Li *Journal of Inequalities and Applications* (2020) 2020:234
https://doi.org/10.1186/s13660-020-02501-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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The solvability of a kind of generalized Riemann–Hilbert problems on function spaces H_*

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Abstract

In this paper, we study a kind of generalized Riemann–Hilbert problems (R-HPs) with several unknown functions in strip domains. We mainly discuss methods of solving R-HPs with two unknown functions and obtain general solutions and conditions of solvability on function spaces H_*. At the end of this paper, we consider in detail the behavior of the solution at ∞ and in different domains. Thus the results in this paper generalize and improve the theory of the classical Riemann–Hilbert problems.

MSC: 30E25; 30G35; 40G35

Keywords: Riemann–Hilbert problems; Singular integral equations; Sokhotski–Plemelj formula; Canonical function; Function spaces H_*

1 Introduction

It is well known that Riemann–Hilbert problems are a powerful mathematical tool widely applied in physics, fracture mechanics, engineering mechanics, engineering and technology, and many other fields [1–3]. Especially, the problem of finding solutions for some kinds of singular integral equations is often transformed to solving Riemann–Hilbert problems [4–15]. In [1, 2] the Riemann–Hilbert problems on an infinite straight line has systematically been studied, and the Riemann–Hilbert problems with unknown function on two parallel lines was further described. So far, the results of the boundary value problems for analytic functions have been mostly confined to the case of only one unknown function.

Motivated by the above researches, the main purpose of this paper is extending the theory to the R-HPs with $n \geq 2$ unknown functions on n parallel straight lines, and we mainly discuss the case $n = 2$. Using the classical boundary value theory and principle of analytic continuation, we investigate the analytic solutions and the conditions of solvability on function spaces H_* (the notation H_* can be found in Sect. 2). At nodal points the asymptotic behavior of a solution of such a problem is discussed in detail. Our method of solving problems is innovate, different from those in classical cases. Meanwhile, this paper also improves some results of [2–4, 7].

2 Definitions and lemmas

In this section, we introduce some definitions and lemmas.

Definition 2.1 Let $F(x)$ be a continuous function in the real number field $\mathbb{R}$. A function $F(x)$ belongs to $\hat{H}$ if the following two conditions are fulfilled:

(1) there exists $B \in \mathbb{R}^+$ such that

$$\left|F(x_1) - F(x_2)\right| \leq B\left|{x_1}^{-1} - {x_2}^{-1}\right| \tag{2.1}$$

on the neighborhood N_∞ of ∞, that is, there exists a sufficiently large $M > 0$ such that (2.1) is satisfied for all $x_1, x_2 \in \mathbb{R}\backslash[-M, M]$.

(2) $F \in H$ on $[-M, M]$, where H is the class of Hölder continuous functions (for the notation H, see [1]).

Definition 2.2 A function F belongs to H_* if it satisfies:

(1) $F \in \hat{H}$, (2) $F \in L^2(\mathbb{R})$ (see [1, 6] for the definition of $L^2(\mathbb{R})$).

With respect to the function spaces H_*, one of its important properties is closedness under pointwise multiplication.

If a function F satisfies the Hölder condition on a neighborhood N_∞ of ∞, then we write $F \in H(N_\infty)$.

Definition 2.3 Let $f(t) \in H_*$. We define its Fourier transform $\mathbb{F}$ and the inverse Fourier transform $\mathbb{F}^{-1}$ as follows:

$$\mathbb{F}f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{itx}\,dt; \qquad \mathbb{F}^{-1}F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(x)e^{-itx}\,dx. \tag{2.2}$$

Lemma 2.1 *Let functions Ω_1 and Ω_2 be analytic in the upper half-plane $\mathbb{C}^+$ and the lower half-plane $\mathbb{C}^-$ except their poles $z_0 = \infty$ and z_k $(k = 1, 2, \ldots, n)$. Suppose that the boundary values of Ω_1 and Ω_2 on $\operatorname{Im} z = 0$ are equal. The main parts of the Laurent expansion of Ω_1 and Ω_2 at $z_0 = \infty$ are*

$$G_0(z) = c_1^0 z + c_2^0 z^2 + \cdots + c_m^0 z^m, \tag{2.3}$$

where $c_m^0 \neq 0$ and c_k^0 $(1 \leq k \leq m)$ are constants. The main parts of the Laurent expansion for Ω_1 and Ω_2 at every pole z_k $(k = 1, 2, \ldots, n)$ are

$$G_k\left(\frac{1}{z - z_k}\right) = \frac{c_1^k}{z - z_k} + \frac{c_2^k}{(z - z_k)^2} + \cdots + \frac{c_p^k}{(z - z_k)^p}, \quad \forall k \in \{1, 2, \ldots, n\}, \tag{2.4}$$

where c_l^k $(1 \leq l \leq p)$ are constants, and $c_p^k \neq 0$, $p \geq 1$. Then Ω_1 and Ω_2 can be represented by the same function Ω in the complex plane $\mathbb{C}$, namely,

$$\Omega(z) = c_0 + G_0(z) + \sum_{k=1}^{n} G_k\left(\frac{1}{z - z_k}\right), \tag{2.5}$$

where c_0 is a complex constant.

Proof We can prove the lemma by using the generalized Liouville theorem [14, 16, 17] and the principle of analytic continuation [1, 2, 7]. □

The following lemmas are obvious facts, and we omit their proofs.

