Journal of Inequalities and Applications
a SpringerOpen Journal

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Extraction new results of common fixed point theorems for (T,α_s,F)-contraction of six mappings in a tripled b-metric space with an application of integral equations

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Abstract

The aim of this work is to usher in tripled b-metric spaces, triple weakly α_s-admissible, triangular partially triple weakly α_s-admissible and their properties for the first time. Also, we prove some theorems about coincidence and common fixed point for six self-mappings. On the other hand, we present a new model, talk over an application of our results to establish the existence of common solution of the system of Volterra-type integral equations in a triple b-metric space. Also, we give some example to illustrate our theorems in the section of main results. Finally, we show an application of primary results.

MSC: α_S-complete tripled b-metric; (T,α_s,F)-contractions; Common fixed point

Keywords: 47H10; 54H25; 37C25

1 Introduction and preliminaries

The Banach contraction principle plays a central part in metric fixed point theory, and a great number of researchers revealed many fruitful generalizations of this resolution in diverse ways. In 1989, Bakhtin investigated the concept of b-metric space [1]. However, Czerwik initiated the study of fixed point of self-mappings in a b-metric space and proved an analogue of Banach's fixed point theorem [2]. Since then, numerous research articles have been published comprising fixed point theorems for several classes of single-valued and multi-valued operators in b-metric spaces (for example, consider [3–6]). In 2012, the concept of F-contraction, which is one of these generalizations, was introduced by Wardowski [7]. He presented that every F-contraction defined in a complete metric space has a unique fixed point. Subsequently, the subject of F-contraction proved to be a milestone in the fixed point theory, and numerous research papers on F-contraction have been published (for instance, see [4, 8–19]). In the same year, Samet et al. investigated the idea of (α,ψ)-contractive and α-admissible mappings and established some significant fixed point solutions for such a variety of functions defined on a complete metric space (for more details, see [20]). Some authors such as Salimi, Latif, Hussain et al. improved the concept of α-admissibility and proved some important (common) fixed point theorems as well (for more information, see [21–24].

Recently, Cosentino and Vetro established a fixed point result for Hardy–Rogers-type F-contraction [25]. Also, Minak, Helvaci, and Altun presented a fixed point result for Ćirić-type generalized F-contraction [26]. In 2018, Nazam, Muhammad, and Postolache investigated some common fixed point results for four self-mappings satisfying such kind of contractions on the α_s-complete b-metric space and applied their conclusion to infer several new and old results, based on the idea of Ćirić-type and Hardy–Rogers-type (α_s, F)-contractions [27].

In this study, motivated by [27] and among these achievements, we are working to stretch out the Ćirić-type and Hardy–Rogers-type (α_s, F)-contractions based on six self-mappings defined on a b-metric space. Also, some common fixed point results for six self-mappings satisfying such kind of contractions are shown in the (T, α_s, F)-complete tripled b-metric space. Consequently, we discuss an application of the main result to show the existence of common solution of the system of Volterra-type integral equations.

Let X be a nonempty set, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}_0^+ = [0, \infty)$, and $s > 1$ be a real constant. Suppose that d_b maps $X \times X \times X$ into $\mathbb{R}_0^+$ somehow that for all x, y, z, and a_i with $i \in \{1, 2, 3, 4\}$ belong to X satisfying the following conditions [9]:

- $d_b(x, y, z) = 0$ if and only if $x = y = z$.
- $d_b(x, y, z) > 0$ if and only if $x \neq y$ or $x \neq z$ or $y \neq z$.
- $d_b(x, y, z) = d_b(x, z, y) = d_b(z, y, x) = d_b(y, x, z) = d_b(z, x, y) = d_b(y, z, x)$.
- $d_b(x, x, y) = d_b(x, y, y)$.
- $d_b(x, x, y) \leq d_b(x, y, z)$, $d_b(x, x, z) \leq d_b(x, y, z)$, $d_b(y, y, z) \leq d_b(x, y, z)$.
- $d_b(x, y, z) \leq s[d_b(x, a_1, a_2) + d_b(y, a_3, a_4) + d_b(z, a_2, a_3)]$.

We say that (X, d_b, s) is a tripled b-metric space.

Example 1.1 Let $X = \mathbb{R}_0^+$. We define $d_b : X^3 \to \mathbb{R}_0^+$ as follows:

$$d_b(x, y, z) = \max\{|x - y|^2, |x - z|^2, |y - z|^2\}.$$

Then (X, d_b, s) is a tripled b-metric space with $s = 2$.

We bring back into reader's mind some definitions and properties of b-metric.

Definition 1.2 (see [2]) Let A be a nonempty set, and let $s > 1$ be a real number. A mapping $d^* : A^2 \to \mathbb{R}_0^+$ is said to be a b-metric if, for all a, b, and $c \in A$, we have:

- $a = b$ if and only if $d^*(a, b) = 0$;
- $d^*(a, b) = d^*(b, a)$;
- $d^*(a, b) \leq s[d^*(a, c) + d^*(c, b)]$.

In this case, the triple (A, d^*, s) is called a b-metric space (with coefficient s).

Remark 1.3 Definition 1.2 allows us to remark that b-metric space is effectually more general than metric space as a b-metric is a metric when $s = 1$. It is worth to mention that the b-metric structure produces some differences to the classical case of metric spaces: the b-metric on a nonempty set M need not be continuous, open balls in such spaces need not be open sets, and so on. The following example describes the significance of a b-metric.

For the notions like convergence, completeness, Cauchy sequence in the setting of b-metric spaces, the reader is referred to Aghajani et al. [28], Czerwik [2], Amini-Harandi [29], Huang et al. [3], Khamsi and Hussain [5]. In line with Wardowski [7], Cosentino et al. [30] investigated a nonlinear function $F:\mathbb{R}^+ \to \mathbb{R}$ complying with the following axioms:

- F is strictly increasing;
- $\lim_{r\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$;
- $\lim_{r\to\infty} r_n = 0$ there exists $a \in (0,1)$ such that $\lim_{r_n\to 0^+} (r_n)^a F(r_n) = 0$;
- $\tau + F(sr_n) \leq F(r_{n-1})$ implies $\tau + F(s^n r_n) \leq F(s^{n-1} r_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$

for all sequence $\{r_n\}$ of positive numbers. We denote the set of all functions satisfying the conditions (F_1), (F_2), (F_3), and (F_4) by $\mathcal{F}_s$.

Example 1.4 (see [30]) Let $F:\mathbb{R}^+ \to \mathbb{R}$ be defined by $F(r) = \ln r$ or $F(r) = r + \ln r$. Then F satisfies in the conditions.

Theorem 1.5 (see [31]) *Let (X,d) be a complete metric space and $T: X \to X$ be a bijective (ξ,α,η)-expansive mapping of type B satisfying the following conditions*:

- *T^{-1} is α-admissible with respect to η;*
- *There exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq \eta(x_0, T^{-1}x_0)$;*
- *T is continuous.*

Then T has a fixed point.

Definition 1.6 (see [32]) Let (X,p_b) be a partial b-metric space with the coefficient $s \geq 1$. A mapping $T: X \to X$ is said to be a generalized $\alpha-\eta-\psi$-Geraghty contractive type mapping if there exist $\psi \in \Psi$, $\alpha,\eta: X\times X \to [0,\infty)$, and $\beta \in \mathcal{F}$ such that

$$\alpha(x,y) \geq \eta(x,y) \quad \text{implies} \quad \psi\bigl(sp_b(Tx,Ty)\bigr) \leq \beta\bigl(\psi\bigl(M_s^T(x,y)\bigr)\bigr)\psi\bigl(M_s^T(x,y)\bigr) \tag{1.1}$$

for all $x,y \in X$, where

$$M_s^T(x,y) = \max\left\{p_b(x,y), p_b(x,Tx), p_b(y,Ty), \frac{p_b(x,Ty)+p_b(y,Tx)}{2s}\right\}.$$

Theorem 1.7 (see [32]) *Let (X,p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let $T: X \to X$ be a generalized $\alpha-\eta-\psi$-Geraghty contractive type mapping. Suppose that the following conditions hold*:

- *T is a triangular α-orbital admissible mapping with respect to η;*
- *There exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$;*
- *$\{x_n\}$ is α-regular with respect to η.*

Then T has a fixed point.

Example 1.8 (see [32]) Let $X = [0,\infty)$ and with the partial b-metric $p_b: X\times X \to [0,\infty)$ defined by $p_b(x,y) = \max\{x,y\}^2$ for all $x,y \in X$. Obviously, (X,p_b) is a partial b-metric space with $s = 2$. Define the mapping $T: X\to X$ given by

$$Tx = \begin{cases} \frac{x}{9} & \text{if } x \in [0,1]; \\ \ln x + 3 & \text{if } x \in (1,\infty). \end{cases}$$

Define $\psi : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,1)$ by $\psi(t) = t$ and

$$\beta(t) = \begin{cases} \frac{e^{-t}}{1+t} & \text{if } x \in (0,\infty); \\ \frac{1}{2} & \text{if } t = 0. \end{cases}$$

Let $\alpha, \eta : X \times X \to [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 6 & \text{if } x \in [0,1]; \\ 0 & \text{if } x \in (1,\infty), \end{cases}$$

and

$$\eta(x,y) = \begin{cases} 2 & \text{if } x \in [0,1]; \\ 1 & \text{if } x \in (1,\infty). \end{cases}$$

Let $\alpha(x,Tx) \geq \eta(x,Tx)$. Thus $x, Tx \in [0,1]$ and so $T^2x = T(Tx) \in [0,1]$, which implies that $\alpha(Tx,T^2x) \geq \eta(Tx,T^2x)$, that is, T is α-orbital admissible with respect to η. Now, let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$, we get that $x,y,Ty \in [0,1]$ and so $\alpha(x,Ty) \geq \eta(x,Ty)$. Therefore T is triangular α-orbital admissible with respect to η. Let $\{x_n\}$ be a sequence such that $\{x_n\}$ is p_b-convergent to z and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Then $\{x_n\} \subseteq [0,1]$ for any $n \in \mathbb{N}$ and so $z \in [0,1]$, from which we have $\alpha(x_n,z) \geq \eta(x_n,z)$. That is, $\{x_n\}$ is α-regular with respect to η. The condition (ii) of Theorem 1.7 is satisfied with $x_1 = 1 \in X$ since $(\alpha(1,T1) = 2 \geq 2 = \eta(1,T1)$. We next prove that T is a generalized α-η-ψ-Geraghty contraction type mapping. Let $x,y \in X$ with $\alpha(x,y) \geq \eta(x,y)$. Thus $x,y \in [0,1]$. Without loss of generality, we may assume that $0 \leq y \leq x \leq 1$. Therefore

$$p_b(Tx,Ty) = \left[\max\left\{\frac{x}{9},\frac{y}{9}\right\}\right]^2 = \frac{x^2}{81}$$

and

$$M_s^T(x,y) = \max\left\{x^2,x^2,y^2,\frac{x^2+[\max\{y,\frac{x}{9}\}]^2}{4}\right\} = x^2.$$

Since $\frac{2}{81} \leq \frac{1}{2e} \leq \frac{e^{-x^2}}{1+x^2}$, we obtain that

$$\begin{aligned} \psi\left(sp_b(Tx,Ty)\right) &= \psi\left(2\frac{x^2}{81}\right) = \frac{2x^2}{81} \leq \frac{e^{-x^2}}{1+x^2} \cdot x^2 \\ &\leq \beta\left(\psi\left(x^2\right)\right)\psi\left(x^2\right) \\ &\leq \beta\left(\psi\left(M_s^T(x,y)\right)\right)\psi\left(M_s^T(x,y)\right). \end{aligned}$$

Thus T is a generalized α-η-ψ-Geraghty contraction type mapping. Hence all the assumptions in Theorem 1.7 are satisfied and thus T has a fixed point which is $x = 0$.

Definition 1.9 (see [27]) Let (M, d^*, s) be a b-metric space, $S : M \to M$ and $\alpha_s : M \times M \to \mathbb{R}_0^+$ be two mappings. The mapping S is said to be α_s-admissible if

$$\alpha_s(r_1, r_2) \geq s^2 \Rightarrow \alpha_s\big(S(r_1), S(r_2)\big) \geq s^2 \quad \text{for all } r_1, r_2 \in M.$$

Theorem 1.10 (see [27]) *Let M be a nonempty set and α_s be as defined in Definition 1.9. Let f, g, S, T be α_s – b-continuous self-mappings defined on an α_s-complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that, for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$\tau + F\big(sd^*\big(f(r_1), g(r_2)\big)\big) \leq F\big(\mathcal{M}_1(r_1, r_2)\big). \tag{1.2}$$

Assume that the pairs (f, S), (g, T) are α_s-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point (say) v in M. Moreover, if $\alpha_s(Sv, Tv) \geq s^2$, then v is a common fixed point of f, g, S, T.

