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On structure of discrete Muckenhoupt and discrete Gehring classes

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Abstract

In this paper, we study the structure of the discrete Muckenhoupt class $\mathcal{A}^p(\mathcal{C})$ and the discrete Gehring class $\mathcal{G}^q(\mathcal{K})$. In particular, we prove that the self-improving property of the Muckenhoupt class holds, i.e., we prove that if $u \in \mathcal{A}^p(\mathcal{C})$ then there exists $q < p$ such that $u \in \mathcal{A}^q(\mathcal{C}_1)$. Next, we prove that the power rule also holds, i.e., we prove that if $u \in \mathcal{A}^p$ then $u^q \in \mathcal{A}^p$ for some $q > 1$. The relation between the Muckenhoupt class $\mathcal{A}^1(\mathcal{C})$ and the Gehring class is also discussed. For illustrations, we give exact values of the norms of Muckenhoupt and Gehring classes for power-low sequences. The results are proved by some algebraic inequalities and some new inequalities designed and proved for this purpose.

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1 Introduction

We fix an interval $\mathbb{I} \subset \mathbb{R}$ and consider subintervals I of $\mathbb{I}$ and denote by $|I|$ the Lebesgue measure of I. A weight w is nonnegative locally integrable function. In the literature a nonnegative measurable weight function w defined on a bounded fixed interval $\mathbb{I}$ is called an $A^p(\mathcal{C})$-Muckenhoupt weight for $1 < p < \infty$ if there exists a constant $\mathcal{C} < \infty$ such that

$$\left(\frac{1}{|I|}\int_I w(t)\,dt\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}(t)\,dt\right)^{p-1} \le \mathcal{C} \tag{1}$$

for every subinterval $I \subset \mathbb{I}$. For a given exponent $p > 1$, we define the A^p-norm of the function w by the following quantity:

$$A^p(w) := \sup_{I \subset \mathbb{I}}\left(\frac{1}{|I|}\int_I w(t)\,dt\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}(t)\,dt\right)^{p-1}, \tag{2}$$

where the supremum is taken over all intervals $I \subset \mathbb{I}$. For a given fixed constant $\mathcal{C} > 1$, if the weight w belongs to $A^p(\mathcal{C})$, then $A^p(w) \le \mathcal{C}$. A weight w satisfying the condition

$$\frac{1}{|I|}\int_I w(x)\,dx \le \mathcal{C}w(x) \quad \text{for every } x \in I \tag{3}$$

is called an $A^1(\mathcal{C})$-Muckenhoupt weight where $\mathcal{C} > 1$. In [25] Muckenhoupt proved the following result.

Lemma 1.1 *If w is a nonincreasing weight satisfying condition (3), then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that*

$$\frac{1}{|I|}\int_I w^p(x)\,dx \le \frac{\mathcal{C}}{\mathcal{C}-p(\mathcal{C}-1)}\left(\frac{1}{|I|}\int_I w(x)\,dx\right)^p. \tag{4}$$

Bojarski, Sbordone, and Wik [3] improved the Muckenhoupt result by excluding the monotonicity condition on the weight w by using the rearrangement ω^* of the function ω over the interval I and established the best constant. In particular, they proved the following lemma.

Lemma 1.2 *If w is a nonincreasing weight and satisfies condition (3) with $\mathcal{C} > 1$, then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that*

$$\frac{1}{|I|}\int_I \omega^p(t)\,dt \le \frac{\mathcal{C}^{1-p}}{\mathcal{C}-p(\mathcal{C}-1)}\left(\frac{1}{|I|}\int_I \omega(s)\,ds\right)^p. \tag{5}$$

In [25] Muckenhoupt also proved the following result.

Lemma 1.3 *If $1 < p < \infty$ and w satisfies the A^p-condition (1) on the interval $\mathbb{I}$, with constant $\mathcal{C}$, then there exist constants q and $\mathcal{C}_1$ depending on p and $\mathcal{C}$ such that $1 < q < p$ and w satisfies the A^q-condition*

$$\left(\frac{1}{|I|}\int_I w(t)\,dt\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{q-1}}(t)\,dt\right)^{q-1} \le \mathcal{C}_1 \tag{6}$$

for every subinterval $I \subset \mathbb{I}$.

In other words, Muckenhoupt's result (see also Coifman and Fefferman [9]) for *self-improving* property states that: if $w \in A^p(\mathcal{C})$ then there exist a constant $\epsilon > 0$ and a positive constant $\mathcal{C}_1$ such that $w \in A^{p-\epsilon}(\mathcal{C}_1)$, and then

$$A^p(\mathcal{C}) \subset A^{p-\epsilon}(\mathcal{C}_1). \tag{7}$$

Muckenhoupt [25] also proved the following result.

Lemma 1.4 *If $1 < p < \infty$ and $w \in A^p(\mathcal{C})$ on the interval $\mathbb{I}$ with a constant $\mathcal{C}$, then there exist constants r and $\mathcal{C}_1$ depending only on p and $\mathcal{C}$ such that $1 < r$ and $w^r \in A^p(\mathcal{C}_1)$.*

Gehring [12, 13] introduced a new class of weights satisfying reverse Hölder's inequalities in connection with the integrability properties of the gradient of quasiconformal mappings. The function w is said to belong to the Gehring class $G^q(\mathcal{K})$ for $q > 1$ with a constant $\mathcal{K} < \infty$ if

$$\left(\frac{1}{|I|}\int_I w^q(x)\,dx\right)^{1/q} \le \mathcal{K}\left(\frac{1}{|I|}\int_I w(x)\,dx\right) \quad \text{for all } I \subset \mathbb{I}. \tag{8}$$

For a given exponent $p > 1$, we define the $G^q(w)$-norm of the weight w by the following quantity:

$$G^q(w) = \sup_I \left[\frac{|I|}{\int_I w(s)\,ds} \left(\frac{1}{|I|} \int_I w^q(s)\,ds \right)^{\frac{1}{q}} \right]^{\frac{q}{q-1}},$$

where the supremum is taken over all $I \subset \mathbb{I}$. For a given fixed constant $\mathcal{K} > 1$, if the weight w belongs to $G^p(\mathcal{K})$, then $G^p(w) \leq \mathcal{K}$.

Remark 1.1 Lemma 1.2 proves that if the weight w belongs to the Muckenhoupt class $A^1(\mathcal{C})$, then w belongs to the Gehring class $G^p(\mathcal{K})$ with $\mathcal{K} = [\mathcal{C}^{1-p}/(\mathcal{C} - p(\mathcal{C}-1))]^{1/p-1}$ for $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$.

