Kim and Majee *Journal of Inequalities and Applications* (2020) 2020:227
https://doi.org/10.1186/s13660-020-02493-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Modified Krasnoselski–Mann iterative method for hierarchical fixed point problem and split mixed equilibrium problem

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Abstract

In this paper, we introduce a modified Krasnoselski–Mann type iterative method for capturing a common solution of a split mixed equilibrium problem and a hierarchical fixed point problem of a finite collection of k-strictly pseudocontractive nonself-mappings. Many of the algorithms for solving the split mixed equilibrium problem involve a step size which depends on the norm of a bounded linear operator. Since the computation of the operator norm is very difficult, we formulate our iterative algorithm in such a way that the implementation of the proposed algorithm does not require any prior knowledge of operator norm. Weak convergence results are established under mild conditions. We also establish strong convergence results for a certain class of hierarchical fixed point and split equilibrium problem. Our results generalize some important results in the recent literature.

MSC: 90C33; 47H10; 49J40

Keywords: Nonexpansive mapping; Averaged mapping; Split mixed equilibrium problem; Fixed point problem

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C and D be two nonempty closed and convex subsets of H_1 and H_2, respectively. A nonself-mapping $T : C \mapsto H_1$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in H_1.$$

If $k = 0$, then T is a nonexpansive nonself-mapping.

For a mapping $T : C \mapsto H_1$, the fixed point problem is to find $x \in C$ such that $x = Tx$. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

Moudafi and Mainge [24] considered the following hierarchical fixed point problem (in short, HFPP) for a nonexpansive self-mapping T with respect to another nonexpansive self-mapping S on C in the following way: Find $x^* \in \mathrm{Fix}(T)$ such that

$$\langle x^* - Sx^*, x^* - x \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.1}$$

Let us denote the solution set of the HFPP (1.1) as $\mathcal{S} = \{x^* \in \mathrm{Fix}(T) : \langle x^* - Sx^*, x^* - x\rangle \leq 0, \forall x \in \mathrm{Fix}(T)\}$.

We can check that x^* is a solution of the HFPP (1.1) if and only if $x^* = P_{\mathrm{Fix}(T)} \circ Sx^*$, where $P_{\mathrm{Fix}(T)}$ is the metric projection of H_1 onto $\mathrm{Fix}(T)$.

By using the definition of the normal cone to $\mathrm{Fix}(T)$, i.e.,

$$N_{\mathrm{Fix}(T)} = \begin{cases} \{u \in H_1 : \langle y - x, u\rangle \leq 0, \forall y \in \mathrm{Fix}(T)\}, & \text{if } x \in \mathrm{Fix}(T), \\ \emptyset, & \text{otherwise}, \end{cases} \tag{1.2}$$

we can easily prove that HFPP (1.1) is equivalent to the variational inclusion

$$0 \in (I - S)x^* + N_{\mathrm{Fix}(T)}x^*.$$

We note that based on the relation $x^* = P_{\mathrm{Fix}(T)} \circ Sx^*$, HFPP (1.1) has an iterative algorithm $x_{n+1} = P_{\mathrm{Fix}(T)} \circ Sx_n$. This iterative method converges if the mapping $P_{\mathrm{Fix}(T)} \circ S$ has a fixed point and S is averaged not just nonexpansive. Disadvantage of this method is that the calculation of $P_{\mathrm{Fix}(T)} \circ S$ is usually not easy. To overcome this, Moudafi [22] introduced an algorithm which uses T itself, rather than $P_{\mathrm{Fix}(T)} \circ S$. Moudafi introduced the iterative method in the following way: For starting point $x_0 \in C$, define $\{x_n\}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\big(\sigma_n Sx_n + (1 - \sigma_n)Tx_n\big), \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two real sequences in $(0, 1)$.

Problems (1.1) are often used in the area of optimization and related fields, such as signal processing and image reconstruction (see [4, 6, 11, 14–20, 25, 27–30, 32–36] and the references therein).

On the other hand, for a bifunction $F : C \times C \to \mathbb{R}$, an equilibrium problem is defined by

$$\text{find } x \in C \text{ such that } F(x, y) \geq 0, \quad \text{for all } y \in C.$$

The solution set of this problem is denoted as $\mathrm{EP}(F, C)$. Equilibrium problem includes variational inequality problem, optimization problem, the Nash equilibrium problem, saddle point problem, complementarity problem, convex differential optimization etc. as special cases (see Blum and Oettli [1]).

In 2009, Marino et al. [21] introduced an iterative method to find common solutions of the following system of equilibrium problem and hierarchical fixed point problem:

$$\begin{cases} \text{find } x^* \in C \text{ such that } F(x^*, y) \geq 0, \quad \forall y \in C; \quad \text{and} \\ x^* \in \mathrm{Fix}(T) \text{ such that } \langle x^* - f(x^*), x^* - y\rangle \leq 0, \quad \forall y \in \mathrm{Fix}(T), \end{cases} \tag{1.4}$$

where F is a bifunction, f is a ρ-contraction and T is a nonexpansive self-mapping.

In 2011, Moudafi [23] introduced and studied the following split equilibrium problem S_pEP: Suppose $F_1 : C \times C \to \mathbb{R}$ and $F_2 : D \times D \to \mathbb{R}$ are two bifunctions and $A : H_1 \to H_2$ is a bounded linear operator. Then the split equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C. \tag{1.5}$$

and $y^* = Ax^* \in D$ solves

$$F_2(y^*, y) \geq 0, \quad \forall y \in D. \tag{1.6}$$

In 2012, Censor et al. [7] introduced and studied the following split variational inequality problem S_pVIP: Find $x^* \in C$ such that

$$\langle g_1(x^*), y - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.7}$$

and $y^* = Ax^* \in Q$ solves

$$\langle g_2(x^*), y - x^* \rangle \geq 0, \quad \forall y \in D, \tag{1.8}$$

where $g_1 : C \to H_1$ and $g_2 : D \to H_2$ are two nonlinear mappings, and $A : H_1 \to H_2$ is a bounded linear operator.

Very recently, Kazmi et al. [13] have introduced and analyzed a Krasnoselski–Mann iteration method for finding a common solution of HFPP (1.1) and the following split mixed equilibrium problem S_pMEP: Find $x^* \in C$ such that

$$F_1(x^*, x) + \langle g_1(x^*), y - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.9}$$

and $y^* = Ax^* \in Q$ solves

$$F_2(y^*, y) + \langle g_2(x^*), y - x^* \rangle \geq 0, \quad \forall y \in D. \tag{1.10}$$

The solution set of S_pMEP is denoted by $\Gamma = \{p \in \mathrm{MEP}(F_1, g_1) : Ap \in \mathrm{MEP}(F_2, g_2)\}$. To solve the S_pMEP (1.9)–(1.10) and HFPP (1.1), Kazmi et al. [13] introduced the following algorithm: For the starting point $x_0 \in C$, define $\{x_n\}$ by

$$\begin{cases} u_n = (1 - \alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1 - \sigma)T x_n), \\ x_{n+1} = U(u_n + \gamma A^*(V - I)Au_n), \quad n \geq 1, \end{cases} \tag{1.11}$$

where S, T are nonexpansive self-mappings on C and the step size $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of the bounded linear operator A.