Remark 1.11 (see [27]) If we suppose that $\alpha_s(v, w) \geq s^2$ for each pair of common fixed point of f, g, S, T, then v is unique. Indeed, if w is another fixed point of f, g, S, T and assuming on the contrary that $d^*(fv, gw) > 0$, then from (1.2) we have

$$F\big(sd^*(v, w)\big) = F\big(sd^*\big(S(v), T(w)\big)\big) \leq F\big(\mathcal{M}_1(v, w)\big) - \tau, \tag{1.3}$$

where

$$\mathcal{M}_1(v, w) = \max\Bigg\{d^*\big(S(v), T(w)\big), d^*\big(f(v), S(v)\big), \\ d^*\big(g(w), T(w)\big) \frac{d^*(S(v), g(w)) + d^*(f(v), T(w))}{2s}\Bigg\}.$$

Thus, by (1.3), we have

$$F\big(sd^*(v, w)\big) < F\big(d^*(v, w)\big),$$

which is a contradiction. Hence, $v = w$ and v is a unique common fixed point of self-mappings f, g, S, T.

Theorem 1.12 (see [27]) *Let f, g, S, T be self-mappings defined on an α_s-regular and α_s-complete metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$, and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that, for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$\tau + F(sd^*\big(f(r_1), g(r_2)\big) \leq F\big(\mathcal{M}_1(r_1, r_2)\big). \tag{1.4}$$

Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point v in M. Moreover, if $\alpha_s(Sv, Tv) \geq s^2$, then v is a coincidence point of f, g, S, T.

Theorem 1.13 (see [27]) *Let f, g, S, T be α_s-continuous self-mappings defined on an α_s-complete b-metric space (M, d^*, s) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that, for all $(r_1, r_2) \in \gamma_{f,g,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $\tau > 0$ such that*

$$\tau + F\bigl(sd^*\bigl(f(r_1), g(r_2)\bigr)\bigr) \le F\bigl(\mathcal{M}_i(r_1, r_2)\bigr) \tag{1.5}$$

holds for one of $i = 2, 3, 4, 5, 6$, where

$$\begin{aligned}\mathcal{M}_2(r_1, r_2) = {} & a_1 d^*\bigl(S(r_1), T(r_2)\bigr) + a_2 d^*\bigl(f(r_1), S(r_1)\bigr) + a_3 d^*\bigl(g(r_2), T(r_2)\bigr) \\ & + a_4\bigl[d^*\bigl(S(r_1), g(r_2)\bigr) + d^*\bigl(f(r_1), T(r_2)\bigr)\bigr]\end{aligned}$$

with $a_i \ge 0$, $i = 1, 2, 3, 4$, such that $a_1 + a_2 + a_3 + 2sa_4 = 1$;

$$\mathcal{M}_3(r_1, r_2) = a_1 d^*\bigl(S(r_1), T(r_2)\bigr) + a_2 d^*\bigl(f(r_1), S(r_1)\bigr) + a_3 d^*\bigl(g(r_2), T(r_2)\bigr),$$

with $a_1 + a_2 + a_3 = 1$;

$$\begin{aligned}\mathcal{M}_4(r_1, r_2) = {} & k \max\bigl\{d^*\bigl(f(r_1), S(r_1)\bigr), d^*\bigl(g(r_2), T(r_2)\bigr)\bigr\} \quad \text{with } k \in [0, 1); \\ \mathcal{M}_5(r_1, r_2) = {} & a_1(r_1, r_2) d^*\bigl(S(r_1), T(r_2)\bigr) + a_2(r_1, r_2) d^*\bigl(f(r_1), S(r_1)\bigr) \\ & + a_3(r_1, r_2) d^*\bigl(g(r_2), T(r_2)\bigr) \\ & + a_4(r_1, r_2)\bigl[d^*\bigl(S(r_1), g(r_2)\bigr)\bigr) + d^*\bigl(f(r_1), T(r_2)\bigr)\bigr]\end{aligned}$$

with $a_i(r_1, r_2)$, $i = 1, 2, 3, 4$ are nonnegative functions such that

$$\begin{aligned}& \sup_{r_1, r_2 \in M}\bigl\{a_1(r_1, r_2) + a_2(r_1, r_2) + a_3(r_1, r_2) + 2sa_4(r_1, r_2)\bigr\} = 1; \\ & \mathcal{M}_6(r_1, r_2) = a_1 d^*\bigl(S(r_1), T(r_2)\bigr) + \frac{a_2 + a_3}{2}\bigl[d^*\bigl(f(r_1), S(r_1)\bigr) + d^*\bigl(g(r_2), T(r_2)\bigr)\bigr] \\ & \qquad + \frac{a_4 + a_5}{2s}\bigl[d^*\bigl(S(r_1), g(r_2)\bigr) + d^*\bigl(f(r_1), T(r_2)\bigr)\bigr]\end{aligned}$$

with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$.

Assume that the pairs (f, S), (g, T) are α_s-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α_s-admissible pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have the coincidence point v in M. Moreover, if $\alpha_s(Sv, Tv) \ge s^2$, then v is a common point of f, g, S, T.

2 Main results

In this section, first we introduce some definitions in a tripled b-metric space (X, d_b) and present several examples.

Definition 2.1 Let (X, d_b, s) be a tripled b-metric space, $T : X \to X$ and $\alpha_s : X^3 \to \mathbb{R}_0^+$ be two mappings. The mapping T is said to be α_s-admissible if $\alpha_s(x, y, z) \ge s^2$, then $\alpha_s(Tx, Ty, Tz) \ge s^2$ for all $x, y, z \in X$.

Definition 2.2 Let (X, d_b, s) be a tripled b-metric space, $T : X \to X$ and $\alpha_s : X^3 \to \mathbb{R}_0^+$ be two mappings. The mapping T is said to be triangular α_s-admissible if

- $\alpha_s(x,y,z) \geq s^2$ implies that $\alpha_s(Tx, Ty, Tz) \geq s^2$ for all $x,y,z \in X$;
- $\alpha_s(x,y,z) \geq s^2$ and $\alpha_s(y,z,w) \geq s^2$ imply $\alpha_s(x,z,w) \geq s^2$for all $x,y,z,w \in X$.

Definition 2.3 Let (X,d_b,s) be a tripled b-metric space, $f,g,h : X \to X$ and $\alpha_s : X^3 \to \mathbb{R}_0^+$ be four mappings. The tripled (f,g,h) is said to be

- triple weakly α_s-admissible if $\alpha_s(f(x), gf(x), hgf(x)) \geq s^2$, $\alpha_s(g(x), hg(x), fhg(x)) \geq s^2$, and $\alpha_s(h(x), fh(x), gfh(x)) \geq s^2$ for all $x \in X$;
- partially weakly α_s-admissible if $\alpha_s(f(x), gf(x), hgf(x)) \geq s^2$ for all $x \in X$.

Definition 2.4 Let (X,d_b,s) be a tripled b-metric space and $f,g,h,\phi : X \to X$ be four mappings such that $f(X) \cup g(X) \cup h(X) \subseteq \phi(X)$. The triple of mappings (f,g,h) is said to be

- triple weakly α_s-admissible with respect to ϕ if and only if $\alpha_s(f(x), g(y), h(z)) \geq s^2$ for all $x \in X$, for all $y \in \phi^{-1}gf(x)$, for all $z \in \phi^{-1}hgf(x)$ and $\alpha_s(h(x), g(y), f(z)) \geq s^2$ for all $x \in X$, for all $y \in \phi^{-1}gh(x)$, for all $z \in \phi^{-1}fgh(x)$ and $\alpha_s(g(x), f(y), h(z)) \geq s^2$ for all $x \in X$, for all $y \in \phi^{-1}fg(x)$, for all $z \in \phi^{-1}hfg(x)$;
- partially triple weakly α_s-admissible with respect to ϕ if and only if

$$\alpha_s\bigl(f(x), g(y), h(z)\bigr) \geq s^2$$

for all $x \in X$, $y \in \phi^{-1}gf(x)$, and $z \in \phi^{-1}hgf(x)$.

Definition 2.5 Let (X,d_b,s) be a tripled b-metric space and $f,g,h,\phi : X \to X$ be four mappings such that $f(X) \cup g(X) \cup h(X) \subseteq \phi(X)$. The triple of mappings (f,g,h) is said to be triangular triple weakly α_s-admissible with respect to ϕ if

- $\alpha_s(h(x), g(y), f(z)) \geq s^2$ for all $x \in X$, for all $y \in \phi^{-1}gf(x)$, $z \in \phi^{-1}hgf(x)$, and

$$\alpha_s\bigl(h(x), g(y), f(z)\bigr) \geq s^2$$

for all $x \in X$, for all $y \in \phi^{-1}gh(x)$, for all $z \in \phi^{-1}fgh(x)$, and $\alpha_s(g(x), f(y), h(z)) \geq s^2$ for all $x \in X$, for all $y \in \phi^{-1}fg(x)$, for all $z \in \phi^{-1}hfg(x)$;
- $\alpha_s(x,y,z) \geq s^2$ and $\alpha_s(y,z,w) \geq s^2$ imply $\alpha_s(x,z,w) \geq s^2$ for all $x,y,z,w \in X$.

Example 2.6 Let $X = \mathbb{R}_0^+$ and

$$d_b(x,y,z) = \max\bigl\{|x-y|^2, |x-z|^2, |y-z|^2\bigr\}$$

for all $x,y,z \in X$. Then (X,d_b,s) is a tripled b-metric with $s = 2$. We define $f(x) = x$, $g(x) = x^{\frac{1}{2}}$, $h(x) = x^{\frac{1}{4}}$, and $S(x) = x^4$ if $x \in [0,1)$ and $f(x) = g(x) = h(x) = S(x) = 1$, whenever $x \in [1,\infty)$ and $\alpha_s : X^3 \to \mathbb{R}_0^+$ as follows:

$$\alpha(x,y,z) = \begin{cases} \max\{4+y-x, 4+z-x, 4+z-x\}, & x,y,z \in [0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then, for all $x \in [0,1)$, $y \in S^{-1}(g(f(x)))$, $z \in S^{-1}(h(g(x)))$, we have $y = x^{\frac{1}{8}}$, $z = x^{\frac{1}{32}}$,

$$\alpha_s\bigl(x, g\bigl(x^{\frac{1}{8}}\bigr), h\bigl(x^{\frac{1}{32}}\bigr)\bigr) = \alpha_s\bigl(x, x^{\frac{1}{16}}, x^{\frac{1}{32\times 4}}\bigr) \geq s^2.$$

Thus the triple of mappings (f, g, h) is triangular weakly α_s-admissible with respect to S. Indeed, if $\alpha_s(x,y,z) \geq s^2$ and $\alpha_s(y,z,w) \geq s^2$, then $\alpha_s(x,z,w) \geq s^2$. Since $y-x \geq 0$ or $z-x \geq 0$ or $z-y \geq 0$ and $z-y \geq 0$ or $w-z \geq 0$ or $w-y \geq 0$. Thus $w-x \geq 0$ or $w-z \geq 0$ or $z-x \geq 0$.

Definition 2.7 Let $f, g, h, \phi : X \to X$ be four self-mappings defined on a tripled b-metric space such that $f(X) \cup g(X) \cup h(X) \subseteq \phi(X)$. The triple of mappings (f, g, h) is said to be triangular triple partially weakly α_s-admissiblewith respect to ϕ if

- $\alpha_s(f(x), g(y), h(z)) \geq s^2$ for all $x \in X$, $y \in \phi^{-1}(g(f(x)))$, $z \in \phi^{-1}(hg(f(x)))$,
- $\alpha_s(x,y,z) \geq s^2$, $\alpha_s(y,z,w) \geq s^2$ imply $\alpha_s(x,z,w) \geq s^2$ for all $x, y, z \in X$.

Definition 2.8 Let (X, d_b, s) be a tripled b-metric space. The tripled b-metric space X is said to be α_s-complete if and only if every Cauchy sequence $\{x_n\}$ in X such that $\alpha_s(x_n, x_{n+1}, x_{n+2}) \geq s^2$ for all $n \in \mathbb{N}$ converges in X. That is,

$$\lim_{n\to\infty} d_b(x_n, x, x) = \lim_{n\to\infty} d_b(x_n, x_n, x) = 0.$$

If X is a complete tripled metric space, then X is also an α_s-complete tripled metric space, but the converse is not true. The following example explains this fact.