For power-low functions Malaksiano in [18, Lemma 2.2] proved that if $I = (0,1)$, $p > 1$, and $\alpha > -1/p$, then

$$\left(G^p\left(y^\alpha\right)\right)^{(p-1)/p} = \frac{1+\alpha}{(1+p\alpha)^{1/p}}. \tag{9}$$

Moreover, if $0 < \alpha < \beta$ and $p > 1$, then $G^p(y^\alpha; (0,1)) < G^p(y^\beta; (0,1))$. Also in [19, Lemma 2.2] Malaksiano proved that if $I = (0,1)$, $q > 1$, and $\alpha \in (-1, q-1)$, then

$$A^q\left(y^\alpha\right) = \frac{(q-1)^{(q-1)}}{(\alpha+1)(q-1-\alpha)^{q-1}}. \tag{10}$$

Moreover, if $0 < \alpha < \beta < q-1$ and $q > 1$, then $A^q(y^{-\alpha}; (0,1)) < A^q(y^{-\beta}; (0,1))$.

In recent years the study of the discrete analogues in harmonic analysis has become an active field of research. For example, the study of regularity and boundedness of discrete operators on l^p analogues for L^p-regularity and higher summability of sequences has been considered by some authors, see for example [2, 15–17, 26, 27] and the references they have cited. Whereas some results from Euclidean harmonic analysis admit an obvious variant in the discrete setting, others do not. The main challenge in such studies is that there are no general methods to study these questions, see for example [5–8, 26, 27, 30–32] and the references cited therein. We confine ourselves to proving the discrete analogue of Muckenhoupt results (Lemmas 1.1 and 1.3) and establish some inclusion properties between the discrete Muckenhoupt class and the discrete Gehring class. For structure and relations between classical Muckenhoupt and Gehring classes (in the integral forms) and their applications, we refer the reader to the papers [1, 3, 10–14, 18, 20–25, 28] and the references cited therein.

Throughout the paper, we assume that $1 < p < \infty$ and $\mathbb{I}$ is a fixed finite interval from $\mathbb{Z}_+$. A discrete weight u defined on $\mathbb{Z}_+ = \{1, 2, \ldots\}$ is a sequence $u = \{u(n)\}_{n=1}^{\infty}$ of nonnegative real numbers. We consider the norm on $l^p(\mathbb{Z}_+)$ of the form

$$\|u\|_{l^p(\mathbb{Z}_+)} := \left(\sum_{n=1}^{\infty} \left|u(n)\right|^p \right)^{1/p} < \infty.$$

A discrete nonnegative weight u belongs to the discrete Muckenhoupt class $\mathcal{A}^1(A)$ on the fixed interval $\mathbb{I} \subset \mathbb{Z}_+$ for $p > 1$ and $A > 1$ if the inequality

$$\frac{1}{|J|}\sum_{k\in J} u(k) \leq Au(k) \tag{11}$$

holds for every subinterval $J \subset \mathbb{I}$ and $|J|$ is the cardinality of the set J. A discrete nonnegative weight u belongs to the discrete Muckenhoupt class $\mathcal{A}^p(A)$ on the interval $\mathbb{I} \subseteq \mathbb{Z}_+$ for $p > 1$ and $A > 1$ if the inequality

$$\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right)\left(\frac{1}{|J|}\sum_{k\in J} u^{\frac{-1}{p-1}}(k)\right)^{p-1} \leq A \tag{12}$$

holds for every subinterval $J \subset \mathbb{I}$. For a given exponent $p > 1$, we define the $\mathcal{A}^p$-norm of the discrete weight u by the following quantity:

$$\mathcal{A}^p(u) := \sup_{J\subset\mathbb{I}}\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in J} u^{\frac{-1}{p-1}}(k)\right)^{p-1}, \tag{13}$$

where the supremum is taken over all intervals $J \subset \mathbb{I}$. Note that by Hölder's inequality $\mathcal{A}^p(u) \geq 1$ for all $1 < p < \infty$, and the following inclusion is true:

$$\text{if } 1 < p \leq q < \infty, \text{ then } \mathcal{A}^1 \subset \mathcal{A}^p \subset \mathcal{A}^q \text{ and } \mathcal{A}^q(u) \leq \mathcal{A}^p(u).$$

For a given exponent $q > 1$ and a constant $\mathcal{K} > 1$, a discrete nonnegative weight u belongs to the discrete Gehring class $\mathcal{G}^q(\mathcal{K})$ (or satisfies a reverse Hölder inequality) on the interval $\mathbb{I}$ if, for every subinterval $J \subseteq \mathbb{I}$, we have

$$\left(\frac{1}{|J|}\sum_{k\in J} u^q(k)\right)^{1/q} \leq \mathcal{K}\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right). \tag{14}$$

For a given exponent $q > 1$, we define the $\mathcal{G}^q$-norm of u as follows:

$$\mathcal{G}^q(u) := \sup_{J\subset\mathbb{I}}\left[\left(\frac{1}{|J|}\sum_{k\in J} u(k)\right)^{-1}\left(\frac{1}{|J|}\sum_{k\in J} u^q(k)\right)^{\frac{1}{q}}\right]^{\frac{q}{q-1}}, \tag{15}$$

where the supremum is taken over all intervals $J \subseteq \mathbb{I}$ and represents the best constant for which the $\mathcal{G}^q$-condition holds true independently on the interval $J \subseteq \mathbb{I}$. Note that by Hölder's inequality $\mathcal{G}^q(u) \geq 1$ for all $1 < q < \infty$, and that the following inclusion is true:

$$\text{if } 1 < p \leq q < \infty, \text{ then } \mathcal{G}^q \subset \mathcal{G}^p \text{ and } 1 \leq \mathcal{G}^p(u) \leq \mathcal{G}^q(u). \tag{16}$$

Our aim in this paper, in the next section, is to prove the discrete analogy of the Muckenhoupt results which include the self-improving property of the Muckenhoupt class and we also prove the transition property due to Bojarski, Sbordone, and Wik [3] with a sharp constant. In particular, we prove that if $u \in \mathcal{A}^p(\mathcal{C})$ then there exists $q < p$ such that $u \in \mathcal{A}^q(\mathcal{C}_1)$

and if $u \in \mathcal{A}^p$ then $u^q \in \mathcal{A}^p$ for some $q > 1$. For the relation between the discrete Muckenhoupt class and the discrete Gehring class, we prove that if $u \in \mathcal{A}^1(\mathcal{C})$ then $u \in \mathcal{G}^q(\mathcal{K})$ with exact values of the exponent q and the constant $\mathcal{K}$. In addition, for illustration, we establish the exact values of the Muckenhoupt norm $\mathcal{A}^q(n^\alpha)$ and the Gehring norm $\mathcal{G}^p(n^\alpha)$ for power-low sequences $\{n^\alpha\}$.

