k+m-1} w_3^{k+\ell-1},$$

$$w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k \left(w_3 y_3 + \frac{w_3}{w_1} i \right) S_{n-k}(w_2 - 1) w_3^{n-k} w_2^{k-1}$$

$$= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k \left(w_3 y_3 + \frac{w_3}{w_2} i \right) S_{n-k}(w_1 - 1) w_3^{n-k} w_1^{k-1}$$

$$= w_2^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} B_k \left(w_1 y_3 + \frac{w_1}{w_2} i \right) S_{n-k}(w_3 - 1) w_1^{n-k} w_3^{k-1} \tag{c-2}$$

$$= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_3-1} B_k \left(w_1 y_3 + \frac{w_1}{w_3} i \right) S_{n-k}(w_2 - 1) w_1^{n-k} w_2^{k-1}$$

$$= w_1^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} B_k \left(w_2 y_3 + \frac{w_2}{w_1} i \right) S_{n-k}(w_3 - 1) w_2^{n-k} w_3^{k-1}$$

$$= w_3^{n-1} \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_3-1} B_k \left(w_2 y_3 + \frac{w_2}{w_3} i \right) S_{n-k}(w_1 - 1) w_2^{n-k} w_1^{k-1},$$

$$(w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} B_n \left(w_3 y_3 + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j \right)$$

$$= (w_2 w_3)^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} B_n \left(w_1 y_3 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right) \tag{c-3}$$

$$= (w_3 w_1)^{n-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} B_n \left(w_2 y_3 + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j \right).$$

Example 3.2 Assume that $n = 4, j = 3$. Here, $\Omega_3 = \Omega_3^{(4)} = \{\bar{1} = \{2, 3, 4\} < \bar{2} = \{1, 3, 4\} < \bar{3} = \{1, 2, 4\} < \bar{4} = \{1, 2, 3\}\}$. In view of our discussion leading up to Theorem 2.2, we may con-

sider only the subsets $\Omega = \Omega_{3i}$ ($i = 0, 1, 2, 3, 4$) of Ω_3 . We let the interested reader write out the identities of symmetry for each of the following cases by using Theorem 2.2.

- (a) $\Omega_{30} = \phi$ ($\Omega_3 - \Omega_{30} = \Omega_3$)
 - (a-1) $\Omega_{30}^{(e)} = \phi$ ($\Omega_{30,A}^{(e)} = \phi$ for each $A \in \Omega_3$), $\Omega_{30}^{(s)} = \phi$.
- (b) $\Omega_{31} = \{\bar{1}\}$ ($\Omega_3 - \Omega_{31} = \{\bar{2}, \bar{3}, \bar{4}\}$)
 - (b-1) $\Omega_{31}^{(e)} = \phi$ ($\Omega_{31,A}^{(e)} = \phi$ for each $A \in \Omega_3 - \Omega_{31}$), $\Omega_{31}^{(s)} = \{\bar{1}\}$,
 - (b-2) $\Omega_{31}^{(e)} = \{\bar{1}\}$ ($\Omega_{31,2}^{(e)} = \Omega_{31,3}^{(e)} = \phi$, $\Omega_{31,4}^{(e)} = \{\bar{1}\}$), $\Omega_{31}^{(s)} = \phi$.
- (c) $\Omega_{32} = \{\bar{1}, \bar{2}\}$ ($\Omega_3 - \Omega_{32} = \{\bar{3}, \bar{4}\}$)
 - (c-1) $\Omega_{32}^{(e)} = \phi$ ($\Omega_{32,A}^{(e)} = \phi$ for each $A \in \Omega_3 - \Omega_{32}$), $\Omega_{32}^{(s)} = \{\bar{1}, \bar{2}\}$,
 - (c-2) $\Omega_{32}^{(e)} = \{\bar{1}\}$ ($\Omega_{32,3}^{(e)} = \phi$, $\Omega_{32,4}^{(e)} = \{\bar{1}\}$), $\Omega_{32}^{(s)} = \{\bar{2}\}$,
 - (c-3-1) $\Omega_{32}^{(e)} = \{\bar{1}, \bar{2}\}$ ($\Omega_{32,3}^{(e)} = \phi$, $\Omega_{32,4}^{(e)} = \{\bar{1}, \bar{2}\}$), $\Omega_{32}^{(s)} = \phi$,
 - (c-3-2) $\Omega_{32}^{(e)} = \{\bar{1}, \bar{2}\}$ ($\Omega_{32,3}^{(e)} = \{\bar{1}\}$, $\Omega_{32,4}^{(e)} = \{\bar{2}\}$), $\Omega_{32}^{(s)} = \phi$.
- (d) $\Omega_{34} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ ($\Omega_3 - \Omega_{34} = \phi$)
 - (d-1) $\Omega_{34}^{(e)} = \phi$, $\Omega_{34}^{(s)} = \Omega_{34}$.