Motivated by the above results, we revisit the problem considered by Kazmi et al. [13]. We introduce and analyze a modified Krasnoselski–Mann type iterative method with the help of averaged mappings for finding a common solution of the HFPP (1.1) of a finite collection of k-strictly pseudocontractive *non-self*-mappings and S_pMEP (1.9)–(1.10). Our work can be seen as an extension of HFPP (1.1) [13, 22] from single nonexpansive self-mapping to a finite collection of k-strictly pseudocontractive nonself-mappings. The authors in [10, 13] have selected the step size γ in the interval $(0, \frac{1}{L})$, where L is the spectral radius of the operator A^*A and A^* is the adjoint of the bounded linear operator A.

It is well known that the computation or an estimate of the spectral radius of a given operator is very difficult at times. This makes the implementation of the proposed algorithm very difficult. In our iterative method, we give an explicit formula to choose the step size, which does not require any prior knowledge of operator norm. We also establish strong convergence results for a certain class of hierarchical fixed point and split equilibrium problem.

We organize the paper in the following way. Some basic definitions and lemmas are given in Sect. 2. In Sect. 3, we present our modified iterative methods for a split mixed equilibrium problem and hierarchical fixed point problem. Finally, we make some remarks to highlight the main contribution of this paper.

2 Preliminaries

In order to prove our main results, we recall some basic definitions and lemmas, which will be needed in the sequel. Let H_1 be a real Hilbert space and C be a nonempty closed convex subset of H_1. Let the symbols "$\rightharpoonup$" and "$\rightarrow$" denote the weak and strong convergence, respectively. We know that in a Hilbert space H_1, the following properties hold:

$$\|x-y\|^2 \leq \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad \forall x,y \in H_1 \tag{2.1}$$

and

$$2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x,y \in H_1. \tag{2.2}$$

It is well known that every nonexpansive mapping $T : H_1 \rightarrow H_1$ satisfies the inequality:

$$\bigl\langle (x-Tx)-(y-Ty), Ty-Tx\bigr\rangle \leq \frac{1}{2}\bigl\|(Tx-x)-(Ty-y)\bigr\|^2, \quad \forall (x,y) \in H_1 \times H_1.$$

Therefore for all $(x,y) \in H_1 \times \mathrm{Fix}(T)$, we get

$$\langle x-Tx, y-Tx\rangle \leq \frac{1}{2}\bigl\|(Tx-x)\bigr\|^2. \tag{2.3}$$

A mapping $T : H_1 \rightarrow H_1$ is said to be monotone if

$$\langle Tx-Ty, x-y\rangle \geq 0, \quad \forall x,y \in H_1.$$

T is said to be α-inverse strongly monotone if there exists a $\alpha > 0$ such that

$$\langle Tx-Ty, x-y\rangle \geq \alpha\|Tx-Ty\|^2, \quad \forall x,y \in H_1.$$

T is said to be firmly nonexpansive if

$$\langle Tx-Ty, x-y\rangle \geq \|Tx-Ty\|^2, \quad \forall x,y \in H_1.$$

A mapping P_C is said to be metric projection from H_1 onto C if for every $x \in H_1$, there exists a unique nearest point in C denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.4}$$

It is well known that P_C is nonexpansive and firmly nonexpansive. Moreover, P_C is characterized by the following property:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall x \in H_1, y \in C. \tag{2.5}$$

A multivalued mapping $M : H_1 \to 2^{H_1}$ is called monotone if for any $x, y \in H_1$

$$\langle u - v, x - y\rangle \geq 0, \quad \forall u \in Mx, v \in My.$$

For a multivalued mapping M, graph(M) is defined by graph(M) := $\{(x, u) \in H_1 \times H_1 : u \in Mx\}$. A multivalued monotone mapping $M : H_1 \to 2^{H_1}$ is said to be maximal monotone if graph(M) is not properly contained in the graph of any other monotone mapping. It is well known that a multivalued monotone mapping is maximal monotone if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in$ graph(M) implies that $u \in Mx$. Let $M : H_1 \to 2^{H_1}$ be a multivalued maximal monotone operator. Then the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \quad \forall x \in H_1,$$

for some $\lambda > 0$, where I is the identity operator on H_1. It is well known that for all $\lambda > 0$ the resolvent operator is single-valued, nonexpansive and firmly nonexpansive. Also, we know that $\mathrm{Fix}(J_\lambda^M) = M^{-1}(0)$.

Definition 2.1 A mapping $T : H_1 \to H_1$ is said to be an averaged mapping if there exists some number $\alpha \in (0, 1)$ such that $T = (1 - \alpha)I + \alpha S$, where $I : H_1 \to H_1$ is the identity mapping and $S : H_1 \to H_1$ is a nonexpansive mapping. An averaged mapping is also nonexpansive and $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.

Lemma 2.2 ([3, 4]) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 \cdots T_N).$$

In particular, for $N = 2$, $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \mathrm{Fix}(T_1 T_2) = \mathrm{Fix}(T_2 T_1)$.

Lemma 2.3 ([37]) *Assume that $S : C \to H_1$ is a k-strictly pseudocontractive mapping. Define a mapping T by $Tx = \alpha x + (1 - \alpha)Sx$ for all $x \in H_1$, where $\alpha \in [k, 1)$. Then T is a nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.

Lemma 2.4 ([37]) *Let $T : C \to H_1$ be a k-strictly pseudocontractive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Then* $\mathrm{Fix}(P_C T) = \mathrm{Fix}(T)$.

Lemma 2.5 (Demiclosedness principle [12]) *Let C be a nonempty closed convex subset of a real Hilbert space H_1 and let $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Assumption A ([1]) Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies the following assumptions:

(i) $F(x,x) \geq 0$, $\forall x \in C$;
(ii) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;
(iii) F is upper hemicontinuous, i.e., for each $x,y,z \in C$,

$$\limsup_{t\to 0} F\bigl(tz + (1 - t)x, y\bigr) \leq F(x,y); \tag{2.6}$$

(iv) For each $x \in C$ fixed, the function $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.6 ([9]) *Assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies Assumption A. For $r > 0$ and $x \in H_1$, define a mapping $T_r^F : H_1 \to C$ as follows*:

$$T_r^F(x) := \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}. \tag{2.7}$$

Then the following properties hold:

(i) *T_r^F is nonempty and single-valued.*
(ii) *T_r^F is firmly nonexpansive, i.e., $\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle$, $\forall x,y \in H_1$.*
(iii) *$\mathrm{Fix}(T_r^F) = \mathrm{EP}(F,C)$.*
(iv) *$\mathrm{EP}(F,C)$ is closed and convex.*

Furthermore, assume that $F_2 : D \times D \to \mathbb{R}$ satisfy the conditions in Assumption A. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to D$ as follows:

$$T_s^{F_2}(w) := \left\{d \in D : F(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \forall e \in D\right\}. \tag{2.8}$$

Then we easily observe that $T_s^{F_2}$ is nonempty, single-valued and firmly nonexpansive. Also, $\mathrm{EP}(F_2,D)$ is closed and convex, and $\mathrm{Fix}(T_s^{F_2}) = \mathrm{EP}(F_2,D)$, where $\mathrm{EP}(F_2,D)$ is the solution of the following equilibrium problem: Find $y^* \in D$ such that

$$F_2\bigl(y^*,y\bigr) \geq 0, \quad \forall y \in D.$$

Lemma 2.7 ([8]) *Let $\{\delta_n\}$ and $\{\gamma_n\}$ be non-negative sequences satisfying $\sum_{n=0}^{\infty} \delta_n < +\infty$ and $\gamma_{n+1} \leq \gamma_n + \delta_n$ for all $n \in \mathbb{N}$. Then $\{\gamma_n\}$ is a convergent sequence.*