2 Main results

Throughout this section, we assume that the sequences in the statements of theorems are nonnegative and assume for the sake of conventions that $0 \cdot \infty = 0$, $0/0 = 0$, $\sum_{s=a}^{b} y(s) = 0$, whenever $a > b$, and

$$\Delta\left(\sum_{s=a}^{k-1} y(s)\right) = y(k), \qquad \sum_{s=a}^{k-1} \Delta y(s) = y(k) - y(a).$$

We fix an interval $\mathbb{I} \subset \mathbb{Z}_+$ and consider $\mathbb{I}$ of the form $\{1, 2, \dots, k, \dots, N\}$ (or $[1, N] \subset \mathbb{Z}_+$). For any weight $u : \mathbb{I} \to \mathbb{R}^+$ which is nonnegative, we define the operator $\mathcal{H}u : \mathbb{I} \to \mathbb{R}^+$ by

$$\mathcal{H}u(k) = \frac{1}{k} \sum_{s=1}^{k} u(s), \quad \text{for all } k \in \mathbb{I}. \tag{17}$$

The following lemma gives some properties of the operator $\mathcal{H}u$ that will be needed later.

Lemma 2.1 *Let $\mathcal{H}u$ be defined as in (17). Then we have the following properties:*
(1). *If u is nonincreasing, then so is $\mathcal{H}u(k)$ and $\mathcal{H}u(k) \geq u(k)$.*
(2). *If u is nondecreasing, then so is $\mathcal{H}u(k)$ and $\mathcal{H}u(k) \leq u(k)$.*

Proof (1). From the definition of $\mathcal{H}$, we see that: If u is nonincreasing, then

$$\mathcal{H}u(k) = \frac{1}{k} \sum_{s=1}^{k} u(s) \geq \frac{1}{k} \sum_{s=1}^{k} u(k) = u(k).$$

Hence, we have by using the above inequality that

$$\Delta\bigl(\mathcal{H}u(k)\bigr) = \frac{ku(k) - \sum_{s=1}^{k} u(s)}{k(k+1)} \leq \frac{\sum_{s=1}^{k} u(s) - \sum_{s=1}^{k} u(s)}{k(k+1)} = 0,$$

thus $\mathcal{H}u(k)$ is nonincreasing. This completes the proof of the first case.

(2). If u is nondecreasing, then

$$\mathcal{H}u(k) = \frac{1}{k} \sum_{s=1}^{k} u(s) \leq \frac{1}{k} \sum_{s=1}^{k} u(k) = u(k).$$

Also, we have by using the above inequality that

$$\Delta\bigl(\mathcal{H}u(k)\bigr) = \frac{ku(k) - \sum_{s=1}^{k-1} u(s)}{k(k+1)} \geq \frac{\sum_{s=1}^{k-1} u(s) - \sum_{s=1}^{k-1} u(s)}{k(k+1)} = 0,$$

thus $\mathcal{H}u(k)$ is nondecreasing. This completes the proof of the second case. The proof is complete. □

Remark 2.1 As a consequence of Lemma 2.1, we notice that if $q > 1$ and u is nonnegative, nonincreasing, then $\mathcal{H}u^q$ is also nonnegative and nonincreasing and $\mathcal{H}u^q \geq u^q$. We also notice from Lemma 2.1 that if $q > 1$ and u is nonnegative and nondecreasing, then $\mathcal{H}u^q$ is also nonnegative and nondecreasing and $\mathcal{H}u^q \leq u^q$.

In the proof of the next lemma, we shall use the notion of the characteristic function χ_J defined on a set J by

$$\chi_J(k) = \begin{cases} 1 & \text{for all elements } k \text{ on } J, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Lemma 2.2 *Let* $1 < q < \infty$, *and let* $u \in \mathcal{A}^q(\mathcal{C})$ *for* $\mathcal{C} > 1$. *Then, for any subset* $J = \{1, 2, \dots, k\}$, *we have that*

$$\frac{1}{k}\sum_{s=1}^{k} u(s) \leq \mathcal{C} \sup_{k \in J} u(k). \tag{19}$$

Proof For any nonnegative sequence $\lambda(k)$ defined on $\mathbb{I}$, we see for any subset $J = \{1, 2, \dots, k\} \subset \mathbb{I}$ that

$$\begin{aligned} \sum_{s=1}^{k} \lambda(s) &= \sum_{s=1}^{k} \lambda(s) u^{1/q}(s) u^{-1/q}(s) \\ &\leq \left(\sum_{s=1}^{k} \lambda^q(s) u(s)\right)^{1/q} \left(\sum_{s=1}^{k} u^{-q'/q}(s)\right)^{1/q'}. \end{aligned}$$

By using $q' = q/(q-1)$, we get that

$$\sum_{s=1}^{k} \lambda(s) \leq \left(\sum_{s=1}^{k} \lambda^q(s) u(s)\right)^{1/q} \left(\sum_{s=1}^{k} u^{-\frac{1}{q-1}}(s)\right)^{\frac{q-1}{q}}.$$

That is,

$$\left(\sum_{s=1}^{k} \lambda(s)\right)^q \leq \left(\sum_{s=1}^{k} \lambda^q(s) u(s)\right) \left(\sum_{s=1}^{k} u^{-\frac{1}{q-1}}(s)\right)^{q-1}.$$

Multiplying both sides by $(1/(k)^q) \sum_{s=1}^{k} u(s)$, we get that

$$\begin{aligned} &\frac{1}{(k)^q}\left(\sum_{s=1}^{k} \lambda(s)\right)^q \sum_{s=1}^{k} u(s) \\ &\quad \leq \left(\sum_{s=1}^{k} \lambda^q(s) u(s)\right) \left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right) \left(\frac{1}{k}\sum_{s=1}^{k} u^{-\frac{1}{q-1}}(s)\right)^{q-1}. \end{aligned} \tag{20}$$

Now, since $u \in \mathcal{A}^q(\mathcal{C})$, for $\mathcal{C} > 1$, we see that

$$\left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right) \left(\frac{1}{k}\sum_{s=1}^{k} u^{\frac{-1}{q-1}}(s)\right)^{q-1} \leq \mathcal{C}. \tag{21}$$

By using (21) in (20), we get that

$$\frac{1}{(k)^q}\left(\sum_{s=1}^{k}\lambda(s)\right)^q\sum_{s=1}^{k}u(s)\le \mathcal{C}\left(\sum_{s=1}^{k}\lambda^q(s)u(s)\right)$$
$$\le \mathcal{C}\sup_{s\in J}u(s)\left(\sum_{s=1}^{k}\lambda^q(s)\right).$$

For $\lambda(s)=\chi_J(s)$, we see that $\sum_{s=1}^{k}u(s)\le \mathcal{C}k\sup_{s\in J}u(s)$. That is,

$$\frac{1}{k}\sum_{s=1}^{k}u(s)\le \mathcal{C}\sup_{k\in J}u(k),$$

which is the desired inequality (19). The proof is complete. □

Remark 2.2 The above lemma can be written as: if $u\in\mathcal{A}^q(\mathcal{C})$ for some $\mathcal{C}>1$, then $\mathcal{H}u(k)\le \mathcal{C}\sup_{k\in J}u(k)$.