Example 2.9 Let $X = \mathbb{R}^+$ and $d_b : X^3 \to \mathbb{R}_0^+$ be the tripled b-metric. Define $\alpha_s : X^3 \to \mathbb{R}_0^+$,

$$\alpha(x,y,z) = \begin{cases} 4\max\{e^{|x-y|}, e^{|y-z|}, e^{|x-z|}\}, & x,y,z \in [0, \frac{5}{2}), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that (X, d_b, S) in not a complete tripled b-metric space, but (X, d_b, s) is an α_s-complete tripled b-metric.

Definition 2.10 Let (X, d_b, s) be a tripled b-metric space. We say that the self-mapping T is an α_s-continuous mapping on (X, d_b, s) if, for given $x \in X$ and sequence $\{x_n\}$,

$$\lim_{n\to\infty} d_b(x_n, x, x) = \lim_{n\to\infty} d_b(x_n, x_n, x) = 0,$$

and $\alpha(x_n, x_{n+1}, x_{n+2}) \geq s^2$ for all $n \in \mathbb{N}$ implies

$$\lim_{n\to\infty} d_b(Tx_n, Tx, Tx) = \lim_{n\to\infty} d_b(Tx_n, Tx_n, Tx) = 0.$$

Example 2.11 Let $X = \mathbb{R}_0^+$ and $d_b : X^3 \to \mathbb{R}_0^+$ for all $x, y, z \in X$, define by $d_b(x,y,z) = \max\{|x-y|^2, |x-z|^2, |y-z|^2\}$ and

$$T(x) = \begin{cases} \sin \pi x, & x \in [0,1], \\ \cos \pi x + 2, & x \in (1, \infty), \end{cases}$$

$$\alpha_s(x,y,z) = \begin{cases} x^2 + y^2 + 4, & x,y,z \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then T is not continuous on X; however, T is α_s-continuous.

Definition 2.12 Let (X, d_b, s) be a tripled b-metric space. The pairs of self-mappings (f, g), (g, h), and (f, h) are said to be α_s-compatible if

$$\lim_{n\to\infty} d_b\bigl(gh(x_n), hg(x_n), g(x_n)\bigr) = 0,$$
$$\lim_{n\to\infty} d_b\bigl(fg(x_n), gf(x_n), f(x_n)\bigr) = 0,$$
$$\lim_{n\to\infty} d_b\bigl(hf(x_n), fh(x_n), h(x_n)\bigr) = 0,$$

or $\lim_{n\to\infty} d_b(gh(x_n), hg(x_n), h(x_n)) = 0$ or $\lim_{n\to\infty} d_b(fg(x_n), gf(x_n), g(x_n)) = 0$ or

$$\lim_{n\to\infty} d_b\bigl(hf(x_n), fh(x_n), f(x_n)\bigr) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s^2$, and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = t$$

for some $t \in X$.

Example 2.13 Let $X = [1, \infty)$ and $d_b : X \times X \times X \to \mathbb{R}_0^+$ be defined by

$$d_b(x, y, z) = \max\bigl\{|x - y|^2, |x - z|^2, |y - z|^2\bigr\}$$

for all $x, y, z \in X$, then $(X, d_b, s = 2)$ is a tripled b-metric space. Define $f(x) = 4$, $g(x) = 16 - 3x$ if $x \in [1, 4]$ and $f(x) = 8$ and $g(x) = 9$ whenever $x \in (4, \infty)$ and

$$\alpha(x, y, z) = \begin{cases} 6, & x, y, z \in [1, 4], \\ 0, & \text{otherwise.} \end{cases}$$

Let us consider $\{x_n\}$ to be a sequence such that $\alpha(x_n, x_{n+1}, x_{n+2}) \geq s^2$, and let

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x),$$

then $x_n = 4$. It is clear that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x) = 4$. We obtain that

$$\begin{aligned} \lim_{n\to\infty} d_b\bigl(fg(x_n), gf(x_n), f(x_n)\bigr) &= \lim_{n\to\infty} d_b\bigl(fg(x_n), gf(x_n), g(x_n)\bigr) \\ &= d_b(4, 4, 4) = 0. \end{aligned}$$

Hence (f, g) is an α_s-compatible pair. Now, if we consider $x_n = 4 - \frac{1}{n}$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = 4.$$

But $\lim_{n\to\infty} gf(x_n) = 4$,

$$\lim_{n\to\infty} fg(x_n) = \lim_{n\to\infty} f\left(16 - 3\left(4 - \frac{1}{n}\right)\right) = \lim_{n\to\infty} f\left(4 + \frac{3}{n}\right) = 8,$$

and $\lim_{n\to\infty} d_b(fg(x_n), gf(x_n), fx_n) \neq 0$. Consequently, (f, g) is not compatible.

Definition 2.14 Let f, g, and T be self-mappings defined on a nonempty set X. If $f(x) = g(x) = T(x)$ for some $x \in X$, then x is called a coincidence point of f, g, and T. Three self-mappings f, g, and T defined on X are said to be weakly compatible if $\{f, g\}$, $\{g, T\}$, and $\{f, T\}$ commute at their coincidence points.

Definition 2.15 Let (X, d_b, s) be a tripled b-metric space. The space (X, d_b, s) is said to be α_s-regular if, for any sequence $\{x_n\}$ in X, the following condition holds: if $x_n \to x$ as $n \to \infty$ and $\alpha_s(x_n, x_{n+1}, x_{n+2}) \geq s^2$ for all $n \in \mathbb{N}$, then $\alpha_s(x_n, x, x) \geq s^2$ and $\alpha_s(x_n, s_n, x) \geq s^2$ for all $n \in \mathbb{N}$.

Now, we are ready to prove our results.

Lemma 2.16 *Let (X, d_b, s) be a tripled b-metric space. If there exist three sequence $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that $\lim_{n\to\infty} d_b(x_n, y_n, z_n) = 0$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = t$ for some $t \in X$, then $\lim_{n\to\infty} z_n = t$.*

Proof By the triangle inequality, we have

$$d_b(z_n, t, t) \leq s\bigl[d_b(z_n, x_n, y_n) + d_b(t, t, t) + d_b(t, y_n, t)\bigr].$$

By taking limit as $n \to \infty$, the result follows. □

Definition 2.17 Let (X, d_b, s) be a tripled b-metric space, $f, g, h, S_1, S_2, S_3 : X \to X$ be self-mappings, and α_s be as defined in Definition 2.1. We define the set λ_{f,g,h,α_s} by

$$\begin{aligned} \lambda_{f,g,h,\alpha_s} = \bigl\{&(\alpha, \beta, \gamma) \in X^3 : \alpha_s\bigl(S_1(\alpha), S_2(\beta), S_3(\gamma)\bigr) \geq s^2, \\ &\text{and } d_b\bigl(f(\alpha), g(\beta), h(\gamma)\bigr) > 0\bigr\}. \end{aligned} \tag{2.1}$$

Let

$$\begin{aligned} &M(\alpha, \beta, \gamma) \\ &\quad = \max\biggl\{d_b\bigl(S_1(\alpha), S_2(\beta), S_3(\gamma)\bigr), d_b\bigl(f(\alpha), S_2(\alpha), S_3(\alpha)\bigr), \\ &\qquad d_b\bigl(g(\beta), S_1(\beta), S_3(\beta)\bigr), d_b\bigl(h(\gamma), S_1(\gamma), S_2(\gamma)\bigr), \\ &\qquad \frac{d_b(S_1(\alpha), g(\beta), h(\gamma)) + d_b(f(\alpha), S_2(\beta), h(\gamma)) + d_b(S_3(\gamma), g(\beta), f(\alpha))}{3s}\biggr\}. \end{aligned} \tag{2.2}$$

The following theorem is one of our main results.

Theorem 2.18 *Let X be a nonempty set and α_s be as defined in Definition 2.1. Let f, g, h, S_1, S_2, S_3 be α_s – b-continuous self-mappings defined an α_s-complete tripled b-metric space (X, d_b, s) such that $f(X) \subseteq S_1(X)$, $g(X) \subseteq S_2(X)$, and $h(X) \subseteq S_3(X)$. Suppose that, for all $(x, y, z) \in \lambda_{f,g,h,\alpha_s}$, there exist $F \in \mathcal{F}_s$ and $r > 0$ such that*

$$r + F\bigl(sd_b\bigl(f(x), g(y), h(z)\bigr)\bigr) \leq F\bigl(M(x, y, z)\bigr). \tag{2.3}$$

Assume that the pairs (f,S_1), (g,S_2), and (h,S_3) are α_s-compatible and the triples (f,g,h), (g,f,h), and (h,g,f) are triangular partially weakly α_s-admissible with respect to S_1, S_2, and S_3, respectively. Then the pairs (f,S_1), (g,S_2), and (h,S_3) have the coincidence fixed point say v in X. Moreover, if $\alpha_s(S_1(v),S_2(v),S_3(v)) \geq s^2$, then v is a common fixed point of f, g, h, S_1, S_2, S_3.

Proof Let $x_0 \in X$ be an arbitrary point. As $f(X) \subseteq S_1(X)$, there exists $x_1 \in X$ such that $f(x_0) = S_1(x_1)$. Since $g(x_1) \in S_2(X)$, we can choose $x_2 \in X$ such that $g(x_1) = S_2(x_2)$. Since $h(x_2) \in S_3(X)$, there exists $x_3 \in X$ such that $h(x_2) = S_3(x_3)$. In general, x_{2n}, x_{2n+1}, and x_{2n+2} are chosen in X such that $f(x_{2n}) = S_1(x_{2n+1})$, $g(x_{2n+1}) = S_2(x_{2n+2})$, and $h(x_{2n+2}) = S_3(x_{2n+3})$. Define a sequence $\{J_n\} \in X$ such that, for all $n \in \mathbb{N}$, $J_{2n+1} = f(x_{2n}) = S_1(x_{2n+1})$, $J_{2n+2} = g(x_{2n+1}) = S_2(x_{2n+2})$, and $J_{2n+2} = h(x_{2n+2}) = S_3(x_{2n+3})$. As $x_1 \in S_1^{-1}(f(x_0))$, $x_2 \in S_2^{-1}(g(x_1))$, $x_3 \in S_3^{-1}(h(x_2))$, and (f,g,h), (h,g,f), and (g,f,h) are triangular partially weakly α_s-admissible triples of mappings with respect to S_1, S_2, and S_3, respectively, we have

$$\alpha_s\bigl(f(x_0),g(x_1),h(x_2)\bigr) = \alpha_s\bigl(S_1(x_1),S_2(x_2),S_3(x_3)\bigr) \geq s^2,$$
$$\alpha_s\bigl(h(x_2),g(x_1),f(x_0)\bigr) = \alpha_s\bigl(S_3(x_3),S_2(x_2),S_1(x_1)\bigr) \geq s^2,$$

and

$$\alpha_s\bigl(g(x_1),f(x_0),h(x_2)\bigr) = \alpha_s\bigl(S_2(x_2),S_1(x_1),S_3(x_3)\bigr) \geq s^2.$$

Continuing this way, we obtain

$$\alpha_s\bigl(S_1(x_{2n+1}),S_2(x_{2n+2}),S_3(x_{2n+3})\bigr) \geq s^2,$$
$$\alpha_s\bigl(S_3(x_{2n+3}),S_2(x_{2n+2}),S_1(x_{2n+1})\bigr) \geq s^2,$$

and $\alpha_s(S_2(x_{2n+2}),S_1(x_{2n+1}),S_3(x_{2n+3})) \geq s^2$. Thus, we have

$$\alpha_s(J_{2n+1},J_{2n+2},J_{2n+3}) \geq s^2,$$
$$\alpha_s(J_{2n+3},J_{2n+2},J_{2n+1}) \geq s^2,$$

and $\alpha_s(J_{2n+2},J_{2n+1},J_{2n+3}) \geq s^2$ for all $n \in \mathbb{N}$. At present, we prove that

$$\lim_{l\to\infty} d_b(J_l,J_{l+1},J_{l+2}) = 0.$$

Set $d_l = d_b(J_l,J_{l+1},J_{l+2})$. Suppose that $d_{l_0} = 0$ for some l_0. Then $J_{l_0} = J_{l_0+1}$. If $l_0 = 2n$, then $J_{2n} = J_{2n+1}$ gives $J_{2n+1} = J_{2n+2}$. Indeed, by contractive condition (2.3), we get

$$\begin{aligned} F\bigl(sd_b(J_{2n+1},J_{2n+2},J_{2n+3})\bigr) &= F\bigl(sd_b\bigl(f(x_{2n}),g(x_{2n+1}),h(x_{2n+2})\bigr)\bigr) \\ &\leq F\bigl(M(x_{2n},x_{2n+1},x_{2n+2})\bigr) - r \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$M(x_{2n},x_{2n+1},x_{2n+2}) = \max\Bigl\{d_b\bigl(S_1(x_{2n}),S_2(x_{2n+1}),S_3(x_{2n+2})\bigr),$$

$$\begin{aligned}
&d_b\big(f(x_{2n}), S_2(x_{2n}), S_3(x_{2n})\big),\\
&d_b\big(g(x_{2n+1}), S_1(x_{2n+1}), S_3(x_{2n+1})\big),\\
&d_b\big(h(x_{2n+2}), S_1(x_{2n+2}), S_2(x_{2n+2})\big),\\
&\frac{1}{3s}\big[d_b\big(S_1(x_{2n}), g(x_{2n+1}), h(x_{2n+2})\big)\\
&+ d_b\big(f(x_{2n}), S_2(x_{2n+1}), h(x_{2n+2})\big)\\
&+ d_b\big(S_3(x_{2n+2}), g(x_{2n+1}), f(x_{2n})\big)\big]\Big\}\\
= \max\Big\{&d_b(J_{2n}, J_{2n+1}, J_{2n+2}), d_b(J_{2n+1}, J_{2n}, J_{2n-1}),\\
&d_b(J_{2n+2}, J_{2n+1}, J_{2n}), d_b(J_{2n+2}, J_{2n+2}, J_{2n+2}),\\
&\frac{1}{3s}\big[d_b(J_{2n}, J_{2n+2}, J_{2n+2}) + d_b(J_{2n+1}, J_{2n+1}, J_{2n+2})\\
&+ d_b(J_{2n+1}, J_{2n+2}, J_{2n+1})\big]\Big\}.
\end{aligned}$$