Definition 2.8 ([2, 24]) A sequence $\{M_n\}$ of maximal monotone mappings defined on H_1 is said to be graph convergent to a multivalued mapping M if $\{\mathrm{graph}(M_n)\}$ converges to graph(M) in the Kuratowski–Painlevé sense, that is,

$$\limsup_{n\to\infty} \operatorname{graph}(M_n) \subset \operatorname{graph}(M) \subset \liminf_{n\to\infty} \operatorname{graph}(M_n).$$

Lemma 2.9 ([9]) *We have the following statements*:

(i) *Let M be a maximal monotone mapping on H_1. Then $\{{t_n}^{-1}M\}$ is graph convergent to $N_{M^{-1}0}$ as $t_n \to 0$ provided that $M^{-1}0 \neq \emptyset$.*

(ii) *Let $\{M_n\}$ be a sequence of maximal monotone mappings on H_1 which is graph convergent to a mapping M defined on H_1. If B is a Lipschitz maximal monotone mapping on H_1, then $\{B + M_n\}$ is graph convergent to $B + M$ and $B + M$ is maximal monotone.*

3 Main results

In this section, we state and prove our main results of the paper. First we will study the weak convergence theorem.

Theorem 3.1 *Let H_1 and H_2 be two Hilbert spaces. Let C and D be nonempty closed and convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Suppose $F_1 : C \times C \to \mathbb{R}$ and $F_2 : D \times D \to \mathbb{R}$ are two bifunctions which satisfy Assumption A and F_2 is upper semicontinuous. Let $g_1 : C \to H_1$ and $g_2 : D \to H_2$ be η_1- and η_2-inverse strongly monotone mappings, respectively. Let $S : C \to C$ be a nonexpansive self-mapping and $\{T_i\}_{i=1}^N : C \to H_1$ be k_i-strictly pseudocontractive nonself-mappings. Assume that*

$$\mathcal{F} = \Gamma \cap \mathcal{S} \neq \emptyset.$$

Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in C, \\ u_n = (1 - \alpha_n)x_n + \alpha_n(\tau_n S x_n + (1 - \tau_n)T_N^n T_{N-1}^n \cdots T_1^n x_n), \\ x_{n+1} = U(u_n + \gamma_n A^*(V - I)Au_n), \quad n \geq 1, \end{cases} \tag{3.1}$$

where $U = T_{r_n}^{F_1}(I - r_n g_1)$, $V = T_{r_n}^{F_2}(I - r_n g_2)$, $T_i^n = (1 - \delta_n^i)I + \delta_n^i P_C(\beta_i I + (1 - \beta_i)T_i)$, $0 \leq k_i \leq \beta_i < 1$, $\delta_n^i \in (0, 1)$ for $i = 1, 2, \ldots, N$ and

$$\gamma_n = \frac{\sigma_n \|(T_{r_n}^{F_2}(I - r_n g_2) - I)Ax_n\|}{\|A^*(T_{r_n}^{F_2}(I - r_n g_2) - I)Ax_n\|}, \quad 0 < a \leq \sigma_n \leq b < 1.$$

Let $\{\alpha_n\}$, $\{\tau_n\}$ be two real sequences in $(0, 1)$ and $\{r_n\} \subset (0, \alpha)$, where $\alpha = 2\min\{\eta_1, \eta_2\}$. Suppose the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \tau_n < \infty$;

(ii) $\lim_{n\to\infty} \frac{\|x_n - u_n\|}{\alpha_n \tau_n} = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0$;

(iv) $\lim_{n\to\infty} |\delta_{n+1}^i - \delta_n^i| = 0$ *for $i = 1, 2, \ldots, N$.*

Then the sequence $\{x_n\}$ converges weakly to $x^* \in \mathcal{F}$

Proof Since our method easily deduce the general case, we only prove the theorem for $N = 2$. Since $g_1 : H_1 \to H_1$ is η_1-strongly monotone mapping, for any $x, y \in H_1$, we get

$$\begin{aligned}\|(I - r_n g_1)x - (I - r_n g_1)y\|^2 &= \|(x - y) - r_n(g_1 x - g_1 y)\|^2 \\ &\leq \|x - y\|^2 - r_n(2\eta_1 - r_n)\|g_1 x - g_1 y\|^2 \\ &\leq \|x - y\|^2.\end{aligned}$$

This means that $(I - r_n g_1)$ is nonexpansive. Similarly, we can show that $(I - r_n g_2)$ is a nonexpansive mapping. Thus, U and V are also nonexpansive mappings. Let $x^* \in \mathcal{F}$. From Lemma 2.2, Lemma 2.3 and Lemma 2.4, we get $x^* = T_2^n T_1^n x^*$. Hence, we have

$$\begin{aligned}\|u_n - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n(\tau_n S x_n + (1 - \tau_n)T_2^n T_1^n x_n) - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n[\tau_n\|S x_n - x^*\| + (1 - \tau_n)\|T_2^n T_1^n x_n - x^*\|] \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n[\tau_n\|x_n - x^*\| + (1 - \tau_n)\|x_n - x^*\|] \\ &\quad + \alpha_n\tau_n\|S x^* - x^*\| \\ &= \|x_n - x^*\| + \alpha_n\tau_n\|S x^* - x^*\|.\end{aligned} \tag{3.2}$$

Also, since $x^* \in \mathcal{F}$, we have $Ux^* = x^*$ and $VAx^* = Ax^*$.

Let $v_n = u_n + \gamma A^*(V - I)Au_n$. Then we have

$$\begin{aligned}\|v_n - x^*\|^2 &= \|u_n + \gamma_n A^*(V - I)Au_n - x^*\|^2 \\ &= \|u_n - x^*\|^2 + \gamma_n^2\|A^*(V - I)Au_n\|^2 \\ &\quad + 2\gamma_n\langle u_n - x^*, A^*(V - I)Au_n\rangle.\end{aligned} \tag{3.3}$$

Since $VAx^* = Ax^*$, we get

$$\begin{aligned}&\langle u_n - x^*, A^*(V - I)Au_n\rangle \\ &\quad = \langle Au_n - Ax^*, (V - I)Au_n\rangle \\ &\quad = \langle Au_n - Ax^* + (V - I)Au_n - (V - I)Au_n, (V - I)Au_n\rangle \\ &\quad = \langle VAu_n - Ax^*, (V - I)Au_n\rangle - \|(V - I)Au_n\|^2 \\ &\quad = \frac{1}{2}[\|VAu_n - Ax^*\|^2 + \|(V - I)Au_n\|^2 - \|Au_n - Ax^*\|^2] \\ &\qquad - \|(V - I)Au_n\|^2 \\ &\quad \leq \frac{1}{2}[\|Au_n - Ax^*\|^2 - \|Au_n - Ax^*\|^2] - \frac{1}{2}\|(V - I)Au_n\|^2 \\ &\quad = -\frac{1}{2}\|(V - I)Au_n\|^2.\end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\|v_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \gamma_n(\|(V - I)Au_n\|^2 - \gamma_n\|A^*(V - I)Au_n\|^2). \tag{3.5}$$

Now

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|^2 &= \left\|U\left(u_n+\gamma_n A^*(V-I)Au_n\right)-x^*\right\|^2\\ &\le \left\|u_n+\gamma_n A^*(V-I)Au_n-x^*\right\|^2\\ &= \left\|u_n-x^*\right\|^2-\gamma_n\left(\left\|(V-I)Au_n\right\|^2-\gamma_n\left\|A^*(V-I)Au_n\right\|^2\right).\end{aligned}\tag{3.6}$$