Lemma 2.3 *Let* $1<q<\infty$ *and u be a nonincreasing weight. If* $u\in\mathcal{A}^q(\mathcal{C})$, *then* $u\in\mathcal{A}^1(\mathcal{C})$.

Proof To prove the lemma, we need to prove that: if

$$\left(\frac{1}{k}\sum_{s=1}^{k}u(s)\right)\left(\frac{1}{k}\sum_{s=1}^{k}u^{\frac{-1}{q-1}}(s)\right)^{q-1}\le \mathcal{C}\quad\text{for all } k\in\mathbb{I}, \tag{22}$$

for some $\mathcal{C}>1$ independent of k, then

$$\frac{1}{k}\sum_{s=1}^{k}u(s)\le \mathcal{C}u(s) \tag{23}$$

for all $1<s\le k$, and $k\in\mathbb{I}$. By using (22) and employing Lemma 3.1 in [29], we get that

$$\frac{1}{k}\sum_{s=1}^{k}u(s)\le \mathcal{C}\exp\left(\frac{1}{k}\sum_{s=1}^{k}\log u(s)\right). \tag{24}$$

Now, by applying property (2) in Lemma 2.1 for the nondecreasing weight $\log u(s)$, we obtain that

$$\frac{1}{k}\sum_{s=1}^{k}u(s)\le \mathcal{C}\exp\left(\frac{1}{k}\sum_{s=1}^{k}\log u(s)\right)\le \mathcal{C}\exp\bigl(\log u(k)\bigr)$$
$$=\mathcal{C}u(k)\le \mathcal{C}u(s)$$

for all $1<s\le k$. The proof is complete. □

Lemma 2.4 *Let* $1<p<\infty$ *and u be a nonnegative weight. Then* $u\in\mathcal{A}^p$ *if and only if* $u^{1-p'}\in\mathcal{A}^{p'}$, *with* $\mathcal{A}^{p'}(u^{1-p'})=[\mathcal{A}^p(u)]^{p'-1}$, *where* p' *is the conjugate of* p.

Proof From the definition of the class $\mathcal{A}^p$, and since $1-p' = 1/(1-p) < 0$, we have for $A > 1$ and all $k \in \mathbb{I}$ that

$$\begin{aligned}
u \in \mathcal{A}^p \quad &\Leftrightarrow \quad \frac{1}{k}\sum_{s=1}^{k} u(s) \le A\left(\frac{1}{k}\sum_{s=1}^{k} u^{\frac{1}{1-p}}(s)\right)^{1-p} \\
&\Leftrightarrow \quad \left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right)^{\frac{1}{1-p}} \ge A^{\frac{1}{1-p}}\frac{1}{k}\sum_{s=1}^{k} u^{\frac{1}{1-p}}(s) \\
&\Leftrightarrow \quad \frac{1}{k}\sum_{s=1}^{k} u^{1-p'} \le A^{p'-1}\left(\frac{1}{k}\sum_{s=1}^{k}\left(u^{1-p'}\right)^{\frac{1}{1-p'}}\right)^{1-p'} \\
&\Leftrightarrow \quad u^{1-p'} \in \mathcal{A}^{p'},
\end{aligned}$$

with $\mathcal{A}^{p'}(u^{1-p'}) = [\mathcal{A}^p(u)]^{p'-1}$. The proof is complete. □

The following lemma will play an important role in proving one of our main results.

Lemma 2.5 *Assume that u is a nonincreasing weight, and let $A(k) = \sum_{s=1}^{k} u(s)$. If $p > 1$, then*

$$\frac{1}{k}\sum_{s=1}^{k}\left[u(s)\frac{A^{p-1}(s)}{s^{p-1}} - \frac{p-1}{p}\frac{(A(s))^p}{s^p}\right] \le \frac{1}{p}\frac{(A(k))^p}{k^p} \tag{25}$$

for all $k \in \mathbb{I}$.

Proof Since u is nonincreasing, then so is $\omega(s) = A(s)/s$, thus we have

$$\begin{aligned}
&u(s)\omega^{p-1}(s) - \frac{p-1}{p}\omega^p(s) \\
&\quad\le u(s)\omega^{p-1}(s-1) - \frac{p-1}{p}\omega^p(s) \\
&\quad= \left[A(s) - A(s-1)\right]\omega^{p-1}(s-1) - \frac{p-1}{p}\omega^p(s) \\
&\quad= \left[s\omega(s) - (s-1)\omega(s-1)\right]\omega^{p-1}(s-1) - \frac{p-1}{p}\omega^p(s) \\
&\quad= s\omega(s)\omega^{p-1}(s-1) - (s-1)\omega^p(s-1) - \frac{p-1}{p}\omega^p(s).
\end{aligned} \tag{26}$$

By applying Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1,$$

with $q = \frac{p}{p-1}$, $a = \omega(s)$ and $b = \omega^{p-1}(s)$, we obtain that

$$\omega(s)\omega^{p-1}(s-1) \le \frac{1}{p}\omega^p(s) + \frac{p-1}{p}\omega^p(s-1). \tag{27}$$

By substituting (27) into (26), we obtain

$$\begin{aligned}
&u(s)\omega^{p-1}(s) - \frac{p-1}{p}\omega^p(s) \\
&\quad \leq s\left[\frac{1}{p}\omega^p(s) + \frac{p-1}{p}\omega^p(s-1)\right] - (s-1)\omega^p(s-1) - \frac{p-1}{p}\omega^p(s) \\
&\quad = \frac{1}{p}\left[(p-1)s\omega^p(s-1) - p(s-1)\omega^p(s-1) - (p-1)\omega^p(s) + (s-1)\omega^p(s-1)\right] \\
&\qquad + \frac{1}{p}\left[s\omega^p(s) - (s-1)\omega^p(s-1)\right] \\
&\quad = -\frac{p-1}{p}\Delta\omega^p(s-1) + \frac{1}{p}\Delta(s-1)\omega^p(s-1).
\end{aligned} \tag{28}$$

Using (28) and since $\frac{1-p}{p} < 0$, then we have

$$\begin{aligned}
&\frac{1}{k}\sum_{s=1}^{k}\left[u(s)\omega^{p-1}(s) - \frac{p-1}{p}\omega^p(s)\right] \\
&\quad \leq \left(\frac{1-p}{p}\right)\frac{1}{k}\sum_{s=1}^{k}\Delta\omega^p(s-1) + \frac{1}{p}\frac{1}{k}\sum_{s=1}^{k}\Delta(s-1)\omega^p(s-1) \\
&\quad = \left(\frac{1-p}{p}\right)\frac{1}{k}\omega^p(k) + \frac{1}{p}\omega^p(k) \leq \frac{1}{p}\omega^p(k).
\end{aligned}$$

Rewriting the last inequality and using $\omega = A(k)/k$, we obtain that

$$\frac{1}{k}\sum_{s=1}^{k}\left[u(s)\frac{A^{p-1}(s)}{s^{p-1}} - \frac{p-1}{p}\frac{A^p(s)}{s^p}\right] \leq \frac{1}{p}\frac{A^p(k)}{k^p},$$

which is the required inequality (25). The proof is complete. □

As a consequence of the above lemma and by the definition of $\mathcal{H}$, and the fact that $(\mathcal{H}u)(k) = A(k)/k$, we obtain the following lemma.