So

$$\begin{aligned}
M(x_{2n}, x_{2n+1}, x_{2n+2}) = \max\Big\{&d_b(J_{2n}, J_{2n+1}, J_{2n+1}), d_b(J_{2n-1}, J_{2n}, J_{2n+1}),\\
&d_b(J_{2n}, J_{2n+1}, J_{2n+2}),\\
&\frac{1}{3s}\big[d_b(J_{2n}, J_{2n+2}, J_{2n+2}) + d_b(J_{2n+1}, J_{2n+1}, J_{2n+2})\\
&+ d_b(J_{2n+1}, J_{2n+1}, J_{2n+2})\big]\Big\}\\
\le \max\big\{&d_b(J_{2n}, J_{2n+1}, J_{2n+2}), d_b(J_{2n-1}, J_{2n}, J_{2n+1}),\\
&d_b(J_{2n}, J_{2n+1}, J_{2n+2}), d_b(J_{2n}, J_{2n+1}, J_{2n+2})\big\}\\
= \max\big\{&d_b(J_{2n}, J_{2n+1}, J_{2n+2}), d_b(J_{2n-1}, J_{2n}, J_{2n+1})\big\}.
\end{aligned}$$

Since $d_b(J_{2n}, J_{2n+1}, J_{2n+2}) = 0$, therefore $M(x_{2n}, x_{2n+1}, x_{2n+2}) = d_b(J_{2n-1}, J_{2n}, J_{2n+1})$. Then

$$F\big(sd_b(J_{2n+1}, J_{2n+2}, J_{2n+3})\big) = F\big(d_b(J_{2n-1}, J_{2n}, J_{2n+1})\big) - r.$$

By (F_1), we have

$$sd_b(J_{2n+1}, J_{2n+2}, J_{2n+3}) \le d_b(J_{2n-1}, J_{2n}, J_{2n+1}) - r.$$

Let $l = 2n$, then we have $sd_b(J_{2n}, J_{2n+1}, J_{2n+2}) \le d_b(J_{2n-2}, J_{2n-1}, J_{2n}) - r$. Thus, for all n,

$$d_b(J_n, J_{n+1}, J_{n+2}) \le \frac{1}{s} d_b(J_{n-1}, J_n, J_{n+1}).$$

That is, a sequence $\{d_b(J_n, J_{n+1}, J_{n+2})\}$ is nonincreasing and $d_b(J_n, J_{n+1}, J_{n+2}) \to 0$ as $n \to \infty$. Hence $\lim_{l\to\infty} d_b(J_l, J_{l+1}, J_{l+2}) = 0$ holds true. Now, suppose that $d_l = d_b(J_l, J_{l+1}, J_{l+2}) > 0$

for each $l \in \mathbb{N}$. We claim that $\lim_{n\to\infty} d_b(J_n, J_{n+1}, J_{n+2}) = -\infty$. Let $l = 2n$. As

$$\alpha_s\big(S_1(x_{2n}), S_2(x_{2n+1}), S_3(x_{2n+2})\big) \geq s^2,$$

$d_b(f(x_{2n}), g(x_{2n}), h(x_{2n+1})) > 0$, so $(x_{2n-1}, x_{2n}, x_{2n+1}) \in \lambda_{f,g,h,\alpha_s}$, by (2.3), we obtain

$$F\big(sd_b(J_{2n}, J_{2n+1}, J_{2n+2})\big) \leq F\big(d_b(J_{2n-1}, J_{2n}, J_{2n+1})\big) - r \tag{2.4}$$

for all $n \in \mathbb{N}$. Similarly, for $\updownarrow = 2n - 1$,

$$F\big(sd_b(J_{2n-1}, J_{2n}, J_{2n+1})\big) \leq F\big(d_b(J_{2n-2}, J_{2n-1}, J_{2n})\big) - r \tag{2.5}$$

for all $n \in \mathbb{N}$. Hence, by (2.4) and (2.5), we have

$$F\big(sd_b(J_n, J_{n+1}, J_{n+2})\big) \leq F\big(d_b(J_{n-1}, J_n, J_{n+1})\big) - r \tag{2.6}$$

for all $n \in \mathbb{N}$. Let $a_n = d_b(J_n, J_{n+1}, J_{n+2})$ for each $n \in \mathbb{N}$. By (2.6) and property (F_4), we have $r + F(s^n a_n) \leq F(s^{n-1} a_{n-1})$ for all $n \in \mathbb{N}$. Continuing this process, we obtain

$$F\big(s^n a_n\big) \leq F(a_n) - nr \tag{2.7}$$

for all $n \in \mathbb{N}$. On taking limit $n \to \infty$ in (2.7), we have $\lim_{n\to\infty} F(s^n a_n) = -\infty$. By property (F_2), we get $\lim_{n\to\infty} s^n a_n = 0$ and (F_2) implies that there exists $k \in (0,1)$ such that $\lim_{n\to\infty}(s^n a_n)^k F(s^n a_n) = 0$. By (2.7), for all $n \in \mathbb{N}$, we obtain

$$\big(s^n a_n\big)^k F\big(s^n a_n\big) - \big(s^n a_n\big)^k F(a_0) \leq -\big(s^n a_n\big)^k nr \leq 0. \tag{2.8}$$

On taking limit $n \to \infty$ in (2.8), we have $\lim_{n\to\infty} n(s^n a_n)^k = 0$. This implies there exists $n_1 \in \mathbb{N}$ such that $n(s^n a_n)^k \leq 1$ for all $n \geq n_1$, or $s^n a_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_1$. To prove $\{J_n\}$ is a Cauchy sequence, by the triangular inequality, we have

$$\begin{aligned}
d_b(x_n, x_m, x_m) &\leq s\big[d_b(x_n, x_{n+1}, x_{n+2}) + d_b(x_m, x_m, x_m), d_b(x_m, x_{m+2}, x_m)\big] \\
&= sd_b(x_n, x_{n+1}, x_{n+2}) + sd_b(x_{n+2}), x_m, x_m) \\
&\leq sd_b(x_n, x_{n+1}, x_{n+2}) + s^2\big[d_b(x_{n+2}, x_{n+3}, x_{n+4}) \\
&\quad + d_b(x_m, x_m, x_m) + d_b(x_m, x_{n+3}, x_{n+1})\big] \\
&= sd_b(x_n, x_{n+1}, x_{n+2}) + s^2 d_b(x_{n+2}, x_{n+3}, x_{n+4}) + s^2 d_b(x_{n+3}, x_m, x_m) \\
&\leq sd_b(x_n, x_{n+1}, x_{n+2}) + s^2 d_b(x_{n+2}, x_{n+3}, x_{n+4}) \\
&\quad + s^3 d_b(x_{n+3}, x_{n+4}, x_{n+5}) + s^3 d_b(x_{n+4}, x_m, x_m).
\end{aligned}$$

Take $m = n + p$, $(n, p \in \mathbb{N})$, then we have

$$\begin{aligned}
d_b(x_n, x_m, x_m) &\leq sd_b(x_n, x_{n+1}, x_{n+2}) + s^2 d_b(x_{n+2}, x_{n+3}, x_{n+4}) \\
&\quad + s^3 d_b(x_{n+3}, x_{n+4}, x_{n+5}) + \cdots + s^{n-1} d_b(x_{n+p-1}, x_{n+p}, x_{n+p})
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{s}{s^n n^{\frac{1}{k}}} + \frac{s^2}{s^{n+2}(n+2)^{\frac{1}{k}}} + \frac{s^3}{s^{n+3}(n+3)^{\frac{1}{k}}} \\
&\quad + \cdots + \frac{s^{p-1}}{s^{n+P-1}(n+p-1)^{\frac{1}{k}}} \\
&= \frac{s^{1-n}}{n^{\frac{1}{k}}} + \frac{s^{-n}}{(n+2)^{\frac{1}{k}}} + \frac{s^{-n}}{(n+3)^{\frac{1}{k}}} + \cdots + \frac{s^{-n}}{(n+p-1)^{\frac{1}{k}}} \\
&= \frac{s^{1-n}}{n^{\frac{1}{k}}} + s^{-n} \sum_{i=2}^{p-1} \frac{1}{(n+i)^{\frac{1}{k}}}.
\end{aligned}$$

Since $\sum_{i=2}^{p-1} \frac{1}{(n+i)^{\frac{1}{k}}}$ is convergent and $s^{-n} \to 0$ as $n \to \infty$, thus we conclude that

$$\lim_{n,m\to\infty} d_b(x_n, x_m, x_m) = 0.$$

This implies that $\{J_n\}$ is a Cauchy sequence in the α_s-complete tripled b-metric space X and

$$\alpha_s(J_n, J_{n+1}, J_{n+2}) \geq s^2,$$

there exists $\nu \in X$ such that

$$\lim_{n\to\infty} d_b(J_{2n+1}, \nu, \nu) = \lim_{n\to\infty} d_b(fx_{2n}, \nu, \nu) = \lim_{n\to\infty} d_b\big(S_1(x_{2n+1}), \nu, \nu\big) = 0.$$

Consequently, $f(x_{2n}) \to \nu$ and $S_1(x_{2n+1}) \to \nu$ as $n \to \infty$. So

$$\lim_{n\to\infty} d_b(J_{2n+1}, \nu, \nu) = \lim_{n\to\infty} d_b(gx_{2n}, \nu, \nu) = \lim_{n\to\infty} d_b\big(S_2(x_{2n+1}), \nu, \nu\big) = 0.$$

Thus $g(x_{2n}) \to \nu$ and $S_2(x_{2n+1}) \to \nu$ as $n \to \infty$. Again, we have

$$\lim_{n\to\infty} d_b(J_{2n}, \nu, \nu) = \lim_{n\to\infty} d_b(hx_{2n}, \nu, \nu) = \lim_{n\to\infty} d_b\big(S_3(x_{2n+1}), \nu, \nu\big) = 0.$$