Lemma 2.2 *If there exists a sufficiently large positive number B such that $F(z) \in H$ for $|z| < B$ and $F(z)$ is analytic for $|z| \geq B$, then $F(z) \in \hat{H}$ on the complex plane $\mathbb{C}$.*

Lemma 2.3 (see [1, 2, 4]) *Let $f(t) \in \{0\}$ and $F(x) = \mathbb{F}f(t)$. The Cauchy-type integral is defined as*

$$\Theta(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(\tau)}{\tau - z}\,d\tau, \quad z \in \mathbb{C}^+ \cup \mathbb{C}^-. \tag{2.6}$$

Then for $z \in \mathbb{C}^+$, we have

$$\Theta^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(\tau)}{\tau - z}\,d\tau = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{itz}\,dt, \tag{2.7}$$

and for $z \in \mathbb{C}^-$, we have

$$\Theta^-(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F(\tau)}{\tau - z}\,d\tau = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{itz}\,dt. \tag{2.8}$$

It is easy to prove that $\Theta^+(z)$ and $\Theta^-(z)$ are analytical in $z \in \mathbb{C}^+$ and $z \in \mathbb{C}^-$, respectively.

3 Problem presentation and solution

We now propose boundary value problems for analytic functions with $n \geq 2$ unknown functions on n parallel lines, and then we discuss the methods of solution of such problems.

Suppose that n lines γ_j $(1 \leq j \leq n)$ are parallel to the X-axis and denote them by $L = \Sigma_{j=1}^{n}\gamma_j$, where $\gamma_j (1 \leq j \leq n)$ take the direction from left to right as the positive direction and can be expressed by $\xi_j = x + iR_j$ $(R_n < \cdots < R_2 < R_1)$, where $x, R_j \in \mathbb{R}$. Our goal is to obtain functions $F_j(z) (1 \leq j \leq n)$ such that $F_j(z)$ are analytic in $R_j < \operatorname{Im} z < R_{j-1} (2 \leq j \leq n)$ and $F_1(z)$ is analytic in $\operatorname{Im} z > R_1$ and $\operatorname{Im} z < R_n$, and the following boundary value conditions are fulfilled:

$$F_j^+(\xi) - A_j(\xi)F_{j+1}^-(\xi) = B_j(\xi), \quad \xi \in \gamma_j, j = 1, 2, \ldots, n, \tag{3.1}$$

when $j = n$, denote $F_{n+1}^-(\xi)$ as $F_1^-(\xi)$, where the given functions $A_j(\xi)$, $B_j(\xi) (1 \leq j \leq n)$ belong to H_* on γ_j. Obviously, R-HP (3.1) can also be written in the following form:

$$\begin{cases} F_1^+(\xi) - A_1(\xi)F_2^-(\xi) = B_1(\xi), & \xi \in \gamma_1, \\ F_2^+(\xi) - A_2(\xi)F_3^-(\xi) = B_2(\xi), & \xi \in \gamma_2, \\ \cdots\cdots\cdots\cdots & \\ F_{n-1}^+(\xi) - A_{n-1}(\xi)F_n^-(\xi) = B_{n-1}(\xi), & \xi \in \gamma_{n-1}, \\ F_n^+(\xi) - A_n(\xi)F_1^-(\xi) = B_n(\xi), & \xi \in \gamma_n. \end{cases} \tag{3.2}$$

It follows from (3.1) that the orders of $F_j(z)(1 \leq j \leq n)$ are equal to each other at ∞. Thus, when the orders of $F_j(z)$ are m at ∞, (3.1) can be denoted as R_m. In fact, the problem R_0 and problem R_{-1} are frequently discussed. On the problem R_0, $F_j(z)$ are supposed to be finite and nonzero at ∞. On problem R_{-1}, $F_j(z)$ are assumed to be zero at ∞. When $A_j(\xi)(1 \leq j \leq n)$ are not zero on L, problem (3.1) is said to be of normal type; otherwise, it is called of nonnormal type or of exception type.

Note that since the positive direction of $\gamma_j(1 \leq j \leq n)$ is the direction from left to right, when the observer moves from left to right on γ_j, the boundary values of left domain of γ_j are positive, that is, the positive boundary values of $F_j(z)(1 \leq j \leq n)$ are the boundary values above γ_j, and the negative boundary values of $F_j(z)$ are those below γ_j.

Without loss of generality, in this paper, we only discuss the case $n = 2$. As for R-HP with $n > 2$ unknown functions on n parallel lines, there is no essential difference for the methods of solution with the case $n = 2$.

When $n = 2$, R-HP (3.1) can be stated as follows.

Problem Assume that $\gamma_1 : \xi = x + i\beta$ and $\gamma_2 : \xi = x + i\alpha$ are two oriented lines, where α and β are real numbers with $\alpha < \beta$. Similarly to the above statement, we take the direction of γ_1 and γ_2 from left to right as the positive direction. We want to obtain functions $F_1(z)$ and $F_2(z)$ such that $F_1(z)$ is analytic in $\{\operatorname{Im} z > \beta\} \cup \{\operatorname{Im} z < \alpha\}$, and $F_2(z)$ is analytic in $\{z : \alpha < \operatorname{Im} z < \beta\}$ and satisfies the following boundary value conditions on γ_1 and γ_2:

$$\begin{cases} F_1^+(\xi) - A_1(\xi)F_2^-(\xi) = B_1(\xi), & \xi \in \gamma_1, \\ F_2^+(\xi) - A_2(\xi)F_1^-(\xi) = B_2(\xi), & \xi \in \gamma_2. \end{cases} \tag{3.3}$$

In fact, (3.3) is the R-HP on two parallel straight lines $\operatorname{Im} z = \beta$ and $\operatorname{Im} z = \alpha$ with $z = \infty$ as a pole, and it is a generalization of the classical R-HP.