From (3.2), (3.3) and (3.4), we get

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|^2 &\le \left(\left\|x_n-x^*\right\|+\alpha_n\tau_n\left\|Sx^*-x^*\right\|\right)^2\\ &\quad -\gamma_n\left(\left\|(V-I)Au_n\right\|^2-\gamma_n\left\|A^*(V-I)Au_n\right\|^2\right).\end{aligned}\tag{3.7}$$

Now, using the definition of γ_n, we get

$$\left\|x_{n+1}-x^*\right\| \le \left\|x_n-x^*\right\|+\alpha_n\tau_n\left\|Sx^*-x^*\right\|.\tag{3.8}$$

Since $\sum_{n=0}^{\infty}\tau_n<\infty$, we have $\sum_{n=0}^{\infty}\alpha_n\tau_n<\infty$. Thus, by using Lemma 2.7 to (3.8), we conclude that $\lim_{n\to\infty}\|x_n-x^*\|$ exists and the value is finite. Hence, $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{v_n\}$.

Now, from (3.7), we get

$$\begin{aligned}&\gamma_n\left(\left\|(V-I)Au_n\right\|^2-\gamma_n\left\|A^*(V-I)Au_n\right\|^2\right)\\ &\quad\le \left(\left\|x_n-x^*\right\|+\alpha_n\tau_n\left\|Sx^*-x^*\right\|\right)^2-\left\|x_{n+1}-x^*\right\|^2\\ &\quad= \left\|x_n-x^*\right\|^2-\left\|x_{n+1}-x^*\right\|^2+\alpha_n^2\tau_n^2\left\|Sx^*-x^*\right\|^2\\ &\qquad+2\alpha_n\tau_n\left\|x_n-x^*\right\|\left\|Sx^*-x^*\right\|.\end{aligned}$$

Since $\lim_{n\to\infty}\tau_n=0$, we get

$$\gamma_n\left(\left\|(V-I)Au_n\right\|^2-\gamma_n\left\|A^*(V-I)Au_n\right\|^2\right)\to 0\quad\text{as } n\to\infty,$$

which by definition of γ_n, implies that

$$\frac{\sigma_n(1-\sigma_n)\|(V-I)Au_n\|^4}{\|A^*(V-I)Au_n\|^2}\to 0\quad\text{as } n\to\infty.$$

Since $0<a\le\sigma_n\le b<1$ and $\|A^*(V-I)Au_n\|$ is bounded, we get

$$\left\|(V-I)Au_n\right\|\to 0\quad\text{as } n\to\infty.$$

Now

$$\left\|A^*(V-I)Au_n\right\|=\left\|A^*\right\|\left\|(V-I)Au_n\right\|\to 0\quad\text{as } n\to\infty.\tag{3.9}$$

So,

$$\|u_n-v_n\|=\left\|A^*(V-I)Au_n\right\|\to 0\quad\text{as } n\to\infty.\tag{3.10}$$

Now, we estimate

$$\begin{aligned}\|x_{n+1}-x_n\| &= \big\|x_{n+1}-x^*-\big(x_n-x^*\big)\big\| \\ &= \big\|x_{n+1}-x^*\big\|^2-\big\|x_n-x^*\big\|^2-2\big\langle x_{n+1}-x_n,x_n-x^*\big\rangle \\ &= \big\|x_{n+1}-x^*\big\|^2-\big\|x_n-x^*\big\|^2-2\big\langle x_{n+1}-p,x_n-x^*\big\rangle \\ &\quad +2\big\langle x_n-p,x_n-x^*\big\rangle,\end{aligned}$$

where p is a weak limit point of $\{x_n\}$. Since $\lim_{n\to\infty}\|x_n-x^*\|$ exists, we get

$$\|x_{n+1}-x_n\|\to 0 \quad \text{as } n\to\infty. \tag{3.11}$$

Since $\liminf_{n\to\infty} r_n>0$, there exists a number $r>0$ such that $r_n>r$. Hence, we get

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^2 &= \big\|Uv_n-Ux^*\big\|^2 \\ &= \big\|T_{r_n}^{F_1}(I-r_ng_1)v_n-T_{r_n}^{F_1}(I-r_ng_1)x^*\big\|^2 \\ &\le \big\|(I-r_ng_1)v_n-(I-r_ng_1)x^*\big\|^2 \\ &\le \big\|v_n-x^*\big\|^2-r_n(2\eta_1-r_n)\big\|g_1v_n-g_1x^*\big\|^2 \\ &\le \big\|v_n-x^*\big\|^2-r(2\eta_1-\alpha)\big\|g_1v_n-g_1x^*\big\|^2,\end{aligned}$$

that is,

$$\begin{aligned}r(2\eta_1-\alpha)\big\|g_1v_n-g_1x^*\big\|^2 &\le \big\|v_n-x^*\big\|^2-\big\|x_{n+1}-x^*\big\|^2 \\ &\le \big\|u_n-x^*\big\|^2-\big\|x_{n+1}-x^*\big\|^2 \\ &\le \big\|x_n-x^*\big\|^2-\big\|x_{n+1}-x^*\big\|^2+\alpha_n^2\tau_n^2\big\|Sx^*-x^*\big\|^2 \\ &\quad +2\alpha_n\tau_n\big\|x_n-x^*\big\|\big\|Sx^*-x^*\big\| \\ &\le \|x_n-x_{n+1}\|\big(\big\|x_n-x^*\big\|+\big\|x_{n+1}-x^*\big\|\big) \\ &\quad +\alpha_n^2\tau_n^2\big\|Sx^*-x^*\big\|^2+2\alpha_n\tau_n\big\|x_n-x^*\big\|\big\|Sx^*-x^*\big\|.\end{aligned}$$

Since $\lim_{n\to\infty}\tau_n=0$, $\|x_n-x^*\|$ is bounded and $r(2\eta_1-\alpha)>0$, we get

$$\lim_{n\to\infty}\big\|g_1v_n-g_1x^*\big\|=0. \tag{3.12}$$

Now, from the firmly nonexpansivity of $T_{r_n}^{F_1}$, we get

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^2 &\le \big\langle (I-r_ng_1)v_n-(I-r_ng_1)x^*,x_{n+1}-x^*\big\rangle \\ &= \frac{1}{2}\big[\big\|(I-r_ng_1)v_n-(I-r_ng_1)x^*\big\|^2+\big\|x_{n+1}-x^*\big\|^2 \\ &\quad -\big\|v_n-x_{n+1}-r_n\big(g_1v_n-g_1x^*\big)\big\|^2\big] \\ &\le \frac{1}{2}\big[\big\|v_n-x^*\big\|^2+\big\|x_{n+1}-x^*\big\|^2-\|v_n-x_{n+1}\|^2\end{aligned}$$

$$\begin{aligned}&+ 2r_n\langle v_n - x_{n+1}, g_1 v_n - g_1 x^*\rangle - r_n^2\|g_1 v_n - g_1 x^*\|^2\Big]\\ &\le \frac{1}{2}\Big[\|v_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 - \|v_n - x_{n+1}\|^2\\ &+ 2r_n\|v_n - x_{n+1}\|\|g_1 v_n - g_1 x^*\|\Big],\end{aligned}$$

this implies that

$$\|x_{n+1} - x^*\|^2 \le \|v_n - x^*\|^2 - \|v_n - x_{n+1}\|^2 + 2r_n\|v_n - x_{n+1}\|\|g_1 v_n - g_1 x^*\|.$$