Hence $h(x_{2n}) \to \nu$ and $S_3(x_{2n+1}) \to \nu$ as $n \to \infty$. Now, since (f, S_1) is an α_s-compatible pair and

$$\alpha_s(J_{2n}, J_{2n+1}, J_{2n+2}) \geq s^2.$$

Therefore, we have $\lim_{n\to\infty} d_b(fS_1(x_{2n}), S_1 f(x_{2n}), x_{2n}) = 0$ and (g, S_2) is an α_s-compatible pair and

$$\alpha_s(J_{2n}, J_{2n+1}, J_{2n+2}) \geq s^2.$$

We have $\lim_{n\to\infty} d_b(gS_2(x_{2n}), S_2 g(x_{2n}), x_{2n}) = 0$ and (h, S_3) is an α_s-compatible pair, we get

$$\lim_{n\to\infty} d_b\big(hS_3(x_{2n}), S_3 h(x_{2n}), x_{2n}\big) = 0.$$

Since $\lim_{n\to\infty} d_b(f(x_{2n}), v, v) = 0$, $\lim_{n\to\infty} d_b(S_1(x_{2n}), v, v) = 0$, and f, S_1 is α_s-continuous. Thus $\lim_{n\to\infty} d_b(S_1 f(x_{2n}), S_1 v, S_1 v) = 0$, $\lim_{n\to\infty} d_b(fS_1(x_{2n}), fv, fv) = 0$, and

$$\lim_{n\to\infty} d_b\bigl(g(x_{2n}), v, v\bigr) = 0,$$

so g, S_2 is α_s-continuous, we have $\lim_{n\to\infty} d_b(S_2 g(x_{2n}), S_2 v, S_2 v) = 0$ and

$$\lim_{n\to\infty} d_b\bigl(gS_2(x_{2n}), gv, gv\bigr) = 0.$$

Again in this way, $\lim_{n\to\infty} d_b(S_3 h(x_{2n}), S_3 v, S_3 v) = 0$ and $\lim_{n\to\infty} d_b(hS_3 g(x_{2n}), hv, hv) = 0$. By the triangle inequality, we have

$$\begin{aligned} d_b\bigl(fv, S_1 v, S_1(x_{2n})\bigr) &\le s\bigl[d_b\bigl(fv, fv, fS_1(x_{2n})\bigr) + d_b\bigl(S_1 v, S_1 f(x_{2n}), S_1 v\bigr) \\ &\quad + d_b\bigl(S_1 x_{2n}, fS_1 x_{2n}, S_1 f(x_{2n})\bigr)\bigr]. \end{aligned} \tag{2.9}$$

Applying limit as $n \to \infty$, we obtain $d_b(fv, S_1 v, v) \le 0$, which yields that $fv = S_1 v = v$. Thus v is a coincidence and common fixed point of f, S_1. Arguing in a similar manner, we can prove that $gv = S_2 v = v$ and $hv = S_1 v = v$. Thus $fv = gv = hv = S_1 v = S_2 v = S_3 v = v$ and v is a common fixed point of f, g, h, S_1, S_2, and S_3. □

Remark 2.19 If we suppose that $\alpha_s(v, w, w) \ge s^2$ for each pair of common fixed points of f, g, h, S_1, S_2, and S_3, then v is unique. Indeed, if w is another fixed point of f, g, h, S_1, S_2, and S_3 and assuming on contrary $d_b(fv, gw, hw) > 0$, then from (2.3) we have

$$F\bigl(d_b(v, w, w)\bigr) = F\bigl(sd_b\bigl(S_1(v), S_2(w), S_3(w)\bigr)\bigr) \le F\bigl(M(v, w, w)\bigr) - r, \tag{2.10}$$

where

$$\begin{aligned} M(v, w, w) = \max\Bigl\{ & d_b\bigl(S_1(v), S_2(w), S_3(w)\bigr), d_b\bigl(f(v), S_2(v), S_3(v)\bigr), \\ & d_b\bigl(g(w), S_1(w), S_3(w)\bigr), d_b\bigl(h(w), S_1(w), S_2(w)\bigr), \\ & \frac{1}{3s}\bigl[d_b\bigl(S_1(v), g(w), h(w)\bigr) \\ & + d_b\bigl(f(v), S_2(w), h(w)\bigr) + d_b\bigl(S_3(w), g(w), f(v)\bigr)\bigr]\Bigr\} \\ = \max\Bigl\{ & d_b(v, w, w), d_b(v, v, v), d_b(w, w, w), d_b(w, w, w), \\ & \frac{1}{3s}\bigl[d_b(v, w, w), d_b(v, w, w) + d_b(w, w, v)\bigr]\Bigr\}. \end{aligned}$$

Thus, by (2.10), we have $F(sd_b(v, w, w)) \le F(d_b(v, w, w)) - r < F(d_b(v, w, w))$, which is a contradiction. Hence $v = w$ and v is a unique common fixed point of self-mappings f, g, h, S_1, S_2, and S_3.

The following example elucidates Theorem 2.18.

Example 2.20 Let $X = \mathbb{R}_0^+$ and $d_b : X \times X \times X \to \mathbb{R}_0^+$ be defined by

$$d_b(x,y,z) = \max\{|x-y|^2, |x-z|^2, |y-z|^2\}$$

for all $x,y,z \in X$. Define $\alpha_s : X \times X \times X \to \mathbb{R}_0^+$ by

$$\alpha_s(x,y,z) = \begin{cases} 4\max\{e^{x-y}, e^{x-z}, e^{y-z}\}, & x \geq y \geq z, \\ 4\max\{e^{y-x}, e^{z-x}, e^{z-y}\}, & x \leq y \leq z. \end{cases}$$

So (S, d_b, s) is an α_s-complete tripled b-metric with $s = 2$. Define the mappings f, g, h, S_1, S_2, and $S_3 : X \to X$ for all $x \in X$ by

$$\begin{aligned} f(x) &= \ln\left(1 + \frac{x}{5}\right), \\ g(x) &= \ln\left(1 + \frac{x}{6}\right), \\ h(x) &= \ln\left(1 + \frac{x}{7}\right), \end{aligned}$$

$S_1(x) = e^{6x} - 1$, $S_2(x) = e^{7x} - 1$, and $S_3(x) = e^{8x} - 1$. Clearly, f, g, h, S_1, S_2, and S_3 are α_s-continuous self-mappings complying with $f(X) = g(X) = h(X) = S_1(X) = S_2(X) = S_3(X)$. We note that the pair (f, S_1) is α_s-compatible. Indeed, let $\{x_n\}$ be a sequence in X satisfying $\alpha_s(x_n, x_{n+1}, x_{n+2}) \geq s^2$ and

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \ln\left(1 + \frac{x_n}{5}\right) = \lim_{n\to\infty} S_1(x_n) = t$$

for some $t \in X$. Then $\lim_{n\to\infty} |f(x_n) - t|^2 = \lim_{n\to\infty} |S_1(x_n) - t|^2 = 0$, equivalently

$$\lim_{n\to\infty} \left|\ln\left(1 + \frac{x_n}{5}\right) - t\right|^2 = \lim_{n\to\infty} \left|e^{6x_n} - 1 - t\right|^2 = 0$$

implies

$$\lim_{n\to\infty} \left|x_n - \left(5e^t - 5\right)\right|^2 = \lim_{n\to\infty} \left|x_n - \frac{\ln(t+1)}{6}\right|^2 = 0.$$

Uniqueness of limit gives that $5e^t - 5 = \frac{\ln(t+1)}{6}$, thus $t = 0$ is only possible solution. Due to *alpha*$_s$-continuity of f and S_1, for $t = 0 \in X$, we have

$$\begin{aligned} &\lim_{n\to\infty} d_b\left(fS_1(x_n), S_1 f(x_n), f(x_n)\right) \\ &\quad = \max\left\{\lim_{n\to\infty} \left|fS_1(x_n) - S_1 f(x_n)\right|^2, \right. \\ &\qquad \left. \lim_{n\to\infty} \left|S_1 f(x_n) - f(x_n)\right|^2, \lim_{n\to\infty} \left|fS_1(x_n) - f(x_n)\right|^2\right\} \\ &\quad = \max\left\{\left|f(t) - S_1(t)\right|^2, \left|S_1(t) - t\right|^2, \left|f(t) - t\right|^2\right\} \\ &\quad = 0. \end{aligned}$$

Similarly, the pair (g, S_2) and (h, S_3) is α_s-compatible. To prove that (f, g, h) is a partially weakly α_s-admissible triple of mappings with respect to $S_!$, let $x \in X$ and $y \in S_1^{-1}(g(f(x)))$, that is, $S_1(y) = g(f(x))$ and

$$e^{6y} - 1 = g\left(\ln\left(1 + \frac{x}{5}\right)\right) = \ln\left(1 + \frac{\ln(1 + \frac{x}{5})}{6}\right).$$

Thus $y = \frac{1}{6}\ln(1 + \ln(1 + \frac{\ln(1+\frac{x}{5})}{6}))$. We have

$$f(x) = \ln\left(1 + \frac{x}{5}\right) \geq g(y) = \ln\left(1 + \frac{y}{6}\right) = \ln\left(1 + \frac{1}{36}\ln\left(1 + \ln\left(1 + \frac{\ln(1 + \frac{x}{5})}{6}\right)\right)\right).$$

We have $z \in S_1^{-1}(hg(f(x)))$, that is, $S_1(z) = hg(f(x))$, $S_1(z) = h(S_1(y))$, $e^z - 1 = \ln(1 + \frac{S_1(y)}{7})$,

$$e^{6z} - 1 = \ln\left(1 + \frac{1}{7}\ln\left(1 + \frac{\ln(1 + \frac{x}{5})}{6}\right)\right),$$

and

$$z = \frac{1}{6}\ln\left(1 + \ln\left(1 + \frac{1}{7}\ln\left(\frac{\ln(1 + \frac{x}{5})}{5}\right)\right)\right).$$

We conclude that

$$\begin{aligned}
g(y) &= \ln\left(1 + \frac{y}{6}\right) = \ln\left(1 + \frac{1}{42}\ln\left(1 + \ln\left(1 + \frac{\ln(1 + \frac{x}{5})}{6}\right)\right)\right) \\
&\geq h(z) = \ln\left(1 + \frac{z}{7}\right) \\
&= \ln\left(1 + \frac{1}{42}\ln\left(1 + \ln\left(1 + \frac{1}{7}\ln\left(1 + \frac{\ln(1 + \frac{x}{5})}{6}\right)\right)\right)\right).
\end{aligned}$$

Thus $\alpha_s(f(x), g(y), h(z)) = 4\max\{e^{x-y}, e^{x-z}, e^{y-z}\} \geq s^2$. In this process, we can prove that (g, f, h) is a partially weakly α_s-admissible triple of mappings with respect to S_2 and (h, g, f) is a partially weakly α_s-admissible triple of mappings with respect S_1. Now, for each $x, y, z \in X$, consider

$$d_b(f(x), g(y), h(z)) = \max\left\{\left|f(x) - g(y)\right|^2, \left|g(y) - h(z)\right|^2, \left|f(x) - h(z)\right|^2\right\},$$

$$\begin{aligned}
\left|f(x) - g(y)\right|^2 &= \left|\ln\left(1 + \frac{x}{5}\right) - \ln\left(1 + \frac{y}{6}\right)\right|^2 \\
&\leq \left(\frac{x}{5} - \frac{y}{6}\right)^2 \\
&= \frac{1}{900}(6x - 5y)^2 \\
&\leq \frac{1}{900}\left(e^{6x} - e^{5y}\right)^2,
\end{aligned}$$

$$\begin{aligned}\left|g(y)-h(z)\right|^2 &= \left|\ln\left(1+\frac{y}{6}\right)-\ln\left(1+\frac{z}{7}\right)\right|^2\\ &\leq \left(\frac{y}{6}-\frac{z}{7}\right)^2\\ &= \frac{1}{1764}(7y-6z)^2\\ &\leq \frac{1}{1764}\left(e^{7y}-e^{6z}\right)^2,\end{aligned}$$

and

$$\begin{aligned}\left|f(x)-h(z)\right|^2 &= \left|\ln\left(1+\frac{x}{5}\right)-\ln\left(1+\frac{z}{7}\right)\right|^2\\ &\leq \left(\frac{x}{5}-\frac{z}{7}\right)^2\\ &= \frac{1}{1225}(7x-5z)^2\\ &\leq \frac{1}{1225}\left(e^{7x}-e^{5z}\right)^2.\end{aligned}$$

Thus

$$\begin{aligned}d_b\big(f(x),g(y),h(z)\big) &\leq \frac{1}{900}\max\left\{\left(e^{6x}-e^{5y}\right)^2,\left(e^{7y}-e^{6z}\right)^2,\left(e^{7x}-e^{5z}\right)^2\right\}\\ &= \frac{1}{900}d_b\big(S_1(x),S_2(y),S_3(z)\big).\end{aligned}$$

Define the function $F:\mathbb{R}^+\to\mathbb{R}$ by $F(x)=\ln x$ for all $x\in\mathbb{R}^+$. Hence, for all $x,y,z\in X$ such that $d_b(f(x),g(y),h(z))>0$, $r=\ln(900)$, we obtain

$$r+F\big(d_b\big(f(x),g(y),h(z)\big)\big)\leq F\big(M(x,y,z)\big).$$

Thus the contractive condition (2.3) is satisfied for all $x,y,z\in X$. Hence, all the hypotheses of Theorem 2.18 are satisfied. Note that f, g, h, S_1, S_2, and S_3 have a unique common fixed point $x=0$.

We have obtained some results from Theorem 2.18, which we express in order.