Here $F_1^+(\xi)$ is the boundary value of the analytic function $F_1(z)$, which is analytic in $\{z : \operatorname{Im} z > \beta\}$, and belongs to H_* on γ_1; $F_1^-(\xi)$ is the boundary value of the analytic function $F_1(z)$, which is analytic in $\{z : \operatorname{Im} z < \alpha\}$, and belongs to H_* on γ_2; $F_2^\pm(\xi)$ are the boundary values of the analytic function $F_2(z)$, which is analytic in $\{z : \alpha < \operatorname{Im} z < \beta\}$, and belong to H_* on γ_1, γ_2, respectively. Since $A_j(\xi), B_j(\xi)$ $(j = 1, 2)$ belong to H_* on γ_j. Thus, for the functions appearing in (3.3), their one-sided limits exist as $x \to \infty$ on γ_1, γ_2, and we have

$$\lim_{x\to+\infty} A_1(i\beta + x) = \lim_{x\to-\infty} A_1(i\beta + x), \qquad \lim_{x\to+\infty} A_2(i\alpha + x) = \lim_{x\to-\infty} A_2(i\alpha + x). \tag{3.4}$$

Similarly to $B_j(\xi)$ $(j = 1, 2)$, we also have

$$B_1(i\beta + \infty) = B_1(i\beta - \infty), \qquad B_2(i\alpha + \infty) = B_2(i\alpha - \infty). \tag{3.5}$$

In this paper, we only solve problem (3.3) in problem R_0. For problem R_m, similar arguments can be done. Since $F_1(z) \in H_*$ on γ_j $(j = 1, 2)$, the $\lim_{z\to\infty} F_1(z)$ exists, that is, both $F_1^+(\infty)$ and $F_1^-(\infty)$ exist. Therefore, on γ_1 and γ_2, we have

$$\lim_{x\to+\infty} F_1^+(i\beta + x) = \lim_{x\to-\infty} F_1^+(i\beta + x), \qquad \lim_{x\to+\infty} F_1^-(i\alpha + x) = \lim_{x\to-\infty} F_1^-(i\alpha + x). \tag{3.6}$$

Similarly, the $\lim_{z\to\infty} F_2(z)$ exists, and so do $F_2^+(\infty)$ and $F_2^-(\infty)$. Then on γ_1 and γ_2, we also have

$$F_2^-(i\beta+\infty)=F_2^-(i\beta-\infty), \qquad F_2^+(i\alpha+\infty)=F_2^+(i\alpha-\infty). \tag{3.7}$$

Hence, $F_1(\infty)$ and $F_2(\infty)$ are finite and nonzero. Here we only consider the case of normal type, that is, $A_j(\xi), A_j^{-1}(\xi)$ $(j=1,2)$ have no zero point on γ_j. For the case of nonnormal type, similar discussions also work (see [13–15]). To solve R-HP (3.3), we define κ_j as follows:

$$\kappa_j=\operatorname{Ind}_{\gamma_j} A_j(\xi)=\frac{1}{2\pi}\bigl[\arg A_j(\xi)\bigr]_{\gamma_j}, \quad j=1,2,$$

and

$$\kappa=\kappa_1+\kappa_2. \tag{3.8}$$

Then we call κ as the index of R-HP (3.3). Set

$$\begin{aligned} \lambda_\infty^{(1)}&=\mu_\infty^{(1)}+i\nu_\infty^{(1)}=\frac{1}{2\pi i}\ln\frac{A_1(+\infty+i\beta)}{A_1(-\infty+i\beta)},\\ \lambda_\infty^{(2)}&=\mu_\infty^{(2)}+i\nu_\infty^{(2)}=\frac{1}{2\pi i}\ln\frac{A_2(+\infty+i\alpha)}{A_2(-\infty+i\alpha)},\\ \gamma_\infty&=\lambda_\infty^{(1)}+\lambda_\infty^{(2)}, \qquad \mu_\infty=\mu_\infty^{(1)}+\mu_\infty^{(2)}. \end{aligned} \tag{3.9}$$

Since $z=\infty$ is a branch point of $\ln A_j(\xi)$ $(j=1,2)$, on the neighborhood $N(\infty)$ of ∞, $\ln A_1(\xi)$ and $\ln A_2(\xi)$ are taken to be continuous branches, respectively, such that $0\le\mu_\infty<1$. Without loss of generality, we take three points z_0, z_1, z_2 such that $\alpha<\operatorname{Im} z_0<\beta$, $\operatorname{Im} z_1>\beta$, $\operatorname{Im} z_2<\alpha$, and we define the functions $Y_1(z)$ and $Y_2(z)$ as follows:

$$Y_1(z)=\begin{cases}\frac{e^{\Gamma_1(z)}}{(z-z_0)^{\kappa_1}}, & \operatorname{Im} z>\beta,\\ \frac{e^{\Gamma_1(z)}}{(z-z_1)^{\kappa_1}}, & \operatorname{Im} z<\beta,\end{cases} \tag{3.10}$$

and

$$Y_2(z)=\begin{cases}\frac{e^{\Gamma_2(z)}}{(z-z_2)^{\kappa_2}}, & \operatorname{Im} z>\alpha,\\ \frac{e^{\Gamma_2(z)}}{(z-z_0)^{\kappa_2}}, & \operatorname{Im} z<\alpha,\end{cases} \tag{3.11}$$

where

$$\begin{aligned} \Gamma_1(z)&=\frac{z-z_0}{2\pi i}\int_{-\infty+i\beta}^{+\infty+i\beta}\frac{\ln E_1(t)}{(t-z)(t-z_0)}\,dt,\\ \Gamma_2(z)&=\frac{z-z_0}{2\pi i}\int_{-\infty+i\alpha}^{+\infty+i\alpha}\frac{\ln E_2(t)}{(t-z)(t-z_0)}\,dt,\\ E_j(t)&=\left(\frac{t-z_0}{t-z_j}\right)^{\kappa_j}A_j(t), \quad j=1,2, \operatorname{Im} z\neq\alpha,\beta, \end{aligned} \tag{3.12}$$

in which we have taken the analytic branches of $\ln E_j(t)$ $(j = 1,2)$, provided that we have chosen