Example 3.3 Assume that $n = 4, j = 2$. Here, $\Omega_2 = \Omega_2^{(4)} = \{34 = \{3, 4\} < 24 = \{2, 4\} < 23 = \{2, 3\} < 14 = \{1, 4\} < 13 = \{1, 3\} < 12 = \{1, 2\}\}$. In view of our discussion leading up to Theorem 2.2, we may consider only the subsets $\Omega = \Omega_{2i}$ ($i = 0, 1, 2, 3, 4, 5, 6$) of Ω_2 . We let the interested reader write out the identities of symmetry for each of the following cases by using Theorem 2.2.

- (a) $\Omega_{20} = \phi$ ($\Omega_2 - \Omega_{20} = \Omega_2$)
 - (a-1) $\Omega_{20}^{(e)} = \phi$ ($\Omega_{20,A}^{(e)} = \phi$ for each $A \in \Omega_2$), $\Omega_{20}^{(s)} = \phi$.
- (b) $\Omega_{21} = \{34\}$ ($\Omega_2 - \Omega_{21} = \{24, 23, 14, 13, 12\}$)
 - (b-1) $\Omega_{21}^{(e)} = \phi$ ($\Omega_{21,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{21}$), $\Omega_{21}^{(s)} = \{34\}$,
 - (b-2) $\Omega_{21}^{(e)} = \{34\}$ ($\Omega_{21,A}^{(e)} = \phi$ for each $A \in \{24, 23, 14, 13\}$, $\Omega_{21,12}^{(e)} = \{34\}$), $\Omega_{21}^{(s)} = \phi$.
- (c) $\Omega_{22} = \{34, 24\}$ ($\Omega_2 - \Omega_{22} = \{23, 14, 13, 12\}$)
 - (c-1) $\Omega_{22}^{(e)} = \phi$ ($\Omega_{22,A}^{(e)} = \phi$ for each $A \in \Omega_2 - \Omega_{22}$), $\Omega_{22}^{(s)} = \{34, 24\}$,
 - (c-2) $\Omega_{22}^{(e)} = \{34\}$ ($\Omega_{22,23}^{(e)} = \Omega_{22,14}^{(e)} = \Omega_{22,13}^{(e)} = \phi$, $\Omega_{22,12}^{(e)} = \{34\}$), $\Omega_{22}^{(s)} = \{24\}$,
 - (c-3-1) $\Omega_{22}^{(e)} = \{34, 24\}$ ($\Omega_{22,23}^{(e)} = \Omega_{22,14}^{(e)} = \Omega_{22,13}^{(e)} = \phi$, $\Omega_{22,12}^{(e)} = \{34, 24\}$), $\Omega_{22}^{(s)} = \phi$,
 - (c-3-2) $\Omega_{22}^{(e)} = \{34, 24\}$ ($\Omega_{22,23}^{(e)} = \Omega_{22,14}^{(e)} = \phi$, $\Omega_{22,13}^{(e)} = \{34\}$, $\Omega_{22,12}^{(e)} = \{24\}$), $\Omega_{22}^{(s)} = \phi$.
- (d) $\Omega_{23} = \{34, 24, 23\}$ ($\Omega_2 - \Omega_{23} = \{14, 13, 12\}$)
 - (d-1) $\Omega_{23}^{(e)} = \phi$ ($\Omega_{23,A}^{(e)} = \phi$ for each $A \in \{14, 13, 12\}$), $\Omega_{23}^{(s)} = \{34, 24, 23\}$,