Corollary 2.21 *Let X be a nonempty set and $\alpha_s: X\times X\times X\to\mathbb{R}_0^+$ be a function. Let (X,d_b,s) be an α_s-complete tripled metric space and f, g, h, S_1, S_2, and S_3 be α_s-continuous self-mappings on (X,d_b,s) such that for all $(x,y,z)\in\lambda_{f,g,h,\alpha_s}$ the inequality*

$$sd_b\big(f(x),g(y),h(z)\big)\leq kM(x,y,z) \tag{2.11}$$

holds. Assume that the pairs (f,S_1), (g,S_2), and (h,S_3) are α_s-compatible and the triples of mappings (f,g,h), (g,f,h), and (h,g,f) are triangular partially weakly α_s-admissible with respect to S_1, S_2, and S_3, respectively. Then the pairs (f,S_1), (g,S_2), and (h,S_3) have the coincidence point v in X. Moreover, if $\alpha_s(S_1v,S_2v,S_3v)\geq s^2$, then v is a common fixed point of f, g, h, S_1, S_2, and S_3.

Proof For all $(x,y,z) \in \lambda_{f,g,h,\alpha_s}$, we have $sd_b(f(x),g(y),h(z)) \le kM(x,y,z)$. It follows that $r + \ln(d_b(f(x),g(y),h(z))) \le \ln(M(x,y,z))$, where $r = \ln(\frac{s}{k}) > 0$. Then the contraction condition (2.11) reduces to (2.3) with $F(x) = \ln x$, and the application of Theorem 2.18 ensures the existence of a fixed point. □

If we set $S = S_1 = S_2 = S_3$ in Theorem 2.18, we obtain the following corollaries.

Corollary 2.22 *Let f, g, h, and S be self-mappings defined on an α_s-complete tripled metric space (X,d_b,s) such that $f(X) \cup g(X) \cup h(X) \subseteq S(X)$ with α_s-continuous. Suppose that, for all $x,y,z \in X$ with $\alpha_s(Tx,Ty,Tz) \ge s^2$, there exist $F \in \mathcal{F}_s$ and $r > 0$ such that $d_b(f(x),g(y),h(z)) > 0$, then*

$$r + F\bigl(sd_b\bigl(f(x),g(y),h(z)\bigr)\bigr) \le F\bigl(M(x,y,z)\bigr),$$

where

$$\begin{aligned} M(x,y,z) = \max\Bigl\{ & d_b\bigl(S(x),S(y),S(z)\bigr), d_b\bigl(f(x),S(x),S(x)\bigr), \\ & d_b\bigl(g(y),S(y),S(y)\bigr), d_b\bigl(h(z),S(z),S(z)\bigr), \\ & \frac{1}{3s}\bigl[d_b\bigl(S(x),g(y),h(z)\bigr) + d_b\bigl(f(x),S(y),h(z)\bigr) \\ & + d_b\bigl(S(z),g(y),f(x)\bigr)\bigr]\Bigr\}. \end{aligned}$$

Assume that either the pair (f,S) is α_s-compatible and f is α_s-continuous or (g,S) is α_s-compatible and g is α_s-continuous, or (h,S) is α_s-compatible and h is α_s-continuous. Then the pairs (f,S), (g,S), and (h,S) have the coincidence point v in X provided that the triple of mappings (f,g,h) is triangular weakly α_s-admissible with respect to S. Moreover, if $\alpha_s(Sv,Sv,Sv) \ge s^2$, then v is a common fixed point of f, g, h, and S.

If we set $S_1 = S_2 = S_3$ and $f = g = h$ in Theorem 2.18, we obtain the following corollary.

Corollary 2.23 *Let f and S be α_s-continuous self-mappings defined on an α_s-complete tripled metric space (X,d_b,s) such that $f(X) \subseteq S(X)$. Suppose that, for all $x,y,z \in X$ with $\alpha_s(Sx,Sy,Sz) \ge s^2$, there exist $F \in \mathcal{F}_s$ and $r > 0$ such that $d_b(f(x),f(y),f(z)) > 0$, then*

$$r + F\bigl(sd_b\bigl(f(x),f(y),f(z)\bigr)\bigr) \le F\bigl(M(x,y,z)\bigr),$$

where

$$\begin{aligned} M(x,y,z) = \max\Bigl\{ & d_b\bigl(S(x),S(y),S(z)\bigr), d_b\bigl(f(x),S(x),S(x)\bigr), \\ & d_b\bigl(f(y),S(y),S(y)\bigr), d_b\bigl(f(z),S(z),S(z)\bigr), \\ & \frac{1}{3s}\bigl[d_b\bigl(S(x),f(y),f(z)\bigr) + d_b\bigl(f(x),S(y),f(z)\bigr) \\ & + d_b\bigl(S(z),f(y),f(x)\bigr)\bigr]\Bigr\}. \end{aligned}$$

Assume that the pair (f,S) is α_s-compatible. Then the mappings f and S have the coincidence fixed point in X provided that fg is a triangular weakly α_s-admissible mapping with respect to S. Moreover, if $\alpha_s(Sv,Sv,Sv) \geq s^2$, then f, S has a common point v.

Corollary 2.24 *Let f,g,h, and S be self-mappings defined on an α_s-regular and α_s-complete tripled metric space (X,d_b,s) such that $f(X),g(X),h(X) \subseteq S(X)$, and $S(X)$ is a closed subset of X. Suppose that, for all $x,y,z \in X$ with $\alpha_s(Sx,Sy,Sz) \geq s^2$, there exist $F \in \mathcal{F}_s$, and $r > 0$ such that $d_b(f(x),g(y),h(z)) > 0$, then $r + F(sd_b(f(x),g(y),h(z))) \leq F(M(x,y,z))$, where*

$$\begin{aligned} M(x,y,z) = \max\Big\{ & d_b\big(S(x),S(y),S(z)\big), d_b\big(f(x),S(x),S(x)\big), \\ & d_b\big(g(y),S(y),S(y)\big), d_b\big(h(z),S(z),S(z)\big), \\ & \frac{1}{3s}\big[d_b\big(S(x),g(y),h(z)\big) + d_b\big(f(x),S(y),h(z)\big) \\ & + d_b\big(S(z),g(y),f(x)\big)\big]\Big\}. \end{aligned}$$

Assume that the pairs (f,S), (g,S), and (h,S) are weakly compatible and the triple of mappings (f,g,h) is triangular weakly α_s-admissible with respect to S. Then the pairs (f,S), (g,S), and (h,S) have the coincidence point v in X. Moreover, if $\alpha_s(Sv,Sv,Sv) \geq s^2$, then v is a coincidence point of f, g, h, and S.

Corollary 2.25 *Let f and S be self-mappings defined on an α_s-regular and α_s-complete tripled metric space (X,d_b,s) such that $f(X) \subseteq S(X)$, and $S(X)$ is a closed subset of X. Suppose that, for all $x,y,z \in X$ with $\alpha_s(Sx,Sy,Sz) \geq s^2$, there exist $F \in \mathcal{F}_s$ and $r > 0$ such that $d_b(f(x),f(y),f(z)) > 0$, then $r + F(sd_b(f(x),f(y),f(z))) \leq F(M(x,y,z))$, where*

$$\begin{aligned} M(x,y,z) = \max\Big\{ & d_b\big(S(x),S(y),S(z)\big), d_b\big(f(x),S(x),S(x)\big), \\ & d_b\big(f(y),S(y),S(y)\big), d_b\big(f(z),S(z),S(z)\big), \\ & \frac{1}{3s}\big[d_b\big(S(x),f(y),f(z)\big) + d_b\big(f(x),S(y),f(z)\big) \\ & + d_b\big(S(z),f(y),f(x)\big)\big]\Big\}. \end{aligned}$$

Assume that the pair (f,S) is weakly compatible and f is a triangular weakly α_s-admissible mapping with respect to S. Then the pair (f,S) has the coincidence point v in X.

Corollary 2.26 *Let f,g, and h be self-mappings defined on a complete tripled metric space (X,d_b,s). Suppose that, for all $x,y,z \in X$ with $\alpha_s(x,y,z) \geq s^2$, there exist $F \in \mathcal{F}_s$ and $r > 0$ such that $d_b(f(x),g(y),h(z)) > 0$, then $r + F(sd_b(f(x),g(y),h(z))) \leq F(M(x,y,z))$, where*

$$\begin{aligned} M(x,y,z) = \max\Big\{ & d_b(x,y,z), d_b\big(f(x),x,x\big), \\ & d_b\big(g(y),y,y\big), d_b\big(h(z),z,z\big), \end{aligned}$$

$$\frac{1}{3s}\big[d_b\big(x,g(y),h(z)\big)+d_b\big(f(x),y,h(z)\big)$$
$$+d_b\big(z,g(y),f(x)\big)\big]\bigg\}.$$

Assume that the triple of mappings (f,g,h) is triangular weakly α_s-admissible. Then f, g, and h have a common fixed point v in X provided that either f or g or h is α_s-continuous, or X is α_s-regular.

Theorem 2.27 *Let f, g, h, S_1, S_2, and S_3 be α_s-continuous self-mappings defined on an α_s-complete tripled b-metric space (X,d_b,s) such that $f(X)\subseteq S_1(X)$, $g(X)\subseteq S_2(X)$, and $h(X)\subseteq S_3(X)$. Suppose that, for all $(x,y,z)\in\lambda_{f,g,h,\alpha_s}$, there exist $F\in\mathcal{F}_s$ and $r>0$ such that*

$$r+F\big(sd_b\big(f(x),g(y),h(z)\big)\big)\le F\big(M_i(x,y,z)\big) \tag{2.12}$$

holds for one of $i=1,2,3,4,5$, where

$$\begin{aligned}M_1(x,y,z)={}&a_1d_b\big(S_1(x),S_2(y),S_3(z)\big)+a_2d_b\big(f(x),S_2(x),S_3(x)\big)\\&+a_3d_b\big(g(y),S_1(y),S_3(y)\big)+a_4d_b\big(h(z),S_1(z),S_2(z)\big)\\&+a_5\big[d_b\big(S_1(x),g(y),h(z)\big)+d_b\big(f(x),S_2(y),h(z)\big)\\&+d_b\big(S_3(z),g(y),f(x)\big)\big]\end{aligned}$$

with $a_i\ge 0$, $i=1,2,3,4,5$, such that $a_1+a_2+a_3+3a_5=s$,

$$\begin{aligned}M_2(x,y,z)={}&a_1d_b\big(S_1(x),S_2(y),S_3(z)\big)+a_2d_b\big(f(x),S_2(x),S_3(x)\big)\\&+a_3d_b\big(g(y),S_1(y),S_3(y)\big)+a_4d_b\big(h(z),S_1(z),S_2(z)\big)\end{aligned}$$

with $a_1+a_2+a_3=s$,

$$\begin{aligned}M_3(x,y,z)={}&k\max\big\{d_b\big(f(x),S_2(x),S_3(x)\big),d_b\big(g(y),S_1(y),S_3(y)\big),\\&d_b\big(h(z),S_1(z),S_2(z)\big)\big\}\end{aligned}$$

with $k\in[0,1)$,

$$\begin{aligned}M_4(x,y,z)={}&a_1(x,y,z)d_b\big(S_1(x),S_2(y),S_3(z)\big)\\&+a_2(x,y,z)d_b\big(f(x),S_2(x),S_3(x)\big)\\&+a_3(x,y,z)d_b\big(g(y),S_1(y),S_3(y)\big)\\&+a_4d_b\big(h(z),S_1(z),S_2(z)\big)\\&+a_5(x,y,z)\big[d_b\big(S_1(x),g(y),h(z)\big)\\&+d_b\big(f(x),S_2(y),h(z)\big)\\&+d_b\big(S_3(z),g(y),f(x)\big)\big]\end{aligned}$$

with $a_i(x,y,z)$, $i = 1,2,3,4,5$, are nonnegative functions such that

$$\sup_{x,y,z\in X}\bigl[a_1(x,y,z) + a_2(x,y,z) + a_3(x,y,z) + 3a_5(x,y,z)\bigr] = s.$$

Suppose that the pairs (f,S_1), (g,S_2), and (h,S_3) are α_s-compatible and the triples of mappings (f,g,h), (g,f,h), and (h,g,f) are triangular partially triple weakly α_s-admissible with respect to S_1, S_2, and S_3, respectively. Then the pairs (f,S_1), (g,S_2), and (h,S_3) have the coincidence point v in X. Moreover, if $\alpha_s(S_1(v),S_2(v),S_3(v)) \geq s^2$, then v is a common fixed point of f, g, h, S_1, S_2, and S_3.