$$\ln \frac{t-z_0}{t-z_j}\bigg|_{t=\infty} = 0, \quad \forall j = 1,2.$$

Note that the integrands appeared in (3.12) belong to H_* on γ_1, γ_2, respectively, and therefore their integrals exist. Due to (3.10) and (3.11), we know that $Y_1^+(z)$ and $Y_1^-(z)$ are analytic in $\operatorname{Im} z > \beta$ and $\operatorname{Im} z < \beta$, respectively; $Y_2^+(z)$ and $Y_2^-(z)$ are analytic in $\operatorname{Im} z > \alpha$ and $\operatorname{Im} z < \alpha$, respectively. Therefore, $Y_1(z)$ and $Y_2(z)$ are sectionally holomorphic functions, and

$$Y_j(z) = O\bigl(|z|^{-\kappa_j}\bigr)(z \to \infty), \quad j = 1,2.$$

It is easy to see that $Y_1(z)$, $Y_2(z)$ are the canonical functions, and one has

$$Y_j^+(t) = A_j(t)Y_j^-(t), \quad \forall j \in \{1,2\}. \tag{3.13}$$

Denote

$$Y(z) = Y_1(z)Y_2(z),$$

it is not difficult to verify that $Y(z)$ satisfies the definition of a canonical function (see [1–3]), thus we call $Y(z)$ as the canonical function of (3.3). Since $A_j(\xi)$ $(j = 1,2)$ belong to H_*, similar to the discussion in [4], one must have, near $z = \infty$,

$$Y(\xi) = \frac{Y_0(\xi)}{\xi^{\gamma_\infty}}, \qquad Y_0(\xi) \in H(N_\infty).$$

Substituting (3.13) into (3.3), we obtain

$$\begin{cases} [Y_1^+(\xi)]^{-1}F_1^+(\xi) - [Y_1^-(\xi)]^{-1}F_2^-(\xi) = B_1(\xi)[Y_1^+(\xi)]^{-1}, & \xi \in \gamma_1; \\ [Y_2^+(\xi)]^{-1}F_2^+(\xi) - [Y_2^-(\xi)]^{-1}F_1^-(\xi) = B_2(\xi)[Y_2^+(\xi)]^{-1}, & \xi \in \gamma_2. \end{cases} \tag{3.14}$$

The first equality in (3.14) is multiplied by $[Y_2^+(\xi)]^{-1}$, and the second one is multiplied by $[Y_1^-(\xi)]^{-1}$, so we get

$$\begin{cases} [Y_1^+(\xi)Y_2^+(\xi)]^{-1}F_1^+(\xi) - [Y_1^-(\xi)Y_2^+(\xi)]^{-1}F_2^-(\xi) = B_1(\xi)[Y_1^+(\xi)Y_2^+(\xi)]^{-1}, \\ \quad \xi \in \gamma_1; \\ [Y_1^-(\xi)Y_2^+(\xi)]^{-1}F_2^+(\xi) - [Y_1^-(\xi)Y_2^-(\xi)]^{-1}F_1^-(\xi) = B_2(\xi)[Y_1^-(\xi)Y_2^+(\xi)]^{-1}, \\ \quad \xi \in \gamma_2. \end{cases} \tag{3.15}$$

Note that in (3.15), we cannot apply the Sokhotski–Plemelj formula [8, 16, 17] directly. In general, when $\kappa = \kappa_1 + \kappa_2 > 0$, $B_1(\xi)[Y_1^+(\xi)Y_2^+(\xi)]^{-1} \notin H_*$, and so does $B_2(\xi)[Y_1^-(\xi)Y_2^+(\xi)]^{-1}$. In order to unify, no matter how κ is chosen, let

$$X_1(z) = \begin{cases} e^{\Gamma_1(z)}, & \operatorname{Im} z > \beta, \\ e^{\Gamma_1(z)}(\frac{z-z_1}{z-z_0})^{-\kappa_1}, & \operatorname{Im} z < \beta, \end{cases} \tag{3.16}$$

and

$$X_2(z) = \begin{cases} e^{\Gamma_2(z)}(\frac{z-z_2}{z-z_0})^{-\kappa_2}, & \operatorname{Im} z > \alpha, \\ e^{\Gamma_2(z)}, & \operatorname{Im} z < \alpha. \end{cases} \tag{3.17}$$

The first and second equalities in (3.15) are multiplied by $(\xi - z_0)^{-\kappa}$, respectively, and thus (3.15) can be transformed to

$$\begin{cases} [X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - [X_1^-(\xi)X_2^+(\xi)]^{-1}F_2^-(\xi) = B_1(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}, \\ \quad \xi \in \gamma_1, \\ [X_1^-(\xi)X_2^+(\xi)]^{-1}F_2^+(\xi) - [X_1^-(\xi)X_2^-(\xi)]^{-1}F_1^-(\xi) = B_2(\xi)[X_1^-(\xi)X_2^+(\xi)]^{-1}, \\ \quad \xi \in \gamma_2. \end{cases} \tag{3.18}$$

Since $X_1(z)$, $X_2(z)$ are bounded and nonzero on γ_1, γ_2, respectively, and $B_1(t) \in H_*$, by (3.16) and (3.17) we get $\frac{B_1(t)}{X_1^+(t)X_2^+(t)} \in H_*$. Therefore we can define the sectionally analytic function

$$\psi_1(z) = \frac{1}{2\pi i} \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(t)[X_1^+(t)X_2^+(t)]^{-1}\,dt}{t-z}, \quad \operatorname{Im} z \neq \beta. \tag{3.19}$$

According to Privalov's theorem (see [1]), we easily see that $\psi_1(z)$ is analytic in $\operatorname{Im} z > \beta$ and $\operatorname{Im} z < \beta$. Applying the Sokhotski–Plemelj formula to $\psi_1(z)$ in (3.19), we have

$$\psi_1^+(\xi) - \psi_1^-(\xi) = B_1(\xi)\big[X_1^+(\xi)X_2^+(\xi)\big]^{-1}, \quad \xi \in \gamma_1. \tag{3.20}$$