That is,

$$\|v_n - x_{n+1}\|^2 \le \|v_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2r_n\|v_n - x_{n+1}\|\|g_1 v_n - g_1 x^*\|.$$

Hence, we have

$$\begin{aligned}\|v_n - x_{n+1}\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n^2\tau_n^2\|Sx^* - x^*\|^2\\ &\quad + 2\alpha_n\tau_n\|x_n - x^*\|\|Sx^* - x^*\| + 2r_n\|v_n - x_{n+1}\|\|g_1 v_n - g_1 x^*\|\\ &\le \|x_n - x_{n+1}\|\big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big) + \alpha_n^2\tau_n^2\|Sx^* - x^*\|^2\\ &\quad + 2\alpha_n\tau_n\|x_n - x^*\|\|Sx^* - x^*\| + 2r_n\|v_n - x_{n+1}\|\|g_1 v_n - g_1 x^*\|.\end{aligned}$$

Using (3.11) and (3.12), we get

$$\lim_{n\to\infty}\|v_n - x_{n+1}\| = 0. \tag{3.13}$$

Now

$$\|x_n - v_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - v_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Again, by using (3.10) and (3.14), we get

$$\|x_n - u_n\| \le \|x_n - v_n\| + \|v_n - u_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.15}$$

Now, we show that $p \in \mathcal{F}$. Since $T_2^n T_1^n$ is an averaged mapping, it is nonexpansive. Using the boundedness of $\{x_n\}$ and nonexpansivity of S, there exists a $K > 0$ such that $\|Sx_n - T_2^n T_1^n x_n\| \le K$ for all $n \ge 0$. Now, we know that

$$\begin{aligned}\|u_n - T_2^n T_1^n x_n\| &= \big\|(1-\alpha_n)x_n + \alpha_n\big(\tau_n Sx_n + (1-\tau_n)T_2^n T_1^n x_n\big) - T_2^n T_1^n x_n\big\|\\ &\le (1-\alpha_n)\|x_n - T_2^n T_1^n x_n\| + \alpha_n\tau_n\|Sx_n - T_2^n T_1^n x_n\|\\ &\le (1-\alpha_n)\|x_n - u_n\| + (1-\alpha_n)\|u_n - T_2^n T_1^n x_n\|\\ &\quad + \alpha_n\tau_n\|Sx_n - T_2^n T_1^n x_n\|,\end{aligned}$$

this implies that

$$\begin{aligned}\alpha_n\|u_n - T_2^n T_1^n x_n\| &\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\tau_n K\\ &\le \|x_n - u_n\| + \alpha_n\tau_n K.\end{aligned}$$

Hence, we have

$$\left\| u_n - T_2^n T_1^n x_n \right\| \le \frac{\|x_n - u_n\|}{\alpha_n} + \tau_n K. \tag{3.16}$$

It follows from the conditions (i)–(ii) that

$$\lim_{n\to\infty} \frac{\|x_n - u_n\|}{\alpha_n} = \lim_{n\to\infty} \tau_n \frac{\|x_n - u_n\|}{\tau_n \alpha_n} = 0.$$

Hence from (3.16), we get

$$\lim_{n\to\infty} \left\| u_n - T_2^n T_1^n x_n \right\| = 0.$$

Now, by using (3.15), we get

$$\left\| x_n - T_2^n T_1^n x_n \right\| \le \|x_n - u_n\| + \left\| u_n - T_2^n T_1^n x_n \right\| \to 0 \quad \text{as } n \to \infty. \tag{3.17}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p$ as $j \to \infty$. Noticing that $\{\delta_n^i\}$ is bounded for $i = 1, 2$, we can assume that $\delta_{n_j}^i \to \delta_\infty^i$ as $j \to \infty$, where $0 < \delta_\infty^i < 1$ for $i = 1, 2$. Define, for $i = 1, 2$,

$$T_i^\infty = \left(1 - \delta_\infty^i\right) I + \delta_\infty^i P_C\left(\beta_i I + (1 - \beta_i) T_i\right).$$

Now, by Lemma 2.3 and Lemma 2.4, $\mathrm{Fix}(P_C(\beta_i I + (1 - \beta_i)T_i)) = \mathrm{Fix}(T_i)$. Again, since $P_C(\beta_i I + (1 - \beta_i)T_i)$ is a nonexpansive mapping, T_i^∞ is averaged and $\mathrm{Fix}(T_i^\infty) = \mathrm{Fix}(T_i)$ for $i = 1, 2$.

Furthermore, since

$$\mathrm{Fix}\left(T_1^\infty\right) \cap \mathrm{Fix}\left(T_2^\infty\right) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \mathrm{Fix}(\mathcal{S}) \neq \emptyset,$$

by Lemma 2.2, we get

$$\mathrm{Fix}\left(T_2^\infty T_1^\infty\right) = \mathrm{Fix}\left(T_1^\infty\right) \cap \mathrm{Fix}\left(T_2^\infty\right) = \mathrm{Fix}(\mathcal{S}).$$

Note that

$$\left\| T_i^{n_j} t - T_i^\infty t \right\| \le \left| \delta_{n_j}^i - \delta_\infty^i \right| \left(\|t\| + \left\| P_C\left(\beta_i t + (1 - \beta_i) T_i(t)\right) \right\| \right).$$

Hence, we get

$$\lim_{j\to\infty} \sup_{t\in B} \left\| T_i^{n_j} t - T_i^\infty t \right\| = 0, \tag{3.18}$$

where B is an arbitrary bounded subset of H_1. Also, we have

$$\begin{aligned}\left\| x_{n_j} - T_2^\infty T_1^\infty x_{n_j} \right\| \le{}& \left\| x_{n_j} - T_2^{n_j} T_1^{n_j} x_{n_j} \right\| + \left\| T_2^{n_j} T_1^{n_j} x_{n_j} - T_2^\infty T_1^{n_j} x_{n_j} \right\| \\ &+ \left\| T_2^\infty T_1^{n_j} x_{n_j} - T_2^\infty T_1^\infty x_{n_j} \right\|\end{aligned}$$

$$\begin{aligned}
&\leq \left\| x_{n_j} - T_2^{n_j} T_1^{n_j} x_{n_j} \right\| + \left\| T_2^{n_j} T_1^{n_j} x_{n_j} - T_2^{\infty} T_1^{n_j} x_{n_j} \right\| \\
&\quad + \left\| T_1^{n_j} x_{n_j} - T_1^{\infty} x_{n_j} \right\| \\
&\leq \left\| x_{n_j} - T_2^{n_j} T_1^{n_j} x_{n_j} \right\| + \sup_{t \in B_1} \left\| T_2^{n_j} t - T_2^{\infty} t \right\| \\
&\quad + \sup_{t \in B_2} \left\| T_1^{n_j} t - T_1^{\infty} t \right\|,
\end{aligned} \tag{3.19}$$

where B_1 is a bounded subset including $\{T_1^{n_j} x_{n_j}\}$ and B_2 is a bounded subset including $\{x_{n_j}\}$. It follows from (3.17), (3.18) and (3.19) that

$$\lim_{j \to \infty} \left\| x_{n_j} - T_2^{\infty} T_1^{\infty} x_{n_j} \right\| = 0.$$

Hence, from Lemma 2.5, we get $p \in \mathrm{Fix}(T_2^{\infty} T_1^{\infty}) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$.