 - (d-2) $\Omega_{23}^{(e)} = \{34\}$ ($\Omega_{23,14}^{(e)} = \Omega_{23,13}^{(e)} = \phi$, $\Omega_{23,12}^{(e)} = \{34\}$), $\Omega_{23}^{(s)} = \{24, 23\}$,
 - (d-3-1) $\Omega_{23}^{(e)} = \{34, 24\}$ ($\Omega_{23,14}^{(e)} = \Omega_{23,13}^{(e)} = \phi$, $\Omega_{23,12}^{(e)} = \{34, 24\}$), $\Omega_{23}^{(s)} = \{23\}$,
 - (d-3-2) $\Omega_{23}^{(e)} = \{34, 24\}$ ($\Omega_{23,14}^{(e)} = \phi$, $\Omega_{23,13}^{(e)} = \{34\}$, $\Omega_{23,12}^{(e)} = \{24\}$), $\Omega_{23}^{(s)} = \{23\}$,
 - (d-4-1) $\Omega_{23}^{(e)} = \{34, 24, 23\}$ ($\Omega_{23,14}^{(e)} = \Omega_{23,13}^{(e)} = \phi$, $\Omega_{23,12}^{(e)} = \{34, 24, 23\}$), $\Omega_{23}^{(s)} = \phi$,
 - (d-4-2) $\Omega_{23}^{(e)} = \{34, 24, 23\}$ ($\Omega_{23,14}^{(e)} = \phi$, $\Omega_{23,13}^{(e)} = \{34\}$, $\Omega_{23,12}^{(e)} = \{24, 23\}$), $\Omega_{23}^{(s)} = \phi$,
 - (d-4-3) $\Omega_{23}^{(e)} = \{34, 24, 23\}$ ($\Omega_{23,14}^{(e)} = \{34\}$, $\Omega_{23,13}^{(e)} = \{24\}$, $\Omega_{23,12}^{(e)} = \{23\}$), $\Omega_{23}^{(s)} = \phi$.
- (e) $\Omega_{24} = \{34, 24, 23, 14\}$ ($\Omega_2 - \Omega_{24} = \{13, 12\}$)
 - (e-1) $\Omega_{24}^{(e)} = \phi$ ($\Omega_{24,13}^{(e)} = \Omega_{24,12}^{(e)} = \phi$), $\Omega_{24}^{(s)} = \{34, 24, 23, 14\}$,
 - (e-2) $\Omega_{24}^{(e)} = \{34\}$ ($\Omega_{24,13}^{(e)} = \phi$, $\Omega_{24,12}^{(e)} = \{34\}$), $\Omega_{24}^{(s)} = \{24, 23, 14\}$,
 - (e-3-1) $\Omega_{24}^{(e)} = \{34, 24\}$ ($\Omega_{24,13}^{(e)} = \phi$, $\Omega_{24,12}^{(e)} = \{34, 24\}$), $\Omega_{24}^{(s)} = \{23, 14\}$,
 - (e-3-2) $\Omega_{24}^{(e)} = \{34, 24\}$ ($\Omega_{24,13}^{(e)} = \{34\}$, $\Omega_{24,12}^{(e)} = \{24\}$), $\Omega_{24}^{(s)} = \{23, 14\}$,
 - (e-4-1) $\Omega_{24}^{(e)} = \{34, 24, 23\}$ ($\Omega_{24,13}^{(e)} = \phi$, $\Omega_{24,12}^{(e)} = \{34, 24, 23\}$), $\Omega_{24}^{(s)} = \{14\}$,
 - (e-4-2) $\Omega_{24}^{(e)} = \{34, 24, 23\}$ ($\Omega_{24,13}^{(e)} = \{34\}$, $\Omega_{24,12}^{(e)} = \{24, 23\}$), $\Omega_{24}^{(s)} = \{14\}$,