Thus the first equality in (3.18) can be reduced to

$$\big[X_1^+(\xi)X_2^+(\xi)\big]^{-1}F_1^+(\xi) - \big[X_1^-(\xi)X_2^+(\xi)\big]^{-1}F_2^-(\xi) = \psi_1^+(\xi) - \psi_1^-(\xi), \quad \xi \in \gamma_1. \tag{3.21}$$

Similarly, we also define the function

$$\psi_2(z) = \frac{1}{2\pi i} \int_{-\infty+i\alpha}^{+\infty+i\alpha} \frac{B_2(t)[X_1^-(t)X_2^+(t)]^{-1}\,dt}{t-z}, \quad \operatorname{Im} z \neq \alpha. \tag{3.22}$$

We again apply the Sokhotski–Plemelj formula to $\psi_2(z)$ in (3.22), and thus the second equality in (3.18) can also be reduced to

$$\big[X_1^-(\xi)X_2^+(\xi)\big]^{-1}F_2^+(\xi) - \big[X_1^-(\xi)X_2^-(\xi)\big]^{-1}F_1^-(\xi) = \psi_2^+(\xi) - \psi_2^-(\xi), \quad \xi \in \gamma_2. \tag{3.23}$$

Combining (3.21) and (3.23), we obtain

$$\begin{cases} [X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - \psi_1^+(\xi) = [X_1^-(\xi)X_2^+(\xi)]^{-1}F_2^-(\xi) - \psi_1^-(\xi), & \xi \in \gamma_1, \\ [X_1^-(\xi)X_2^+(\xi)]^{-1}F_2^+(\xi) - \psi_2^+(\xi) = [X_1^-(\xi)X_2^-(\xi)]^{-1}F_1^-(\xi) - \psi_2^-(\xi), & \xi \in \gamma_2. \end{cases} \tag{3.24}$$

Thus we need only to discuss problem (3.24) instead of (3.3). Adding $-\psi_2^+(\xi)$ in the first equality of (3.24) and subtracting $\psi_1^-(\xi)$ in the second one, we get

$$\begin{cases} [X_1^+(\xi)X_2^+(\xi)]^{-1}F_1^+(\xi) - \psi_1^+(\xi) - \psi_2^+(\xi) \\ \quad = [X_1^-(\xi)X_2^+(\xi)]^{-1}F_2^-(\xi) - \psi_1^-(\xi) - \psi_2^+(\xi), \quad \xi \in \gamma_1, \\ [X_1^-(\xi)X_2^+(\xi)]^{-1}F_2^+(\xi) - \psi_2^+(\xi) - \psi_1^-(\xi) \\ \quad = [X_1^-(\xi)X_2^-(\xi)]^{-1}F_1^-(\xi) - \psi_2^-(\xi) - \psi_1^-(\xi), \quad \xi \in \gamma_2. \end{cases} \tag{3.25}$$

We denote the left side of the first equality in (3.25) by $\Phi_1^+(z)$, whereas the right side is denoted by $\Phi_1^-(z)$, that is,

$$\begin{aligned} \Phi_1^+(z) &= \bigl[X_1^+(z)X_2^+(z)\bigr]^{-1}F_1^+(z) - \psi_1^+(z) - \psi_2^+(z), \\ \Phi_1^-(z) &= \bigl[X_1^-(z)X_2^+(z)\bigr]^{-1}F_2^-(z) - \psi_1^-(z) - \psi_2^+(z). \end{aligned} \tag{3.26}$$

We define the following function $\Phi_1(z)$:

$$\Phi_1(z) = \Phi_1^+(z) \quad \text{when } \operatorname{Im} z > \beta; \qquad \Phi_1(z) = \Phi_1^-(z) \quad \text{when } \operatorname{Im} z < \beta. \tag{3.27}$$

By the boundary value conditions (3.25) we know that $\Phi_1^+(z) = \Phi_1^-(z)$ on $\operatorname{Im} z = \beta$.

We first give explicit solutions of $\Phi_1^+(z)$ and $\Phi_1^-(z)$ in $\operatorname{Im} z > \beta$ and $\operatorname{Im} z < \beta$, respectively. In $\operatorname{Im} z > \beta$, since $[X_1^+(z)]^{-1}$ and $[X_2^+(z)]^{-1}$ are analytic, $[X_1^+(z)X_2^+(z)]^{-1}$ is also analytic. Since $\psi_j^+(z)$ $(j = 1, 2)$ are analytic, it follows that $\Phi_1^+(z)$ is analytic, so the $\lim_{z\to\infty} \Phi_1^+(z)$ exists. For convenience, we assume that

$$\Phi_1^+(z) = D_1(z),$$

and thus

$$\bigl[X_1^+(z)X_2^+(z)\bigr]^{-1}F_1^+(z) - \psi_1^+(z) - \psi_2^+(z) = D_1(z), \quad \operatorname{Im} z > \beta, \tag{3.28}$$

where $D_1(z)$ is analytic in $\operatorname{Im} z > \beta$, and the $\lim_{z\to\infty} D_1(z)$ exists.