Now, we show that $x^* \in \mathcal{S}$. It follows from (3.1) that

$$u_n - x_n = \alpha_n \bigl(\tau_n (S x_n - x_n) + (1 - \tau_n) \bigl(T_2^n T_1^n x_n - x_n \bigr) \bigr),$$

and hence,

$$\frac{1}{\alpha_n \tau_n} (x_n - u_n) = \left((I - S) x_n + \left(\frac{1 - \tau_n}{\tau_n} \right) \bigl(I - T_2^n T_1^n \bigr) x_n \right). \tag{3.20}$$

By using Lemma 2.9(i), it follows that the operator sequence $\{(\frac{1-\tau_n}{\tau_n})(I - T_2^n T_1^n)\}$ is graph convergent to $N_{\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)}$, and hence, from Lemma 2.9(ii), it follows that the operator sequence $\{(I - S) + (\frac{1-\tau_n}{\tau_n})(I - T_2^n T_1^n)\}$ is graph convergent to $(I - S) + N_{\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)}$. Now, by replacing n by n_j and letting the limit in (3.20) and considering the fact that $\lim_{n \to \infty} \frac{1}{\alpha_n \tau_n} \| x_n - u_n \| = 0$ and the graph of $(I - S) + N_{\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)}$ is weakly–strongly closed, we get

$$0 \in (I - S) p + N_{\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)} p,$$

that is, $p \in \mathcal{S}$.

We now show that $p \in \Gamma$. Since $u_n = U(v_n) = T_{r_n}^{F_1} (I - r_n g_1)(v_n)$, we have

$$F_1(x_{n+1}, y) + \langle g_1 v_n, y - x_{n+1} \rangle + \frac{1}{r_n} \langle y - x_{n+1}, x_{n+1} - v_n \rangle \geq 0, \quad \forall y \in C.$$

From the monotonicity of F_1, we get

$$\langle g_1 v_n, y - x_{n+1} \rangle + \frac{1}{r_n} \langle y - x_{n+1}, x_{n+1} - v_n \rangle \geq F_1(y, x_{n+1}), \quad \forall y \in C.$$

Replacing n with n_j in the above inequality, we get

$$\langle g_1 v_{n_j}, y - x_{n_j+1} \rangle + \frac{1}{r_{n_j}} \langle y - x_{n_j+1}, x_{n_j+1} - v_{n_j} \rangle \geq F_1(y, x_{n_j+1}), \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Since $y \in C$ and $p \in C$, we get $y_t \in C$ and hence $F_1(y_t, p) \leq 0$. So, from the above inequality, we get

$$\begin{aligned}
\langle y_t - x_{n+1}, g_1 y_t\rangle &\geq \langle y_t - x_{n+1}, g_1 y_t\rangle - \langle y_t - x_{n+1}, g_1 v_t\rangle \\
&\quad - \frac{1}{r_{n_j}}\left\langle y_t - x_{n_j+1}, \frac{x_{n_j+1} - v_{n_j}}{r_{n_j}}\right\rangle + F_1(y_t, x_{n_j+1}) \\
&= \langle y_t - x_{n+1}, g_1 y_t - g_1 x_{n_j+1}\rangle + \langle y_t - x_{n_j+1}, g_1 x_{n_j+1} - g_1 v_t\rangle \\
&\quad - \frac{1}{r_{n_j}}\left\langle y_t - x_{n_j+1}, \frac{x_{n_j+1} - v_{n_j}}{r_{n_j}}\right\rangle + F_1(y_t, x_{n_j+1}).
\end{aligned}$$

Since $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ have the same asymptotic behavior and $x_{n_j} \rightharpoonup p$, there exist subsequences $\{u_{n_j}\}$ of $\{u_n\}$ and $\{v_{n_j}\}$ of $\{v_n\}$ such that $u_{n_j} \rightharpoonup p$ and $v_{n_j} \rightharpoonup p$. Since $\lim_{j\to\infty}\|x_{n_j+1} - v_{n_j}\| = 0$ and f_1 is Lipschitz continuous, we have $\lim_{j\to\infty}\|g_1 x_{n_j+1} - g_1 v_{n_j}\| = 0$. Since $\liminf_{n\to\infty} r_n > 0$, there exists a number $r > 0$ such that $\liminf_{n\to\infty} r_n = r$. Hence, we have

$$\begin{aligned}
\lim_{j\to\infty}\frac{\|x_{n_j+1} - v_{n_j}\|}{r_{n_j}} &\leq \frac{\lim_{j\to\infty}\|x_{n_j+1} - v_{n_j}\|}{\liminf_{n\to\infty} r_{n_j}} \\
&= \frac{1}{r}\lim_{j\to\infty}\|x_{n_j+1} - v_{n_j}\| \\
&= 0.
\end{aligned}$$

Furthermore, from the monotonicity of g_1 and lower semicontinuity of F_1, we have

$$\langle y_t - p, g_1 y_t\rangle \geq F_1(y_t, p),$$

as $j \to \infty$. And also, from the convexity of F_1, we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) \\
&\leq tF_1(y_t, y) + (1-t)\langle y_t - p, g_1 y_t\rangle \\
&\leq tF_1(y_t, y) + (1-t)t\langle y - p, g_1 y_t\rangle.
\end{aligned}$$

Hence, $0 \leq F_1(y_t, y) + (1-t)\langle y - p, g_1 y_t\rangle$. Letting $t \to 0_+$, for each $y \in C$, we have

$$F_1(p, y) + \langle y - p, g_1 p\rangle \geq 0.$$

This implies that $p \in \mathrm{Sol}(\mathrm{MEP}\,(1.9))$.

Now, we need to show that $Ap \in \mathrm{Sol}(\mathrm{MEP}\,(1.10))$. Since A is bounded linear operator, we have $Ax_{n_j} \rightharpoonup Ap$. Now, setting $d_{n_j} = Au_{n_j} - VAu_{n_j}$, we get $d_{n_j} \to 0$ and $Au_{n_j} - d_{n_j} = VAu_{n_j}$. Therefore, from Lemma 2.6, we have

$$\begin{aligned}
&F_2(Au_{n_j} - d_{n_j}, z) + \bigl\langle g_2 Au_{n_j}, z - (Au_{n_j} - d_{n_j})\bigr\rangle \\
&\quad + \frac{1}{r_{n_j}}\bigl\langle z - (Au_{n_j} - d_{n_j}), Au_{n_j} - d_{n_j} - Au_{n_j}\bigr\rangle \geq 0, \quad \forall z \in D.
\end{aligned}$$

Since F_2 is upper semicontinuous in the first argument, taking lim sup to the above inequality as $j \to \infty$ and using $\liminf_{n\to\infty} r_n > 0$, we get

$$F_2(Ap, z) + \langle z - Ap, g_2 Ap \rangle \geq 0, \quad \forall z \in D,$$

which implies that $Ap \in \mathrm{Sol}(\mathrm{MEP}\,(1.10))$. This shows that $p \in \Gamma$ and thus $p \in \mathcal{F}$. It follows from the existence of $\lim_{n\to\infty} \|x_n - p\|$ and the fact that Hilbert space satisfies Opial's conditions, the sequence $\{x_n\}$ has only one weak limit point, and hence $\{x_n\}$ converges weakly to $p \in \mathcal{F}$. □