- (e-5-1) $\Omega_{24}^{(e)} = \{34, 24, 23, 14\}$ ($\Omega_{24,13}^{(e)} = \phi, \Omega_{24,12}^{(e)} = \{34, 24, 23, 14\}$), $\Omega_{24}^{(s)} = \phi$,
- (e-5-2) $\Omega_{24}^{(e)} = \{34, 24, 23, 14\}$ ($\Omega_{24,13}^{(e)} = \{34\}, \Omega_{24,12}^{(e)} = \{24, 23, 14\}$), $\Omega_{24}^{(s)} = \phi$,
- (e-5-3) $\Omega_{24}^{(e)} = \{34, 24, 23, 14\}$ ($\Omega_{24,13}^{(e)} = \{34, 24\}, \Omega_{24,12}^{(e)} = \{23, 14\}$), $\Omega_{24}^{(s)} = \phi$.
- (f) $\Omega_{25} = \{34, 24, 23, 14, 13\}$ ($\Omega_2 - \Omega_{25} = \{12\}$)
 - (f-1) $\Omega_{25}^{(e)} = \phi$ ($\Omega_{25,12}^{(e)} = \phi$), $\Omega_{25}^{(s)} = \{34, 24, 23, 14, 13\}$,
 - (f-2) $\Omega_{25}^{(e)} = \{34\}$ ($\Omega_{25,12}^{(e)} = \{34\}$), $\Omega_{25}^{(s)} = \{24, 23, 14, 13\}$,
 - (f-3) $\Omega_{25}^{(e)} = \{34, 24\}$ ($\Omega_{25,12}^{(e)} = \{34, 24\}$), $\Omega_{25}^{(s)} = \{23, 14, 13\}$,
 - (f-4) $\Omega_{25}^{(e)} = \{34, 24, 23\}$ ($\Omega_{25,12}^{(e)} = \{34, 24, 23\}$), $\Omega_{25}^{(s)} = \{14, 13\}$,
 - (f-5) $\Omega_{25}^{(e)} = \{34, 24, 23, 14\}$ ($\Omega_{25,12}^{(e)} = \{34, 24, 23, 14\}$), $\Omega_{25}^{(s)} = \{13\}$,
 - (f-6) $\Omega_{25}^{(e)} = \{34, 24, 23, 14, 13\}$ ($\Omega_{25,12}^{(e)} = \{34, 24, 23, 14, 13\}$), $\Omega_{25}^{(s)} = \phi$.
- (g) $\Omega_{26} = \{34, 24, 23, 14, 13, 12\}$ ($\Omega_2 - \Omega_{26} = \phi$)
 - (g-1) $\Omega_{26}^{(e)} = \phi, \Omega_{26}^{(s)} = \Omega_{26}$.

4 Conclusion

Identities of symmetry in two variables for Bernoulli polynomials and power sums, which had been shown by using suitable symmetric identities, were derived by employing a completely different tool in [12], namely the p -adic Volkenborn integrals. Not much later, it was observed in [10] that the identities in two variables can be extended to those in three variables. We recalled that the abundant symmetries of identities in three variables shed new light even on the existing identities in two variables. Namely, some further identities of symmetry can be discovered by specializing one of the three variables as 1.