Lemma 2.6 *Assume that u is a nonincreasing weight, and let $\mathcal{H}u$ be defined as in (17). If $p > 1$, then*

$$\frac{1}{k}\sum_{s=1}^{k}\left[u(s)\big(\mathcal{H}u(s)\big)^{p-1} - \frac{(p-1)}{p}\big(\mathcal{H}u(s)\big)^p\right] \leq \frac{1}{p}\big(\mathcal{H}u(k)\big)^p$$

for all $k \in \mathbb{I}$.

Theorem 2.1 *Let $u : \mathbb{I} \to \mathbb{R}^+$ be a nonnegative and nonincreasing weight. If*

$$\mathcal{H}u(k) \leq \mathcal{C}u(k) \quad \text{for some } \mathcal{C} > 1 \text{ and all } k \in \mathbb{I}, \tag{29}$$

then, for $r \in [1, \mathcal{C}/(\mathcal{C}-1))$, we have that

$$\mathcal{H}\big(u(k)\big)^r \leq A\big[\mathcal{H}u(k)\big]^r \quad \text{for all } k \in \mathbb{I}, \tag{30}$$

where A is given by

$$A := \frac{\mathcal{C}^{1-r}}{r-(r-1)\mathcal{C}}. \tag{31}$$

Proof From the definition of $\mathcal{H}u(k)$ and Lemma 2.5 with $p = r > 1$, we see that

$$\frac{1}{k}\sum_{s=1}^{k}\left[u(s)\big(\mathcal{H}u(s)\big)^{r-1} - \frac{r-1}{r}\big(\mathcal{H}u(s)\big)^{r}\right] \leq \frac{1}{r}\big(\mathcal{H}u(k)\big)^{r}. \tag{32}$$

Define the function

$$\Phi(\eta) = \gamma\eta^{r-1} - \frac{r-1}{r}\eta^{r} \quad \text{for every } \gamma > 0 \text{ and } \eta \geq \gamma. \tag{33}$$

By noting that, for $\eta \geq \gamma$, we have

$$\begin{aligned}\Phi'(\eta) &= \gamma(r-1)\eta^{r-2} - (r-1)\eta^{r-1}(p-\eta)\\ &\leq (r-1)\eta^{r-1} - (r-1)\eta^{r-1} \leq 0.\end{aligned}$$

That is, $\Phi(\eta)$ is decreasing for $\eta \geq \gamma$. From Lemma 2.1, we see that

$$\mathcal{H}u(s) \geq u(s).$$

Now, by taking that $\gamma = u(s)$, $\beta = \mathcal{H}u(s)$ and $\theta = \mathcal{C}u(k)$, we see that $\gamma \leq \beta \leq \theta$, and then we have

$$\Phi(\gamma) \geq \Phi(\beta) \geq \Phi(\theta) \quad \text{for } \gamma \leq \beta \leq \theta.$$

This implies, by using (33), that

$$\begin{aligned}&u(s)\big(\mathcal{H}u(s)\big)^{r-1} - \frac{r-1}{r}\big(\mathcal{H}u(s)\big)^{r}\\ &\quad\geq u(s)\big(\mathcal{C}u(s)\big)^{r-1} - \frac{r-1}{r}\big(\mathcal{C}u(s)\big)^{r}\\ &\quad= \mathcal{C}^{r-1}\big(u(s)\big)^{r} - \frac{r-1}{r}\mathcal{C}^{r}\big(u(s)\big)^{r} = \mathcal{C}^{r-1}\left[1 - \frac{r-1}{r}\mathcal{C}\right]\big(u(s)\big)^{r}.\end{aligned} \tag{34}$$

By combining (32) and (34), we get that

$$\mathcal{C}^{r-1}\left[\frac{r-(r-1)\mathcal{C}}{r}\right]\frac{1}{k}\sum_{s=1}^{k}\big(u(s)\big)^{r} \leq \frac{1}{r}\big(\mathcal{H}u(k)\big)^{r}.$$

This implies that

$$\frac{1}{k}\sum_{s=1}^{k}\big(u(s)\big)^{r} \leq \frac{\mathcal{C}^{1-r}}{r-(r-1)\mathcal{C}}\big(\mathcal{H}u(k)\big)^{r}.$$

The proof is complete. □

Remark 2.3 Theorem 2.1 is a discrete version of Lemma 1.2 and proves that if $u \in \mathcal{A}^1(\mathcal{C})$ then $u \in \mathcal{G}^r(A)$ for $r \in [1, \mathcal{C}/(\mathcal{C}-1))$ and a constant A given by (31).

Theorem 2.2 *Let u be a nondecreasing weight. If $1 < p < \infty$ and $u \in \mathcal{A}^p(\mathcal{C})$, then there exist constants q and $\mathcal{C}_1$ depending on p and $\mathcal{C}$ such that $1 < q < p$ and $u \in \mathcal{A}^q(\mathcal{C}_1)$.*

Proof Since $u \in \mathcal{A}^p(\mathcal{C})$, then it satisfies the condition

$$\left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right)\left(\frac{1}{k}\sum_{s=1}^{k} u^{\frac{-1}{p-1}}(s)\right)^{p-1} \leq \mathcal{C}. \tag{35}$$

From Lemma 2.4, we see also that $u^{1-p'}$ satisfies the $\mathcal{A}^{p'-1}$-condition

$$\left(\frac{1}{k}\sum_{s=1}^{k} u^{1-p'}(s)\right)\left(\frac{1}{k}\sum_{s=1}^{k} \left(u^{1-p'}(s)\right)^{\frac{1}{1-p'}}\right)^{p'-1} \leq A^{p'-1}.$$