The following consequence is a weak convergence theorem for computing a common solution of a mixed equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 3.2 *Let H_1 be a Hilbert spaces. Let C be a nonempty closed and convex subset of H_1. Suppose $F_1 : C \times C \to \mathbb{R}$ is a bifunction which satisfy Assumption A. Let $g_1 : C \to H_1$ be η_1-inverse strongly monotone mapping. Let $S : C \to C$ be a nonexpansive self-mapping and $\{T_i\}_{i=1}^N : C \to H_1$ be k_i-strictly pseudocontractive nonself-mappings. Assume that $\mathcal{F} = \mathrm{MEP}(F_1, g_1) \cap \mathcal{S} \neq \emptyset$. Define a sequence $\{x_n\}$ as follows*:

$$\begin{cases} x_0 \in C, \\ u_n = (1-\alpha_n)x_n + \alpha_n(\tau_n S x_n + (1-\tau_n)T_N^n T_{N-1}^n \cdots T_1^n x_n), \\ x_{n+1} = T_{r_n}^{F_1}(I - r_n g_1)(u_n), \quad n \geq 1, \end{cases} \tag{3.21}$$

where $T_i^n = (1-\delta_n^i)I + \delta_n^i P_C(\beta_i I + (1-\beta_i)T_i)$, $0 \leq k_i \leq \beta_i < 1$, $\delta_n^i \in (0,1)$ for $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}$, $\{\tau_n\}$ be two real sequences in $(0,1)$ and $\{r_n\} \subset (0, 2\eta_1)$. Suppose the following conditions (i)–(iv) *of Theorem 3.1 are satisfied. Then the sequence $\{x_n\}$ converges weakly to $x^* \in \mathcal{F}$*

Proof Taking $A = 0$ in Theorem 3.1, the conclusion of Corollary 3.2 is followed. □

In the above theorem, the sequence generated by algorithm (3.1) converges weakly to a common solution of a split mixed equilibrium problem and a hierarchical fixed point problem.

In this section, our other aim is to prove a strong convergence theorem for a hierarchical fixed point problem and split equilibrium problem for some special cases.

We consider the following hierarchical fixed point and split equilibrium problem: Find $x^* \in \Gamma \cap \mathcal{S}$ such that

$$\langle x^* - Sx^*, x^* - y \rangle \leq 0, \quad \forall y \in \Gamma \cap \mathcal{S}, \tag{3.22}$$

where $\mathcal{S} = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ and Γ is the solution set of the split equilibrium problem (1.5)–(1.6). When S is a contraction mapping, a special case of nonexpansive mappings, we prove a strong convergence theorem for the above problem. We need the following results for our study.

Lemma 3.3 ([8]) *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption A and $T_r^{F_1}$ be defined as in Lemma 2.6. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\left\| T_{r_2}^{F_1} y - T_{r_1}^{F_1} x \right\| \le \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \left\| T_{r_2}^{F_1} y - y \right\|.$$

Lemma 3.4 ([26]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X. Let $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the condition $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integer $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Lemma 3.5 ([31]) *Let $\{\alpha_n\}$ be a sequence of non-negative real numbers such that*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Now we are in a position to state our second main result for strong convergence.

Theorem 3.6 *Let H_1 and H_2 be two Hilbert spaces. Let C and D be nonempty closed and convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : D \times D \to \mathbb{R}$ be two bifunctions satisfying Assumption A and F_2 be upper semicontinuous. Let $S : C \to C$ be a contraction mapping with coefficient $\rho \ge 0$ and $\{T_i\}_{i=1}^N : C \to H_1$ be k_i-strictly pseudocontractive nonself-mappings. Assume that the solution set of problem (3.22) is nonempty. Define a sequence $\{x_n\}$ as follows:*

$$\begin{cases} x_0 \in C, \\ u_n = U(x_n + \gamma A^*(V - I)Ax_n), \quad n \ge 1, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\tau_n S x_n + (1-\tau_n)T_N^n T_{N-1}^n \cdots T_1^n u_n), \end{cases} \tag{3.23}$$

where $U = T_{r_n}^{F_1}$, $V = T_{r_n}^{F_2}$, $T_i^n = (1-\delta_n^i)I + \delta_n^i P_C(\beta_i I + (1-\beta_i)T_i)$, $0 \le k_i \le \beta_i < 1$, and $\delta_n^i \in (0,1)$ for $i = 1, 2, \ldots, N$. Also let $\gamma \in (0, \frac{1}{L})$ where L is the spectral radius of the operator A^*A and A^* is the adjoint operator of A. Let $\{\alpha_n\}$ and $\{\tau_n\}$ be two real sequences in $(0,1)$. Suppose the following conditions are satisfied:

(i) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(ii) $0 \le \alpha_n \le b < 1$ *for some* $b \in (0,1)$ *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty} \tau_n = 0$ *and* $\sum_{n=0}^{\infty} \tau_n = \infty$;

(iv) $\lim_{n\to\infty} |\delta_{n+1}^i - \delta_n^i| = 0$ *for* $i = 1, 2, \ldots, N$.

Then the sequence $\{x_n\}$ converges strongly to $p \in \Gamma \cap \mathcal{S}$, which is the unique solution of the following variational inequality:

$$\langle p - Sp, p - y \rangle \le 0, \quad \forall y \in \Gamma \cap \mathcal{S}. \tag{3.24}$$

Proof We will prove the theorem for $N = 2$. Then our method can be easily extended to the general case. First we show that the sequence $\{x_n\}$ is bounded.

Let x^* be a solution of problem (3.22). Then $T_{r_n}^{F_1}x^* = x^*$ and $T_{r_n}^{F_2}Ax^* = Ax^*$. So, by (3.23) we get

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - x^*\|^2 \\ &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}x^*\|^2 \\ &\le \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\| \\ &\le \|x_n - x^*\|^2 + \gamma^2\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 \\ &+ 2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle. \end{aligned} \tag{3.25}$$

Then

$$\begin{aligned}\|u_n - x^*\|^2 \le{}& \|x_n - x^*\|^2 + \gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, A^*A(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &+ 2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle. \end{aligned} \tag{3.26}$$

Now, we have

$$\begin{aligned}\gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, A^*A(T_{r_n}^{F_2} - I)Ax_n\rangle &\le L\gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= L\gamma^2\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned} \tag{3.27}$$

Denoting $\Lambda := 2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle$ and using (2.3), we have

$$\begin{aligned}\Lambda &= 2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - x^*), (T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n - x^*) + (T_{r_n}^{F_2} - I)Ax_n - (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\ &= 2\gamma\{\langle (T_{r_n}^{F_2} - I)Ax^*, (T_{r_n}^{F_2} - I)Ax_n\rangle - \|(T_{r_n}^{F_2} - I)Ax_n\|^2\} \\ &= 2\gamma\left\{\frac{1}{2}\|(T_{r_n}^{F_2} - I)Ax_n\|^2 - \|(T_{r_n}^{F_2} - I)Ax_n\|^2\right\} \\ &= -\gamma\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned} \tag{3.28}$$

Using (3.26), (3.27) and (3.28), we get

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 + \gamma(L\gamma - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \tag{3.29}$$