Here, in this paper, we generalized the results in three variables to those in an arbitrary number of variables in a suitable setting. We proved our main result, Theorem 2.2, and illustrated our result with some examples. As was noted in [10] and recalled in Example 3.1, the number of distinct identities of symmetry in three variables is not always $6 = |S_3|$, but it is 1, 2, 3, or 6, because it is equal to the quotient $|S_3|/|H|$, where H is a subgroup of S_3 . It is an interesting problem to determine the possible numbers of distinct symmetries in our case. We leave this as a challenging problem for the interested reader. Similar results for other special polynomials together with the corresponding suitable power sums will appear in forthcoming papers.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

TK and DSK conceived of the framework and structured the whole paper; DSK and TK wrote the paper; JK and HYK checked the results of the paper and typed the paper; DSK and TK completed the revision of the article. All authors have read and agreed to the published version of the manuscript.

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References

1. Araci, S., Duran, U., Acikgoz, M.: A study on a class of q -Euler polynomials under the symmetric group of degree n . *J. Nonlinear Sci. Appl.* **9**(8), 5196–5201 (2016)
2. Deeba, E., Rodriguez, D.: Stirling's and Bernoulli numbers. *Am. Math. Mon.* **98**, 423–426 (1991)
3. Gessel, I.: Solution to problem E3237. *Am. Math. Mon.* **96**, 364 (1989)
4. He, Y.: Symmetric identities for Carlitz's q -Bernoulli numbers and polynomials. *Adv. Differ. Equ.* **2013**, 246 (2013)
5. Jang, L.-C., Kim, D. S., Kim, T., Lee, H.: p -Adic integral on \mathbb{Z}_p associated with degenerate Bernoulli polynomials of the second kind. *Adv. Differ. Equ.* **2020**, 278 (2020)
6. Kim, D.S., Kim, T.: Three variable symmetric identities involving Carlitz-type q -Euler polynomials. *Math. Sci.* **8**(4), 147–152 (2014)
7. Kim, D.S., Lee, N., Na, J., Park, K.H.: Abundant symmetry for higher-order Bernoulli polynomials (I). *Adv. Stud. Contemp. Math. (Kyungshang)* **23**(3), 461–482 (2013)
8. Kim, D.S., Lee, N., Na, J., Park, K.H.: Abundant symmetry for higher-order Bernoulli polynomials (II). *Proc. Jangjeon Math. Soc.* **16**(3), 359–378 (2013)
9. Kim, D.S., Park, K.H.: Identities of symmetry for Euler polynomials arising from quotients of fermionic integrals invariant under S_3 . *J. Inequal. Appl.* **2010**, Article ID 851521 (2010)
10. Kim, D.S., Park, K.H.: Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under S_3 . *Appl. Math. Comput.* **219**(10), 5096–5104 (2013)
11. Kim, T.: q -Volkenborn integration. *Russ. J. Math. Phys.* **9**(3), 288–299 (2002)
12. Kim, T.: Symmetry p -adic invariant integral on \mathbb{Z}_p for Bernoulli and Euler polynomials. *J. Adv. Differ. Equ. Appl.* **14**, 1267–1277 (2008)
13. Kim, T., Dolgy, D.V., Kim, D.S.: Some identities of q -Bernoulli polynomials under symmetry group S_3 . *J. Nonlinear Convex Anal.* **16**(9), 1869–1880 (2015)
14. Kim, T., Kim, D. S., Dolgy, D.V., Pyo, S.-S.: Identities of symmetry for type 2 q -Bernoulli polynomials under symmetric group S_3 . *J. Nonlinear Convex Anal.* **21**(9), 1973–1979 (2020)
15. Schikhof, W.H.: *Ultrametric Calculus: An Introduction to p -Adic Analysis*. Cambridge University Press, Cambridge (2006)
16. Simsek, Y.: Complete sum of products of (h, q) -extension of Euler polynomials and numbers. *J. Differ. Equ. Appl.* **16**(11), 1331–1348 (2010)
17. Tuentner, H.: A symmetry of power sum polynomials and Bernoulli numbers. *Am. Math. Mon.* **108**, 258–261 (2001)
18. Yang, S.: An identity of symmetry for the Bernoulli polynomials. *Discrete Math.* **308**, 550–554 (2008)

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