Proof In line with the beginning part of Theorem 2.18, for all $(x,y,z) \in \lambda_{f,g,h,\alpha_s}$ for some $F \in \mathcal{F}_s$ and $r > 0$, from contractive condition (2.12) we get

$$\begin{aligned} F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) &= F\bigl(sd_b\bigl(f(x_{2n}),g(x_{2n+1}),h(x_{2n+2})\bigr)\bigr) \\ &\leq F\bigl(M_1(x_{2n},x_{2n+1},x_{2n+2})\bigr) - r \end{aligned} \tag{2.13}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_1(x_{2n},x_{2n+1},x_{2n+2}) &= a_1 d_b\bigl(S_1(x_{2n}),S_2(x_{2n+1}),S_3(x_{2n+2})\bigr) \\ &\quad + a_2 d_b\bigl(f(x_{2n}),S_2(x_{2n}),S_3(x_{2n})\bigr) \\ &\quad + a_3 d_b\bigl(g(x_{2n+1}),S_1(x_{2n+1}),S_3(x_{2n+1})\bigr) \\ &\quad + a_4 d_b\bigl(h(x_{2n+2}),S_1(x_{2n+2}),S_2(x_{2n+2})\bigr) \\ &\quad + a_5\bigl[d_b\bigl(S_1(x_{2n}),g(x_{2n+1}),h(x_{2n+2})\bigr) \\ &\quad + d_b\bigl(f(x_{2n}),S_2(x_{2n+1}),h(x_{2n+2})\bigr) \\ &\quad + d_b\bigl(S_3(x_{2n+2}),g(x_{2n+1}),f(x_{2n})\bigr)\bigr] \\ &= a_1 d_b(J_{2n},J_{2n+1},J_{2n+1}) + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}) \\ &\quad + a_3 d_b(J_{2n+2},J_{2n+1},J_{2n}) + a_4 d_b(J_{2n+2},J_{2n+2},J_{2n+2}) \\ &\quad + a_5\bigl[d_b(J_{2n},J_{2n+2},J_{2n+2}) + d_b(J_{2n+1},J_{2n+1},J_{2n+2}) \\ &\quad + d_b(J_{2n+1},J_{2n+2},J_{2n+1})\bigr] \\ &\leq a_1 d_b(J_{2n},J_{2n+1},J_{2n+1}) + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}) \\ &\quad + a_3 d_b(J_{2n+2},J_{2n+1},J_{2n}) \\ &\quad + a_5\bigl[3d_b(J_{2n},J_{2n+1},J_{2n+2})\bigr] \\ &= (a_1 + a_3 + 3a_5)d_b(J_{2n},J_{2n+1},J_{2n+2}) \\ &\quad + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}). \end{aligned}$$

Now from (2.13) we have

$$\begin{aligned} F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) &= F\bigl((a_1 + a_3 + 3a_5)d_b(J_{2n},J_{2n+1},J_{2n+2}) \\ &\quad + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1})\bigr) - r. \end{aligned} \tag{2.14}$$

Since F is strictly increasing, (2.14) implies

$$sd_b(J_{2n},J_{2n+1},J_{2n+2}) \leq (a_1 + a_3 + 3a_5)d_b(J_{2n},J_{2n+1},J_{2n+2}) \\ + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}).$$

So

$$(s - a_1 - a_3 - 3a_5)d_b(J_{2n},J_{2n+1},J_{2n+2}) \leq a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}).$$

Hence

$$d_b(J_{2n},J_{2n+1},J_{2n+2}) \leq \frac{a_2}{s - a_1 - a_3 - 3a_5} d_b(J_{2n-1},J_{2n},J_{2n+1}).$$

Since $a_1 + a_2 + a_3 + 3a_5 = s$, therefore $d_b(J_{2n},J_{2n+1},J_{2n+2}) \leq d_b(J_{2n-1},J_{2n},J_{2n+1})$. Thus from (2.14) we obtain

$$F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) \leq F\bigl(d_b(J_{2n-1},J_{2n},J_{2n+1})\bigr) - r \tag{2.15}$$

for all $n \in \mathbb{N}$. Similarly,

$$F\bigl(sd_b(J_{2n-1},J_{2n},J_{2n+1})\bigr) \leq F\bigl(d_b(J_{2n-2},J_{2n-1},J_{2n})\bigr) - r \tag{2.16}$$

for all $n \in \mathbb{N}$. Hence, from (2.15) and (2.16), we have

$$F\bigl(sd_b(J_n,J_{n+1},J_{n+2})\bigr) = F\bigl(d_b(J_{n-1},J_n,J_{n+1})\bigr) - r. \tag{2.17}$$

Inequality (2.17) leads to remark that $\{x_n\}$ is a Cauchy sequence, and the remaining part of the proof can easily be followed from the finishing part of the proof of Theorem 2.18. For $M_2(x,y,z)$, in line with the beginning part of the proof of Theorem 2.18, for all $(x,y,z) \in \lambda_{f,g,h,\alpha_s}$, for some $F \in \mathcal{F}_s$, and $r > 0$, from contractive condition (2.11), we get

$$\begin{aligned} F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) &= F\bigl(sd_b\bigl(f(x_{2n}),g(x_{2n+1}),h(x_{2n+2})\bigr)\bigr) \\ &\leq F\bigl(M_2(x_{2n},x_{2n+1},x_{2n+2})\bigr) - r \end{aligned} \tag{2.18}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} M_2(x_{2n},x_{2n+1},x_{2n+2}) &= a_1 d_b(J_{2n},J_{2n+1},J_{2n+1}) + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}) \\ &\quad + a_3 d_b(J_{2n+2},J_{2n+1},J_{2n}) \\ &\quad + a_4 d_b(J_{2n+2},J_{2n+2},J_{2n+2}) \\ &\leq a_1 d_b(J_{2n},J_{2n+1},J_{2n+2}) + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}) \\ &\quad + a_3 d_b(J_{2n},J_{2n+1},J_{2n+2}) \\ &= (a_1 + a_3)d_b(J_{2n},J_{2n+1},J_{2n+2}) \\ &\quad + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}). \end{aligned}$$

From (2.18), we have

$$F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) \le F\bigl((a_1+a_3)d_b(J_{2n},J_{2n+1},J_{2n+2})) \\ + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}) - r. \tag{2.19}$$

Since F is strictly increasing, (2.19) implies

$$sd_b(J_{2n},J_{2n+1},J_{2n+2}) \le (a_1+a_2)d_b(J_{2n},J_{2n+1},J_{2n+2}) \\ + a_2 d_b(J_{2n+1},J_{2n},J_{2n-1}),$$

so $(s-a_1-a_3)d_b(J_{2n},J_{2n+1},J_{2n+2}) \le a_2 d_b(J_{2n-1},J_{2n},J_{2n+1})$. Hence

$$d_b(J_{2n},J_{2n+1},J_{2n+2}) \le \frac{a_2}{s-a_1-a_3} d_b(J_{2n-1},J_{2n},J_{2n+1}).$$

Thus, from (2.19), we obtain

$$F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) \le F\bigl(d_b(J_{2n-1},J_{2n},J_{2n+1})\bigr) - r \tag{2.20}$$

for all $n \in \mathbb{N}$. Similarly,

$$F\bigl(sd_b(J_{2n-1},J_{2n},J_{2n+1})\bigr) \le F\bigl(d_b(J_{2n-2},J_{2n-1},J_{2n})\bigr) - r \tag{2.21}$$

for all $n \in \mathbb{N}$. Hence, from (2.20) and (2.21), we have

$$F\bigl(sd_b(J_n,J_{n+1},J_{n+2})\bigr) \le F\bigl(d_b(J_{n-1},J_n,J_{n+1})\bigr) - r. \tag{2.22}$$

Inequality (2.22) leads to remark that $\{J_n\}$ is a Cauchy sequence, and the remaining part of the proof can easily be followed from the finishing part of the proof of Theorem 2.18. For $M_3(x,y,z)$, in line with the beginning part of the proof of Theorem 2.18, for all $(x,y,z) \in \lambda_{f,g,h,\alpha_s}$, for some $F \in \mathcal{F}_s$, and $r > 0$, from contractive condition (2.12), we get

$$F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) = F\bigl(sd_b\bigl(f(x_{2n}),g(x_{2n+1}),h(x_{2n+2})\bigr)\bigr) \\ \le F\bigl(M_3(x_{2n},x_{2n+1},x_{2n+2})\bigr) - r \tag{2.23}$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$M_3(x_{2n},x_{2n+1},x_{2n+2}) = k\max\bigl\{d_b(J_{2n-1},J_{2n},J_{2n+1}), d_b(J_{2n+2},J_{2n+1},J_{2n}), 0\bigr\} \\ = k\max\bigl\{d_b(J_{2n-1},J_{2n},J_{2n+1}), d_b(J_{2n+2},J_{2n+1},J_{2n})\bigr\}.$$

If

$$\max\bigl\{d_b(J_{2n-1},J_{2n},J_{2n+1}), d_b(J_{2n+2},J_{2n+1},J_{2n})\bigr\} = d_b(J_{2n+2},J_{2n+1},J_{2n}),$$

then from (2.23) we have $F(sd_b(J_{2n},J_{2n+1},J_{2n+2})) \le F(kd_b(J_{2n},J_{2n+1},J_{2n+2})) - r$. Since F is strictly increasing, we have $sd_b(J_{2n},J_{2n+1},J_{2n+2}) < kd_b(J_{2n},J_{2n+1},J_{2n+2})$. It is a contradiction.

Thus we have

$$F\bigl(sd_b(J_{2n},J_{2n+1},J_{2n+2})\bigr) \le F\bigl(kd_b(J_{2n-1},J_{2n},J_{2n+1})\bigr) - r,$$

and $sd_b(J_{2n},J_{2n+1},J_{2n+2}) \le kd_b(J_{2n-1},J_{2n},J_{2n+1})$. So

$$d_b(J_{2n},J_{2n+1},J_{2n+2}) \le \frac{k}{s} d_b(J_{2n-1},J_{2n},J_{2n+1}).$$

The emaining part of the proof can easily be followed from the proof of Theorem 2.18. Similar arguments hold from $M_4(x,y,z)$. □

Theorem 2.28 *Let f, g, h, S_1, S_2, and S_3 be self-mappings defined on a complete tripled b-metric space (X,d_b,s) such that $f(X) \subseteq S_1(X)$, $g(X) \subseteq S_2(X)$, and $h(X) \subseteq S_3(X)$. If there exist $F \in \mathcal{F}_s$ and $r > 0$ such that $d_b(f(x),g(y),h(z)) > 0$, then*

$$r + F\bigl(sd_b\bigl(f(x),g(y),h(z)\bigr)\bigr) \le F\bigl(M(x,y,z)\bigr)$$

for all $x,y,z \in X$. Then f, g, h, S_1, S_2, and S_3 have a unique common fixed point in X provided that S_1, S_2, and S_3 are continuous and pairs (f,S_1), (g,S_2), and (h,S_3) are compatible.

Proof The arguments follow the same lines as in the proof of Theorem 2.18. □

3 Application to a system of integral equations

Let $X = C([0,1],\mathbb{R})$ be the space of all continuous real-valued functions defined on $[0,1]$. Let $d_b : X \times X \times X \to \mathbb{R}_0^+$ be defined

$$d_b(u,v,w) = \max\Bigl\{\sup_{t\in[0,1]} \bigl|u(t) - v(t)\bigr|^2, \sup_{t\in[0,1]} \bigl|u(t) - w(t)\bigr|^2, \sup_{t\in[0,1]} \bigl|v(t) - w(t)\bigr|^2\Bigr\}$$

for all $u,v,w \in C([0,1],\mathbb{R})$, and define $\alpha_s : X \times X \times X \to \mathbb{R}_0^+$ by $\alpha_s(u,v,w) = s^2$ for all $u,v,w \in X$. Obviously, (X,d_b,s) is an α_s-complete tripled b-metric space. We will apply Theorem 2.18 to show the existence of a common solution of the system of Volterra-type integral equations given by

$$\begin{aligned} u(t) &= p(t) + \int_0^t K\bigl(t,r,S_1\bigl(u(t)\bigr)\bigr)\,dr, \\ v(t) &= p(t) + \int_0^t J\bigl(t,r,S_2\bigl(v(t)\bigr)\bigr)\,dr, \\ w(t) &= p(t) + \int_0^t I\bigl(t,r,S_3\bigl(w(t)\bigr)\bigr)\,dr \end{aligned} \tag{3.1}$$

for all $t \in [0,1]$, where $p : [0,1] \to \mathbb{R}$ is a continuous function and $K,J,I : [0,1] \times [0,1] \times X \to \mathbb{R}$ are lower semi-continuous operators. Now, we prove the following theorem to ensure the existence of solution for the system of integral equations.