In $\operatorname{Im} z < \beta$, when $\kappa \geq 0$, $[X^-(z)]^{-1}$ and $[X_1^-(z)X_2^+(z)]^{-1}$ have a pole z_0 of order κ; when $\kappa < 0$, $[X_1^-(z)X_2^+(z)]^{-1}$ has no singularity, but $X_1^-(z)X_2^+(z)$ has a pole z_0 of order $|\kappa|$. Hence, using Lemmas 2.1 and 2.2, the generalized Liouville theorem, and the principle of analytic continuation, it follows that

$$\Phi_1^-(z) = \frac{q_k(z)}{(z - z_0)^\kappa}, \tag{3.29}$$

where we have set

$$q_\kappa(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_\kappa z^\kappa$$

with arbitrary constants c_j $(0 \leq j \leq \kappa)$. When $\kappa > -1$, $q_\kappa(z)$ is a polynomial of degree no more than κ; when $\kappa \leq -1, q_\kappa(z) \equiv 0$. Due to (3.26), (3.28), and (3.29), we have

$$\begin{aligned} \bigl[X_1^+(z)X_2^+(z)\bigr]^{-1}F_1^+(z) - \psi_1^+(z) - \psi_2^+(z) &= \Phi_1(z), \quad \operatorname{Im} z > \beta, \\ \bigl[X_1^-(z)X_2^+(z)\bigr]^{-1}F_2^-(z) - \psi_1^-(z) - \psi_2^+(z) &= \Phi_1(z), \quad \operatorname{Im} z < \beta, \end{aligned} \tag{3.30}$$

that is,

$$\begin{aligned} &F_1^+(z) = X_1^+(z)X_2^+(z)\bigl[\psi_1^+(z) + \psi_2^+(z) + \Phi_1(z)\bigr], \quad \operatorname{Im} z > \beta, \\ &F_2^-(z) = X_1^-(z)X_2^+(z)\bigl[\psi_1^-(z) + \psi_2^+(z) + \Phi_1(z)\bigr], \quad \operatorname{Im} z < \beta, \end{aligned} \tag{3.31}$$

where

$$\Phi_1(z) = \begin{cases} D_1(z), & \operatorname{Im} z > \beta, \\ \frac{q_k(z)}{(z-z_0)^\kappa}, & \operatorname{Im} z < \beta. \end{cases}$$

Similarly, the left side of the second equality in (3.25) is denoted by $\Phi_2^+(z)$, whereas the right side is denoted by $\Phi_2^-(z)$, namely,

$$\begin{aligned} &\Phi_2^+(z) = \bigl[X_1^-(z)X_2^+(z)\bigr]^{-1}F_2^+(z) - \psi_2^+(z) - \psi_1^-(z); \\ &\Phi_2^-(z) = \bigl[X_1^-(z)X_2^-(z)\bigr]^{-1}F_1^-(z) - \psi_2^-(z) - \psi_1^-(z). \end{aligned} \tag{3.32}$$

We also have

$$\Phi_2(z) = \begin{cases} \Phi_2^+(z), & \operatorname{Im} z > \alpha, \\ \Phi_2^-(z), & \operatorname{Im} z < \alpha. \end{cases} \tag{3.33}$$

In $\operatorname{Im} z < \alpha$, since $[X_j^-(z)]^{-1}$ and $\psi_j^-(z)$ $(j = 1, 2)$ are analytic, $\Phi_2^-(z)$ is also analytic. Similarly to the above discussion, we may denote $\Phi_2^-(z) = D_2(z)$, where $D_2(z)$ is analytic in $\operatorname{Im} z < \alpha$, and the $\lim_{z\to\infty} D_2(z)$ exists.

In $\operatorname{Im} z > \alpha$, note that when $\kappa \geq 0$, $[X_1^-(z)X_2^+(z)]^{-1}$ has a pole $z = z_0$ of order κ; when $\kappa < 0$, $[X_1^-(z)X_2^+(z)]^{-1}$ has no singularity, but $X_1^-(z)X_2^+(z)$ has a pole $z = z_0$ of order $|\kappa|$. Therefore we also obtain

$$\Phi_2^+(z) = \frac{q_k(z)}{(z - z_0)^\kappa}, \tag{3.34}$$

where $q_k(z)$ is as before. From the previous discussions we know that

$$\begin{aligned} &F_1^-(z) = X_1^-(z)X_2^-(z)\bigl[\psi_1^-(z) + \psi_2^-(z) + \Phi_2(z)\bigr], \quad \operatorname{Im} z < \alpha, \\ &F_2^+(z) = X_1^-(z)X_2^+(z)\bigl[\psi_1^-(z) + \psi_2^+(z) + \Phi_2(z)\bigr], \quad \operatorname{Im} z > \alpha, \end{aligned} \tag{3.35}$$

where

$$\Phi_2(z) = \begin{cases} D_2(z), & \operatorname{Im} z < \alpha, \\ \frac{q_k(z)}{(z-z_0)^\kappa}, & \operatorname{Im} z > \alpha. \end{cases}$$

Combining (3.31) and (3.35), R-HP (3.3) has the general solutions given by the formulas

$$\begin{aligned} &F_1^+(z) = X_1^+(z)X_2^+(z)\bigl[\psi_1^+(z) + \psi_2^+(z) + \Phi(z)\bigr], \quad \operatorname{Im} z > \beta, \\ &F_1^-(z) = X_1^-(z)X_2^-(z)\bigl[\psi_1^-(z) + \psi_2^-(z) + \Phi(z)\bigr], \quad \operatorname{Im} z < \alpha, \\ &F_2(z) = X_1^-(z)X_2^+(z)\bigl[\psi_1^-(z) + \psi_2^+(z) + \Phi(z)\bigr], \quad \alpha < \operatorname{Im} z < \beta, \end{aligned} \tag{3.36}$$

in which we have put

$$\Phi(z)=\begin{cases} D_1(z), & \operatorname{Im} z>\beta, \\ D_2(z), & \operatorname{Im} z<\alpha, \\ \frac{q_k(z)}{(z-z_0)^\kappa}, & \alpha<\operatorname{Im} z<\beta, \end{cases} \tag{3.37}$$

where $q_k(z)$ is a polynomial of degree no more than κ as before, and $D_1(z)$ and $D_2(z)$ are analytic in $\operatorname{Im} z>\beta$ and $\operatorname{Im} z<\alpha$, respectively.