Since $1-p' = -1/(p-1)$ and u is nondecreasing, we see that $u^{-\frac{1}{p-1}}$ is nonincreasing. Now, applying Lemmas 2.2 and 2.1, we see that

$$\left(\frac{1}{k}\sum_{s=1}^{k} \left(u(s)\right)^{-\frac{r}{p-1}}\right) \leq A\left[\frac{1}{k}\sum_{s=1}^{k} u^{-\frac{1}{p-1}}(s))\right]^{r} \tag{36}$$

for $r \in (1, r_0)$, with a constant A. Combining (35) and (36), we have that

$$\begin{aligned}
&\left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right)\left(\frac{1}{k}\sum_{s=1}^{k} u^{-\frac{r}{p-1}}(s)\right)^{(p-1)/r} \\
&\quad \leq A^{(p-1)/r}\left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right)\left[\frac{1}{k}\sum_{s=1}^{k} u^{-\frac{1}{p-1}}(s))\right]^{p-1} \\
&\quad \leq A^{(p-1)/r}\mathcal{C}.
\end{aligned}$$

This shows that u satisfies the $\mathcal{A}^q$-condition, where $q = 1 + (p-1)r$ and $\mathcal{C}_1 = A^{(p-1)/r}\mathcal{C}$. It is immediate that q and $\mathcal{C}_1$ depend only on $\mathcal{C}$ and p. The proof is complete. □

Theorem 2.3 *Let u be a nondecreasing weight on $\mathbb{I}$ with $|\mathbb{I}| = 2^r$ for $r \in \mathbb{Z}_+$. If $1 < p < \infty$ and u satisfies the $\mathcal{A}^p$-condition (35) with constant $\mathcal{C}$, then there exist constants q and $\mathcal{C}_1$ depending on p and $\mathcal{C}$ such that $1 < q$ and u^q satisfies the $\mathcal{A}^p$-condition with constant $\mathcal{C}_1$.*

Proof In [4] Böttcher and Seybold proved that if u satisfies the $\mathcal{A}^p$-condition (35) with constant $\mathcal{C}$, then there exists a constant $m > 1$ and $\mathcal{C}_1 < \infty$ depending only on p such that

$$\frac{1}{k}\sum_{s=1}^{k} u^m(s) \leq \mathcal{C}_1\left(\frac{1}{k}\sum_{s=1}^{k} u(s)\right)^{m} \tag{37}$$

for all $m > 1$ and all even natural numbers k. Now, by combining (36) and (37), we see that

$$\begin{aligned}&\left(\frac{1}{k}\sum_{s=1}^{k}u^{m}(s)\right)^{1/m}\left(\frac{1}{k}\sum_{s=1}^{k}u^{-\frac{r}{p-1}}(s)\right)^{(p-1)/r}\\&\quad\le \mathcal{C}_1^{1/m}A^{(p-1)/r}\mathcal{C}\left(\frac{1}{k}\sum_{s=1}^{k}u(s)\right)\left(\frac{1}{k}\sum_{s=1}^{k}u(s)\right)^{-1}.\end{aligned}$$

That is,

$$\left(\frac{1}{k}\sum_{s=1}^{k}u^{m}(s)\right)^{1/m}\left(\frac{1}{k}\sum_{s=1}^{k}u^{-\frac{r}{p-1}}(s)\right)^{(p-1)/r}\le \mathcal{C}_1^{1/m}A^{(p-1)/r}\mathcal{C}. \tag{38}$$

No, let $q = \min\{r, m\}$, then Hölder's inequality implies that

$$\left[\frac{1}{k}\sum_{s=1}^{k}|u^{p}(s)\right]^{\frac{1}{p}}\le\left[\frac{1}{k}\sum_{s=1}^{k}|u^{q}(s)\right]^{\frac{1}{q}}\quad\text{for } 1<p<q,$$

and then (38) shows that

$$\left(\frac{1}{k}\sum_{s=1}^{k}u^{q}(s)\right)^{1/q}\left(\frac{1}{k}\sum_{s=1}^{k}u^{-\frac{q}{p-1}}(s)\right)^{(p-1)/q}\le\mathcal{L},$$

where $\mathcal{L} = \mathcal{C}_1^{1/m}A^{(p-1)/r}\mathcal{C}$. Taking the q^{th} power, we get the desired result for $p > 1$. The proof is complete. □

One of the basic special formulas in the differential calculus is the power rule $(d/dt)t^k = kt^{k-1}$. Unfortunately, the difference of a power is complicated and not very useful since

$$\Delta t^n = (t+1)^n - t^n = \sum_{k=0}^{n-1}\binom{n}{k}t^k.$$

In the following, we show how we can use the difference calculus to prove the property of the parameter of Muckenhoupt and Gehring classes for power-low sequences.

Lemma 2.7

(i). *If* $p > 1$ *and* $-1 < \lambda < p - 1$, *then the norm* $\mathcal{A}^p(n^\lambda) = \Phi(p, \lambda)$, *where*

$$\Phi(p,\lambda)=\begin{cases}\frac{2^{-\lambda}}{(1+\lambda)}(\frac{p-1}{p-\lambda-1})^{p-1}, & \textit{if } \lambda<0,\\ \frac{2^{\lambda}}{(1+\lambda)}(\frac{p-1}{p-\lambda-1})^{p-1}, & \textit{if } \lambda>0.\end{cases} \tag{39}$$

(ii). *If* $0 < \lambda < \beta$, *then* $\mathcal{A}^p(n^{-\lambda}) < \mathcal{A}^p(n^{-\beta})$.

Proof Consider the interval $J = [a, N] \subset \mathbb{I}$. From the definition of the norm of $\mathcal{A}^p(u)$, we see that

$$\mathcal{A}^p(n^\lambda) := \sup_{[a,N]\subset\mathbb{I}}\left(\frac{1}{N-a}\sum_{n=a}^{N-1}n^{\lambda}\right)\left(\frac{1}{N-a}\sum_{n=a}^{N-1}n^{\frac{\lambda}{1-p}}\right)^{p-1}.$$

Now, we determine the summations in the right-hand side. We start by the summation $\sum_{n=a}^{N-1} n^{\lambda}$. Noting that $-1 < \lambda < p-1$ and since $p > 1$, we see that $0 < 1+\lambda < p$ and

$$0 < 1 + \frac{\lambda}{1-p} < \frac{p}{p-1}.$$

We consider different cases:

(1). The case when $\lambda < 0$. In this case we have that $1 + \lambda < 1$, and we see that $\Delta n^{1+\lambda} = (n+1)^{1+\lambda} - n^{1+\lambda}$. By using the inequality

$$py^{p-1}(y-z) \leq y^p - z^p \leq pz^{p-1}(y-z) \quad \text{for } y \geq z > 0 \text{ and } 0 < p < 1, \tag{40}$$

we see that

$$\begin{aligned} 2^{\lambda}(1+\lambda)n^{\lambda} &\leq (1+\lambda)(n+1)^{\lambda} \leq \Delta n^{1+\lambda} \\ &= (n+1)^{1+\lambda} - n^{1+\lambda} \leq (1+\lambda)n^{\lambda}. \end{aligned}$$