Theorem 3.1 *Let $X = C([0,1],\mathbb{R})$ and define the mappings $f, g, h : X \to X$ by*

$$f\big(u(t)\big) = p(t) + \int_0^t K\big(t, r, S_1\big(u(t)\big)\big)dr,$$
$$g\big(v(t)\big) = p(t) + \int_0^t J\big(t, r, S_2\big(v(t)\big)\big)dr,$$
$$h\big(w(t)\big) = p(t) + \int_0^t I\big(t, r, S_3\big(w(t)\big)\big)dr$$

for all $t \in [0,1]$. Assume that the following conditions are satisfied.

- *There exists a continuous function $\phi_i : X \to \mathbb{R}_0^+$, $i = 1,2,3$, such that*

$$\big|K(t,r,S_1) - J(t,r,S_2)\big| \le \phi_1(r)\big|S_1\big(u(t)\big) - S_2\big(v(t)\big)\big|,$$
$$\big|K(t,r,S_1) - I(t,r,S_3)\big| \le \phi_2(r)\big|S_1\big(u(t)\big) - S_3\big(w(t)\big)\big|,$$
$$\big|J(t,r,S_2) - I(t,r,S_3)\big| \le \phi_3(r)\big|S_2\big(v(t)\big) - S_3\big(w(t)\big)\big|$$

for each $t, r \in [0,1]$ and S_1, S_2, and $S_3 \in X$;

- *There exists $\tau > 0$ such that*

$$\int_0^t \phi_1(r)dr, \int_0^t \phi_2(r)dr, \int_0^t \phi_3(r)dr \le \sqrt{\frac{e^{-\tau}}{s}}.$$

Then the system of integral Eqs. (3.1) has a solution.

Proof By assumptions (i) and (ii), we have

$$\begin{aligned} d_b\big(f\big(u(t)\big), g\big(v(t)\big), h\big(w(t)\big)\big) = \max\Big\{ &\sup_{t\in[0,1]} \big|f\big(u(t)\big) - g\big(v(t)\big)\big|^2, \\ &\sup_{t\in[0,1]} \big|g\big(v(t)\big) - h\big(w(t)\big)\big|^2, \\ &\sup_{t\in[0,1]} \big|f\big(u(t)\big) - h\big(w(t)\big)\big|^2\Big\}, \end{aligned}$$

where

$$\begin{aligned} \sup_{t\in[0,1]} \big|f\big(u(t)\big) - g\big(v(t)\big)\big|^2 &= \left(\sup_{t\in[0,1]} \int_0^t \big|K\big(t,r,S_1\big(u(t)\big)\big) - J\big(t,r,S_2\big(v(t)\big)\big)\big|dr\right)^2 \\ &\le \left(\sup_{t\in[0,1]} \int_0^t \phi_1(r)\big|S_1\big(u(t)\big) - S_2\big(v(t)\big)\big|dr\right)^2 \\ &\le \left(\sqrt{\sup_{t\in[0,1]} \big|S_1\big(u(t)\big) - S_2\big(v(t)\big)\big|^2} \int_0^t \phi_1(r)dr\right)^2 \\ &= \sup_{t\in[0,1]} \big|S_1\big(u(t)\big) - S_2\big(v(t)\big)\big|^2 \left(\int_0^t \phi_1(r)dr\right)^2, \end{aligned}$$

$$\sup_{t\in[0,1]} \big|g\big(v(t)\big) - h\big(w(t)\big)\big|^2 \le \sup_{t\in[0,1]} \big|S_2\big(v(t)\big) - S_3\big(w(t)\big)\big|^2 \left(\int_0^t \phi_2(r)dr\right)^2,$$

$$\sup_{t\in[0,1]}\left|f\left(u(t)\right)-h\left(w(t)\right)\right|^2\le\sup_{t\in[0,1]}\left|S_1\left(u(t)\right)-S_3\left(w(t)\right)\right|^2\left(\int_0^t\phi_3(r)dr\right)^2.$$

Consequently, we have

$$\begin{aligned}d_b\left(f\left(u(t)\right),g\left(v(t)\right),h\left(w(t)\right)\right)&=\frac{e^{-\tau}}{s}\max\Big\{\sup_{t\in[0,1]}\left|S_1\left(u(t)\right)-S_2\left(v(t)\right)\right|^2,\\&\quad\sup_{t\in[0,1]}\left|S_2\left(v(t)\right)-S_3\left(w(t)\right)\right|^2,\sup_{t\in[0,1]}\left|S_1\left(u(t)\right)-S_3\left(w(t)\right)\right|^2\Big\}\\&=\frac{e^{-\tau}}{s}d_b\left(S_1\left(u(t)\right),S_2\left(v(t)\right),S_3\left(w(t)\right)\right)\\&\le\frac{e^{-\tau}}{s}M\left(u(t),v(t),Sw(t)\right).\end{aligned}$$

Thus, we obtain

$$sd_b\left(f\left(u(t)\right),g\left(v(t)\right),h\left(w(t)\right)\right)\le e^{-\tau}M\left(u(t),v(t),w(t)\right),$$

which implies that

$$\tau+\ln\left(sd_b\left(f\left(u(t)\right),g\left(v(t)\right),h\left(w(t)\right)\right)\right)\le\ln\left(M\left(u(t),v(t),w(t)\right)\right).$$

For $F(r)=\ln r$, all the hypotheses of Theorem 2.28 are satisfied. Hence the system of integral equations has a unique common solution. □

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this manuscript, and they read and approved the final manuscript.

Publisher's Note

Received: 17 June 2020 Accepted: 21 October 2020 Published online: 31 October 2020

References

1. Bakhtin, I.: The contraction principle in quasi metric spaces. Funkc. Anal. Prilozh. **30**, 26–37 (1989)
2. Czerwik, S.: Contraction mappings in *b*-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**(1), 5–11 (1993) https://dml.cz/handle/10338.dmlcz/120469
3. Huang, H., Vujakovic, J., Radenovic, S.: A note on common fixed point theorems for isotone increasing mappings in ordered *b*-metric spaces. J. Nonlinear Sci. Appl. **8**, 808–815 (2015). https://doi.org/10.22436/jnsa.008.05.31
4. Kamran, T., Postolache, M., Ali, M., Kiran, Q.: Feng and Liu type *f*-contraction in *b*-metric spaces with application to integral equations. J. Math. Anal. **7**(5), 18–27 (2016) http://ilirias.com/jma
5. Khamsi, M., Hussain, N.: KKM mappings in metric type spaces. Nonlinear Anal., Theory Methods Appl. **73**(9), 3123–3129 (2010). https://doi.org/10.1016/j.na.2010.06.084
6. Shatanawi, W., Pitea, A., Lazovic, R.: Contraction conditions using comparison functions on *b*-metric spaces. Fixed Point Theory Appl. **2014**, 135 (2014). https://doi.org/10.1186/1687-1812-2014-135
7. Wardowski, D.: Fixed point theory of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012). https://doi.org/10.1186/1687-1812-2012-94
8. Acar, O., Altun, I.: Multivalued *f*-contractive mappings with a graph and some fixed point results. Publ. Math. **88**(3–4), 305–317 (2016). https://doi.org/10.5486/PMD.2016.7308

9. Ali, M., Kamran, T.: Multivalued f-contractions and related fixed point theorems with an application. Filomat **30**(14), 3779–3793 (2016) http://journal.pmf.ni.ac.rs/filomat/index.php/filomat/article/view/1993
10. Ali, M., Kamran, T., Postolache, M.: Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem. Nonlinear Anal., Model. Control **22**(1), 17–30 (2017). https://doi.org/10.15388/NA.2017.1.2
11. Altun, I., Minak, G., Olgun, M.: Fixed points of multivalued nonlinear f-contractions on complete metric spaces. Nonlinear Anal., Model. Control **21**(2), 201–210 (2016). https://doi.org/10.15388/NA.2016.2.4
12. Durmaz, G., Minak, G., Altun, I.: Fixed points of ordered f-contractions. Hacet. J. Math. Stat. **45**(1), 15–21 (2016). https://doi.org/10.15672/HJMS.20164512482
13. Hussain, A., Nazam, M., Arshad, M.: Connection of Ćirić type f-contraction involving fixed point on closed ball. Gazi Univ. J. Sci. **30**(1), 283–291 (2017) http://gujs.gazi.edu.tr/article/view/5000186644
14. Nazam, M., Arshad, M., Abbas, M.: Existence of common fixed points of improved f-contraction on partial metric spaces. Appl. Gen. Topol. **18**(2), 277–287 (2014). https://doi.org/10.4995/agt.2017.6776
15. Piri, H., Kumam, P.: Some fixed point theorems concerning f-contraction in complete metric spaces. Fixed Point Theory Appl. **2014**, 210 (2014). https://doi.org/10.1186/1687-1812-2014-210
16. Secelean, N.: Iterated function systems consisting of f-contractions. Fixed Point Theory Appl. **2013**, 277 (2013). https://doi.org/10.1186/1687-1812-2013-277
17. Singh, D., Chauhan, V., Altun, I.: Common fixed point of a power graphic (f,ψ)-contraction pair on partial b-metric spaces with application. Nonlinear Anal., Model. Control **22**(5), 662–678 (2017). https://doi.org/10.15388/NA.2017.5.6
18. Vetro, C., Vetro, F.: The Class of F-Contraction Mappings with a Measure of Noncompactness. Springer, Singapore (2017). https://doi.org/10.1007/978-981-10-3722-17
19. Vetro, F.: f-Contractions of Hardy-Rogers type and application to multistage decision processes. Nonlinear Anal., Model. Control **21**(4), 531–546 (2017). https://doi.org/10.15388/NA.2016.4.7
20. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl. **75**(4), 2154–2165 (2012). https://doi.org/10.1016/j.na.2011.10.014
21. Salimi, P., Latif, A., Hussain, N.: Modified (α,ψ)-contractive mappings with applications. Fixed Point Theory Appl. **2013**, 151 (2013). https://doi.org/10.1186/1687-1812-2013-151
22. Hussain, N., Arshad, M., Shoaib, A., Uddin, F.: Common fixed point results for (α,ψ)-contractions on a metric space endowed with graph. J. Inequal. Appl. **136**(1), 1–14 (2014). https://doi.org/10.1186/1029-242X-2014-136
23. Hussain, N., Kutbi, M., Salimi, P.: Fixed point theory in α-complete metric spaces with applications. Abstr. Appl. Anal. **2014** Article ID 280817, 11 pages (2014). https://doi.org/10.1155/2014/280817
24. Hussain, N., Vetro, C., Vetro, F.: Fixed point results for α-implicit contractions with application to integral equations. Nonlinear Anal., Model. Control **21**(3), 362–378 (2016). https://doi.org/10.15388/NA.2016.3.5
25. Cosentino, M., Vetro, P.: Fixed point results for f-contractive mappings of Hardy–Rogers type. Filomat **28**, 715–722 (2014). https://doi.org/10.2298/fil1404715c
26. Minak, G., Helvaci, A., Altun, I.: Ćirić type generalized f-contractions on complete metric spaces and fixed point results. Filomat **28**(6), 1143–1151 (2014) http://journal.pmf.ni.ac.rs/filomat/index.php/filomat/article/view/99
27. Nazam, M., Muhammad, A., Postolache, M.: Coincidence and common fixed point theorems for four mappings satisfying $(\alpha\alpha_s,f)$-contraction. Nonlinear Anal., Model. Control **23**(4), 664–690 (2018). https://doi.org/10.15388/NA.2018.5.3
28. Aghajani, A., Abbas, M., Roshan, J.: Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math. Slovaca **64**(4), 941–960 (2014). https://doi.org/10.2478/s12175-014-0250-6
29. Amini-Harandi, A.: Fixed point theory for quasi-contraction mappings in b-metric spaces. Appl. Math. Lett. **15**(2), 351–358 (2014) http://www.math.ubbcluj.ro/nodeacj/sfptcj.html
30. Cosentino, M., Jleli, M., Samet, B., Vetro, C.: Solvability of integrodifferential problem via fixed point theory in b-metric spaces. Fixed Point Theory Appl. **2015**, 70 (2015). https://doi.org/10.1186/s13663-015-0317-2
31. Farajzadeh, A., Kaewcharoen, A., Lohawech, P.: On fixed point theorems for (ξ,α,η)-expansive mappings in complete metric spaces. Int. J. Pure Appl. Math. **102**(1), 129–146 (2015). https://doi.org/10.12732/ijpam.v102i1.13
32. Farajzadeh, A., Noytaptim, C., Kaewcharoen, A.: Some fixed point theorems for generalized α-η-ψ-Geraghty contractive type mappings in partial b-metric spaces. J. Inform. Math. Sci. **3**(10), 455–478 (2018). https://doi.org/10.26713/jims.v10i3.583