Now we can formulate the main results with respect to the solutions of R-HP (3.3).

Theorem 3.1 *R-HP (3.3) with two unknown functions $F_1(z)$ and $F_2(z)$ on two parallel lines has solutions in $\{z:\alpha<\operatorname{Im} z<\beta\}$ and $\{\operatorname{Im} z>\beta\}\cup\{\operatorname{Im} z<\alpha\}$, respectively. Its general solutions can be expressed by (3.36), where $X_j(z)$ $(j=1,2)$ are defined by (3.16) and (3.17), and $\psi_1(z)$, $\psi_2(z)$ are given by (3.19) and (3.22), respectively. When $\kappa>-1$, $q_\kappa(z)$ is a polynomial of degree no more than κ; when $\kappa\leq -1$, $q_\kappa(z)=0$. In conclusion, the degree of freedom of the solution is equal to $\kappa+1$.*

If $F_1(\infty)=F_2(\infty)=0$, then (3.3) gives a solution in problem R_{-1}. In this case, we have the following conclusion.

Theorem 3.2 *Under the conditions $F_1(\infty)=F_2(\infty)=0$, R-HP (3.3) has a solution if and only if $B_1(\infty)=B_2(\infty)=0$. In such a case, a solution of (3.3) is similar to (3.36), and the only difference lies in that $q_\kappa(z)$ should be substituted by $q_{\kappa-1}(z)$ in (3.36). However, the degree of freedom of the solution for (3.3) is equal to κ.*

At the end of this section, we give an important example in practical application. In problem (3.3), we suppose that

$$\begin{aligned} &A_1(z)=A_2(z)=1, \qquad B_1(z)=\frac{1}{2+z^2}, \qquad B_2(z)=\frac{1}{1+z^2}, \\ &\gamma_1: z=0, \qquad \gamma_2: z=x+i, \quad -\infty<x<+\infty. \end{aligned} \tag{3.38}$$

Then (3.3) can be transformed to the following form:

$$\begin{cases} F_1^+(z)-F_2^-(z)=\frac{1}{2+z^2}, & z\in\gamma_1, \\ F_2^+(z)-F_1^-(z)=\frac{1}{1+z^2}, & z\in\gamma_2. \end{cases} \tag{3.39}$$

Without loss of generality, we assume that $z_0=\frac{1}{2}i$, $z_1=\frac{3}{2}i$, $z_2=-\frac{1}{2}i$. Then we have $\kappa_1=\kappa_2=0$ and $\kappa=0$. Therefore we get

$$\Gamma_j(t)=0, \qquad X_j(z)=1, \quad j=1,2. \tag{3.40}$$

Similarly to the discussion in [1–3], from (3.19), (3.22), and Lemma 2.3 we have

$$
\begin{aligned}
&\text{when } \operatorname{Im} z > 1,\\
&\psi_1^+(z) = \frac{1}{2\pi i}\int_{-\infty+i}^{+\infty+i} \frac{dt}{(2+t^2)(t-z)}\\
&\qquad = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i}^{+\infty+i} \frac{e^{-i\tau z}}{2+\tau^2}\,d\tau = \frac{i}{2\sqrt{2}(z+\sqrt{2}i)},\\
&\text{when } \operatorname{Im} z < 1,\\
&\psi_1^-(z) = \frac{1}{2\pi i}\int_{-\infty+i}^{+\infty+i} \frac{dt}{(2+t^2)(t-z)}\\
&\qquad = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i}^{+\infty+i} \frac{e^{-i\tau z}}{2+\tau^2}\,d\tau = \frac{i}{2\sqrt{2}(z-\sqrt{2}i)},\\
&\text{when } \operatorname{Im} z > 0,\\
&\psi_2^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)(t-z)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{e^{-i\tau z}}{1+\tau^2}\,d\tau = \frac{i}{2(z+i)},\\
&\text{when } \operatorname{Im} z < 0,\\
&\psi_2^-(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dt}{(1+t^2)(t-z)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{e^{-i\tau z}}{1+\tau^2}\,d\tau = \frac{i}{2(z-i)}.
\end{aligned}
\tag{3.41}
$$

Then we obtain the following solutions of problem (3.39):

$$
\begin{aligned}
&F_1^+(z) = \frac{i}{2\sqrt{2}(z+\sqrt{2}i)} + \frac{i}{2(z+i)} + D(z), \quad \operatorname{Im} z > 1,\\
&F_1^-(z) = \frac{i}{2\sqrt{2}(z-\sqrt{2}i)} + \frac{i}{2(z-i)} + D(z), \quad \operatorname{Im} z < 0,\\
&F_2(z) = \frac{i}{2\sqrt{2}(z-\sqrt{2}i)} + \frac{i}{2(z+i)} + D(z), \quad 0 < \operatorname{Im} z < 1,
\end{aligned}
\tag{3.42}
$$

where $D(z)$ is an analytic function in the whole complex plane $\mathbb{C}$. We can easily verify that (3.42) are solutions of problem (3.39).

4 The conditions of solvability of R-HP (3.3)

Now we are concerned about solution (3.36) and the conditions of solvability of R-HP (3.3).