So that

$$\frac{1}{N-a}\sum_{n=a}^{N-1} n^{\lambda} \leq \frac{1}{N-a}\frac{1}{2^{\lambda}(1+\lambda)}\left(N^{1+\lambda} - a^{1+\lambda}\right). \tag{41}$$

Now, we determine the summation $(\sum_{n=a}^{N-1} n^{\frac{\lambda}{1-p}})^{p-1}$. Since $\lambda < 0$, we see that $1 + \frac{\lambda}{1-p} > 1$ and then by using the inequality

$$\gamma z^{\gamma-1}(y-z) \leq y^{\gamma} - z^{\gamma} \leq \gamma y^{\gamma-1}(y-z) \quad \text{for } y \geq z > 0, \gamma \geq 1 \text{ or } \gamma < 0, \tag{42}$$

we see that

$$\begin{aligned} \left(1 + \frac{\lambda}{1-p}\right) n^{\frac{\lambda}{1-p}} &\leq \Delta n^{1+p\lambda} = (n+1)^{1+\frac{\lambda}{1-p}} - n^{1+\frac{\lambda}{1-p}} \\ &\leq \left(1 + \frac{\lambda}{1-p}\right)(n+1)^{\frac{\lambda}{1-p}}, \end{aligned}$$

and hence we obtain

$$\left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^{\frac{\lambda}{1-p}}\right)^{p-1} \leq \left(\frac{p-1}{p-\lambda-1}\right)^{p-1} \frac{(N^{1+\frac{\lambda}{1-p}} - a^{1+\frac{\lambda}{1-p}})^{p-1}}{(N-a)^{p-1}}. \tag{43}$$

By combining (41) and (43), we see that

$$\begin{aligned} \mathcal{A}^p\left(n^{\lambda}\right) &= \frac{1}{2^{\lambda}(1+\lambda)}\left(\frac{p-1}{p-\lambda-1}\right)^{p-1} \\ &\quad \times \sup_{[a,N]\subset\mathbb{I}} \frac{(N^{1+\lambda} - a^{1+\lambda})}{(N-a)^p}\left(N^{1+\frac{\lambda}{1-p}} - a^{1+\frac{\lambda}{1-p}}\right)^{p-1}. \end{aligned} \tag{44}$$

(2). The case when $\lambda > 0$. In this case, we have that $\lambda + 1 > 1$, and then we apply inequality (42) to get that

$$(1+\lambda)n^{\lambda} \leq \Delta n^{1+\lambda} = (n+1)^{1+\lambda} - n^{1+\lambda} \leq (1+\lambda)(1+n)^{\lambda}.$$

This implies that

$$\frac{1}{N-a}\sum_{k=a}^{N-1} n^{\lambda} \leq \frac{1}{N-a}\frac{1}{(1+\lambda)}\left(N^{1+\lambda} - a^{1+\lambda}\right). \tag{45}$$

Since $\lambda > 0$, we see that $1 + \frac{\lambda}{1-p} < 1$, then by applying inequality (40), we get that

$$2^{\frac{\lambda}{1-p}}\left(1 + \frac{\lambda}{1-p}\right)n^{\frac{\lambda}{1-p}} \leq \left(1 + \frac{\lambda}{1-p}\right)(n+1)^{\frac{\lambda}{1-p}} \leq \Delta n^{1+\frac{\lambda}{1-p}},$$

and so we get that

$$\left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^{\frac{\lambda}{1-p}}\right)^{p-1} \leq \frac{2^{\lambda}}{(N-a)^{p-1}}\left(\frac{p-1}{p-\lambda-1}\right)^{p-1}\left(N^{1+\frac{\lambda}{1-p}} - a^{1+\frac{\lambda}{1-p}}\right)^{p-1}. \tag{46}$$

By combining (45) and (46), we have that

$$\begin{aligned} \mathcal{A}^{p}\left(n^{\lambda}\right) = {} & \frac{2^{\lambda}}{(1+\lambda)}\left(\frac{p-1}{p-\lambda-1}\right)^{p-1} \\ & \times \sup_{[a,N)\subset\mathbb{I}} \frac{(N^{1+\lambda} - a^{1+\lambda})}{(N-a)^{p}}\left(N^{1+\frac{\lambda}{1-p}} - a^{1+\frac{\lambda}{1-p}}\right)^{p-1}. \end{aligned} \tag{47}$$

From (44) and (47), we see that

$$\mathcal{A}^{p}\left(n^{\lambda}\right) = \Phi(p,\lambda) \sup_{[a,N)\subset\mathbb{I}} \frac{(N^{1+\lambda} - a^{1+\lambda})}{(N-a)^{p}}\left(N^{1+\frac{\lambda}{1-p}} - a^{1+\frac{\lambda}{1-p}}\right)^{p-1}.$$

Denote $t = N/a > 1$, we see that

$$\begin{aligned} & \left(N^{1+\lambda} - a^{1+\lambda}\right)(N-a)^{-p}\left(N^{1+\frac{\lambda}{1-p}} - a^{1+\frac{\lambda}{1-p}}\right)^{p-1} \\ & \quad = \left(t^{1+\lambda} - 1\right)(t-1)^{-p}\left(t^{1+\frac{\lambda}{1-p}} - 1\right)^{p-1}. \end{aligned}$$

We define

$$\zeta(t,p,\lambda) = \left(t^{1+\lambda} - 1\right)(t-1)^{-p}\left(t^{1+\frac{\lambda}{1-p}} - 1\right)^{p-1}$$

for $t > 1$, $p > 1$, and $-1 < \lambda < p - 1$. Now, by using Lemma 2.2 in [19], we see that $\sup_{t>1} \zeta(t,p,\lambda) = 1$ for all fixed $p > 1$ and $-1 < \lambda < p - 1$. This gives us that $\mathcal{A}^{p}(n^{\lambda}) = \Phi(p,\lambda)$, which proves statement (i).

By noting that the function

$$F(x) = \frac{2^{x}}{(1+x)}\left(\frac{1-p}{x+1-p}\right)^{p-1}$$

is a decreasing function for $x > 0$, we have that $F(-\lambda) < F(-\beta)$ if $0 < \lambda < \beta$. This completes the proof of (ii). The proof is complete. □

Lemma 2.8

(i). *If* $p > 1$ *and* $\alpha > -1/p$, *then the norm* $(\mathcal{G}^p(n^\alpha))^{\frac{p-1}{p}} := \Psi(p,\alpha)$, *where*

$$\Psi(p,\alpha) = \begin{cases} \frac{2^{-\alpha}(1+\alpha)}{(1+p\alpha)^{1/p}}, & \text{if } \alpha < 0, \\ \frac{2^{\alpha}(1+\alpha)}{(1+p\alpha)^{1/p}}, & \text{if } \alpha > 0. \end{cases} \tag{48}$$

(ii). *If* $0 < \alpha < \beta$, *then* $(\mathcal{G}^p(n^\alpha))^{\frac{p-1}{p}} < (\mathcal{G}^p(n^\beta))^{\frac{p-1}{p}}$.

Proof Consider the interval $I = [a,N] \subset J$. From the definition of the norm of $\mathcal{G}^q(u)$, we see that

$$\left(\mathcal{G}^p\left(n^\alpha\right)\right)^{\frac{p-1}{p}} := \sup_{[a,N]\subset\mathbb{I}} \left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^\alpha\right)^{-1} \left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^{p\alpha}\right)^{\frac{1}{p}}.$$

Now, we determine the summations in the right-hand side. We start by the summation

$$\sum_{k=a}^{N-1} n^\alpha.$$

When $\alpha < 0$, we note that $-1/p < \alpha < 0$. Since $p > 1$, we see that $0 < 1 + \alpha < 1$. In this case, we get by employing inequality (40) that

$$(1+\alpha)(n+1)^\alpha \le \Delta n^{1+\alpha} = (n+1)^{1+\alpha} - n^{1+\alpha} \le (1+\alpha)n^\alpha.$$

So that

$$\frac{1}{N-a}\sum_{k=a}^{N-1} n^\alpha \ge \frac{1}{(1+\alpha)}\frac{1}{N-a}\left(N^{1+\alpha} - a^{1+\alpha}\right). \tag{49}$$

Now, we determine the summation

$$\sum_{k=a}^{N-1} n^{p\alpha}.$$

By noting that $-1/p < \alpha < 0$, we see $0 < 1 + p\alpha < 1$, and then by employing inequality (40), we see that

$$\Delta n^{1+p\alpha} \ge (n+1)^{1+p\alpha} - n^{1+p\alpha} \ge (1+p\alpha)(n+1)^{p\alpha} \ge 2^{p\alpha}(1+p\alpha)n^{p\alpha},$$

and so we obtain

$$\left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^{p\alpha}\right)^{\frac{1}{p}} \le \frac{1}{2^\alpha(1+p\alpha)^{1/p}}\frac{1}{(N-a)^{1/p}}\left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}. \tag{50}$$

By combining (49) and (50), we see that

$$\left(\mathcal{G}^p\left(n^\alpha\right)\right)^{\frac{p-1}{p}} = \frac{1+\alpha}{2^\alpha(1+p\alpha)^{1/p}} \times \sup_{[a,N)\subset\mathbb{I}} \left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}(N-a)^{1-\frac{1}{p}}\left(N^{1+\alpha} - a^{1+\alpha}\right)^{-1}. \tag{51}$$

Now, we consider the case when $\alpha > 0$. In this case, we see that $\alpha + 1 > 1$, and then by applying inequality (42) we obtain

$$\Delta n^{1+\alpha} = (n+1)^{1+\alpha} - n^{1+\alpha} \leq 2^\alpha(1+\alpha)n^\alpha.$$

This implies that

$$\sum_{k=a}^{N-1} n^\alpha \geq \frac{1}{2^\alpha(1+\alpha)}\left(N^{1+\alpha} - a^{1+\alpha}\right).$$

By combining the two cases, we have

$$\left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^\alpha\right)^{-1} \leq 2^\alpha(1+\alpha)(N-a)\left(N^{1+\alpha} - a^{1+\alpha}\right)^{-1}. \tag{52}$$

Now, since $\alpha > 0$, then $1 + p\alpha > 1$, and in this case, we see that $\Delta n^{1+p\alpha} > (1+p\alpha)n^{p\alpha}$, and so we get that

$$\left(\sum_{k=a}^{N-1} n^{p\alpha}\right)^{\frac{1}{p}} \leq \frac{1}{(1+p\alpha)^{1/p}}\left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}.$$

So we have that

$$\left(\frac{1}{N-a}\sum_{k=a}^{N-1} n^{p\alpha}\right)^{\frac{1}{p}} \leq \frac{1}{(1+p\alpha)^{1/p}}\frac{1}{(N-a)^{1/p}}\left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}. \tag{53}$$

By combining(52) and (53), we have that

$$\left(\mathcal{G}^p\left(n^\alpha\right)\right)^{\frac{p-1}{p}} = \frac{2^\alpha(1+\alpha)}{(1+p\alpha)^{1/p}} \sup_{[a,N)\subset\mathbb{I}} \left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}(N-a)^{1-\frac{1}{p}}\left(N^{1+\alpha} - a^{1+\alpha}\right)^{-1}. \tag{54}$$

From (48) and (54), we see that

$$\left(\mathcal{G}^p\left(n^\alpha\right)\right)^{\frac{p-1}{p}} = \Psi(p,\alpha) \sup_{[a,N)\subset\mathbb{I}} \left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}(N-a)^{1-\frac{1}{p}}\left(N^{1+\alpha} - a^{1+\alpha}\right)^{-1}.$$

Denote $t = N/a > 1$, we see that

$$\begin{aligned}&\left(N^{1+p\alpha} - a^{1+p\alpha}\right)^{1/p}(N-a)^{1-\frac{1}{p}}\left(N^{1+\alpha} - a^{1+\alpha}\right)^{-1}\\ &\quad = (t-1)^{1-\frac{1}{p}}\left(t^{1+p\alpha} - 1\right)^{1/p}\left(t^{1+\alpha} - 1\right)^{-1}.\end{aligned}$$

We define

$$\zeta(t,p,\alpha) = (t-1)^{1-\frac{1}{p}}\left(t^{1+p\alpha}-1\right)^{1/p}\left(t^{1+\alpha}-1\right)^{-1}$$

for $t > 1$, $p > 1$, and $\alpha > -1/p$. Now, by using Lemma 2.2 in [18], we see that

$$\sup_{t>1}\zeta(t,p,\alpha) = 1$$

for all fixed $p > 1$ and $\alpha > -1/p$. This gives us that

$$\left(\mathcal{G}^p\left(n^\alpha\right)\right)^{\frac{p-1}{p}} = \Psi(p,\alpha),$$

which proves statement (i). By noting that the function

$$F(x) = \frac{2^x(1+x)}{(1+px)^{1/p}}$$

is an increasing function for $x > 0$, we have that $F(\alpha) < F(\beta)$ if $0 < \alpha < \beta$. This completes the proof of (ii). The proof is complete. □

3 Conclusion

In this paper, we studied the structure of the discrete cases of the well-known Muckenhoupt class $\mathcal{A}^p(C)$ and Gehring class $\mathcal{G}^q(K)$. We established exact values of the norms of the discrete Muckenhoupt and Gehring classes for power-low sequences. The relations between the two classes have also been discussed. In fact we have proved that if the weight w belongs to the Muckenhoupt class $\mathcal{A}^1(C)$, then it belongs to the same Gehring classes $\mathcal{G}^q(K)$ for some q obtained from a solution of an algebraic equation.

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