(1) It follows from the discussion in Sect. 3 that when $\kappa < 0$, z_0 is a $|\kappa|$th-order pole of $X_1^-(z)X_2^+(z)$. To guarantee that (3.3) is solvable, the following $-\kappa$ conditions are required:

$$
\int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(t)(t-z)^j\,dt}{X_1^+(t)X_2^+(t)} = -\int_{-\infty+i\alpha}^{+\infty+i\alpha} \frac{B_2(t)(t-z)^j\,dt}{X_1^-(t)X_2^+(t)}, \quad j \in \{-1,-2,\dots,\kappa\}.
\tag{4.1}
$$

(2) The cases of the solutions in $\alpha < \operatorname{Im} z < \beta$. For $z \in \{z : \alpha < \operatorname{Im} z < \beta\}$, if $A_1(z)$ and $A_2^{-1}(z)$ have no zero points, then $F_1(z)$ and $F_2(z)$ can be obtained by (3.36). Otherwise, if $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ are common zero points of $A_1(z)$ and $A_2^{-1}(z)$ of orders $s_1, s_2, \dots, s_n$, respec-

tively, then we must have

$$F_2^{(j)}(z)|_{z=\vartheta_k} = 0, \quad 1 \le k \le n, 1 \le j \le s_k. \tag{4.2}$$

Therefore we obtain:

(a) when $\kappa \ge 0$, the following equations with the unknown elements $c_0, c_1, \ldots, c_k$ have solutions:

$$\begin{aligned}\left[\frac{q_k(z)}{(z-z_0)^\kappa}\right]^{(j)}_{z=\vartheta_k} &= \frac{j!}{2\pi i}\int_{-\infty+i\alpha}^{+\infty+i\alpha} \frac{B_2(\xi)[X_1^-(\xi)X_2^+(\xi)]^{-1}\,d\xi}{(\xi-\vartheta_k)^{j+1}(\xi-z_0)} \\ &\quad - \frac{j!}{2\pi i}\int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}\,d\xi}{(\xi-\vartheta_k)^{j+1}(\xi-z_0)};\end{aligned} \tag{4.3}$$

(b) when $\kappa < 0$, the following equalities must be satisfied:

$$\int_{-\infty+i\alpha}^{+\infty+i\alpha} \frac{B_2(\xi)[X_1^-(\xi)X_2^+(\xi)]^{-1}\,d\xi}{(\xi-\vartheta_k)^{j+1}(\xi-z_0)} = \int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(\xi)[X_1^+(\xi)X_2^+(\xi)]^{-1}\,d\xi}{(\xi-\vartheta_k)^{j+1}(\xi-z_0)}, \tag{4.4}$$

where $j = 0, 1, 2, \ldots, s_k$, $k = 1, 2, \ldots, n$, and $c_0, c_1, \ldots, c_k$ are the coefficients of $q_k(z)$.

(3) The behavior of solution at $z = \infty$. If $z = \infty$ is an ordinary node, then $0 < \mu_\infty < 1$. It follows from $\psi_1(\infty) = \psi_2(\infty) = 0$ that near $z = \infty$,

$$\psi_j(\xi) = \psi_j^*(\xi)\xi^{-\mu_j^*}, \quad 0 < \mu_j^* < \mu_\infty^{(j)}, \qquad \psi_j^*(\xi) \in H, \quad j = 1, 2. \tag{4.5}$$

In (3.36), since $F_2(\xi)$ is bounded, $F_2(\infty)$ is taken as a finite value. Note that $0 < \mu_\infty < 1$. If $\mu_\infty > \frac{1}{2}$, then we easily see that

$$X_1^-(\xi)X_2^+(\xi)q_\kappa(\xi)(\xi-z_0)^{-\kappa} = O\bigl(|\xi|^{-\mu_\infty}\bigr) \quad (\xi \to \infty) \tag{4.6}$$

and

$$X_1^-(\xi)X_2^+(\xi)\bigl(\psi_1(\xi) - \psi_2(\xi)\bigr) = O\bigl(|\xi|^{-\mu_\infty+\varepsilon}\bigr) \quad (\xi \to \infty), \tag{4.7}$$

where ε is a sufficiently small positive number such that $\mu_\infty - \varepsilon > \frac{1}{2}$.

If $\mu_\infty \le \frac{1}{2}$, when $\kappa \ge 0$, the coefficient e_k of κth-power item in $q_k(z)$ should be taken as

$$e_k = \frac{1}{2\pi i}\int_{-\infty+i\beta}^{+\infty+i\beta} \frac{B_1(\xi)\,d\xi}{X_1^+(\xi)X_2^+(\xi)(\xi-z_0)} + \frac{1}{2\pi i}\int_{-\infty+i\alpha}^{+\infty+i\alpha} \frac{B_2(\xi)\,d\xi}{X_1^-(\xi)X_2^+(\xi)(\xi-z_0)}; \tag{4.8}$$

when $\kappa < 0$, (4.1) is required to fulfill.

If $z = \infty$ is a special node, that is, $\mu_\infty = 0$, then we can transform it into the case $\mu_\infty \le \frac{1}{2}$ as an ordinary node, and similar arguments can be done (see [16–19]).

5 Conclusions

In this paper, we propose one kind of R-HP with several unknown functions in strip domains. By using the methods of the classical boundary value problems for analytic functions, we obtain the exact solution, defined by integrals, of problem (3.3) and the conditions of solvability on function spaces H_*. Our method is different from those of the classical R-HP and is novel and effective.

R-HP (3.1) has important applications in practical problems, such as elastic mechanics, heat conduction, and electrostatics. Many problems, such as piezoelectric material, voltage magnetic materials, and functional gradient materials, can often attribute the problem to find the solutions of (3.1). Therefore the solving method of (3.1) has important meaning not only in applications, but also in the theory of resolving the equation itself. This paper mainly deals with the solvability and explicit solutions of the R-HP with two unknown functions. Indeed, it is possible to solving the problem mentioned above in Clifford analysis, which is similar to that in [20–27]. We omit further discussion.

Acknowledgements
The author is thankful to the editor and anonymous referees for their valuable comments and suggestions.

Funding
Research is supported by the National Natural Science Foundation of China (Grant No. 11971015).

Availability of data and materials
Not applicable.

Competing interests
The author declares that no competing interests exist.

Author's contributions
Author read and approved the final manuscript.

Publisher's Note

Received: 17 April 2020 Accepted: 20 October 2020 Published online: 29 October 2020

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