From the definition of γ, we obtain

$$\|u_n - x^*\| \le \|x_n - x^*\|. \tag{3.30}$$

Now

$$\begin{aligned}&\|x_{n+1} - x^*\| \\ &\quad = \|(1 - \alpha_n)(x_n - x^*) + \alpha_n[\tau_n(Sx_n - x^*) + (1 - \tau_n)(T_2^nT_1^nu_n - x^*)]\| \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\bigl[\tau_n\|Sx_n - x^*\| + (1-\tau_n)\|T_2^n T_1^n u_n - x^*\|\bigr] \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\bigl[\tau_n\|Sx_n - Sx^*\| + (1-\tau_n)\|u_n - x^*\|\bigr] \\
&\quad + \alpha_n\tau_n\|Sx^* - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\bigl[\tau_n\rho\|x_n - x^*\| + (1-\tau_n)\|x_n - x^*\|\bigr] \\
&\quad + \alpha_n\tau_n\|Sx^* - x^*\| \\
&\le \bigl(1-(1-\rho)\alpha_n\tau_n\bigr)\|x_n - x^*\| + \alpha_n\tau_n\|Sx^* - x^*\| \\
&\le \max\left\{\|x_n - x^*\|, \frac{\|Sx^* - x^*\|}{(1-\rho)}\right\} \\
&\le \cdots \le \max\left\{\|x_0 - x^*\|, \frac{\|Sx^* - x^*\|}{(1-\rho)}\right\}.
\end{aligned}
\tag{3.31}
$$

Hence the sequence $\{x_n\}$ is bounded.

Now, we show that

$$
\lim_{x\to\infty}\|x_{n+1} - x_n\| = 0.
$$

Let us consider $y_n = \tau_n Sx_n + (1-\tau_n)T_2^n T_1^n u_n$. Then

$$
\begin{aligned}
\|y_{n+1} - y_n\| \le{}& \tau_{n+1}\|Sx_{n+1} - Sx_n\| + |\tau_{n+1} - \tau_n|\|Sx_n - T_2^n T_1^n u_n\| \\
&+ (1-\tau_{n+1})\|T_2^{n+1}T_1^{n+1}u_{n+1} - T_2^n T_1^n u_n\|.
\end{aligned}
\tag{3.32}
$$

In addition, we have

$$
\begin{aligned}
&\|T_2^{n+1}T_1^{n+1}u_{n+1} - T_2^n T_1^n u_n\| \\
&\quad\le \|T_2^{n+1}T_1^{n+1}u_{n+1} - T_2^n T_1^n u_{n+1}\| + \|T_2^n T_1^n u_{n+1} - T_2^n T_1^n u_n\| \\
&\quad\le \|T_2^{n+1}T_1^{n+1}u_{n+1} - T_2^{n+1}T_1^n u_{n+1}\| \\
&\qquad + \|T_2^{n+1}T_1^n u_{n+1} - T_2^n T_1^n u_{n+1}\| + \|u_{n+1} - u_n\| \\
&\quad\le \|T_1^{n+1}u_{n+1} - T_1^n u_{n+1}\| + \|T_2^{n+1}T_1^n u_{n+1} - T_2^n T_1^n u_{n+1}\| \\
&\qquad + \|u_{n+1} - u_n\|.
\end{aligned}
\tag{3.33}
$$

It follows from the definition T_i^n that

$$
\begin{aligned}
&\|T_1^{n+1}u_{n+1} - T_1^n u_{n+1}\| \\
&\quad= \|\bigl(1-\delta_{n+1}^1\bigr)u_{n+1} + \delta_{n+1}^1 P_C\bigl(\beta_1 I + (1-\beta_1)T_1\bigr)u_{n+1} \\
&\qquad - \bigl(1-\delta_n^1\bigr)u_{n+1} + \delta_n^1 P_C\bigl(\beta_1 I + (1-\beta_1)T_1\bigr)u_{n+1}\| \\
&\quad\le |\delta_{n+1}^1 - \delta_n^1|\bigl(\|u_{n+1}\| + \|P_C\bigl(\beta_1 I + (1-\beta_1)T_1\bigr)u_{n+1}\|\bigr).
\end{aligned}
$$

Since $\lim_{n\to+\infty}|\delta_{n+1}^1 - \delta_n^1| = 0$, and $\{u_n\}$, $\{P_C(\beta_1 I + (1-\beta_1)T_1)u_n\}$ are bounded, we get

$$
\lim_{n\to+\infty}\|T_1^{n+1}u_{n+1} - T_1^n u_{n+1}\| = 0.
\tag{3.34}
$$

Similarly, we have

$$\begin{aligned}
&\left\| T_2^{n+1} T_1^n u_{n+1} - T_2^n T_1^n u_{n+1} \right\| \\
&\quad \le \left|\delta_{n+1}^2 - \delta_n^2\right| \left(\left\| T_1^n u_{n+1} \right\| + \left\| P_C\left(\beta_2 I + (1-\beta_2) T_2\right) T_1^n u_{n+1} \right\| \right),
\end{aligned}$$

from which it follows that

$$\lim_{n\to+\infty} \left\| T_2^{n+1} T_1^{n+1} u_{n+1} - T_2^n T_1^n u_{n+1} \right\| = 0. \tag{3.35}$$

Since

$$u_n = T_{r_n}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right)$$

and

$$u_{n+1} = T_{r_{n+1}}^{F_1}\left(x_{n+1} + \gamma A^*\left(T_{r_{n+1}}^{F_2} - I\right) A x_{n+1}\right),$$

it follows from Lemma 3.3 that

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\quad = \left\| T_{r_{n+1}}^{F_1}\left(x_{n+1} + \gamma A^*\left(T_{r_{n+1}}^{F_2} - I\right) A x_{n+1}\right) - T_{r_n}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\| \\
&\quad \le \left\| T_{r_{n+1}}^{F_1}\left(x_{n+1} + \gamma A^*\left(T_{r_{n+1}}^{F_2} - I\right) A x_{n+1}\right) - T_{r_{n+1}}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\| \\
&\qquad + \left\| T_{r_{n+1}}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) - T_{r_n}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\| \\
&\quad \le \left\| \left(x_{n+1} + \gamma A^*\left(T_{r_{n+1}}^{F_2} - I\right) A x_{n+1}\right) - \left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\| \\
&\qquad + \left| \frac{r_{n+1} - r_n}{r_{n+1}} \right| \left\| T_{r_{n+1}}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) - \left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\| \\
&\quad \le \left\| x_{n+1} - x_n - \gamma A^* A (x_{n+1} - x_n) \right\| + \gamma \|A\| \left\| T_{r_{n+1}}^{F_2} A x_{n+1} - T_{r_n}^{F_2} A x_n \right\| + \delta_n \\
&\quad \le \left\{ \|x_{n+1} - x_n\|^2 - 2\gamma \|A x_{n+1} - A x_n\|^2 + \gamma^2 \|A\|^4 \|x_{n+1} - x_n\|^2 \right\}^{\frac{1}{2}} \\
&\qquad + \gamma \|A\| \left\{ \|A x_{n+1} - A x_n\| + \left| \frac{r_{n+1} - r_n}{r_{n+1}} \right| \left\| T_{r_{n+1}}^{F_2} A x_{n+1} - A x_n \right\| \right\} + \delta_n \\
&\quad \le \left(1 - 2\gamma \|A\|^2 + \gamma^2 \|A\|^4\right)^{\frac{1}{2}} \|x_{n+1} - x_n\| + \gamma \|A\|^2 \|x_{n+1} - x_n\| + \gamma \|A\| \sigma_n + \delta_n \\
&\quad = \left(1 - \gamma \|A\|^2\right) \|x_{n+1} - x_n\| + \gamma \|A\|^2 \|x_{n+1} - x_n\| + \gamma \|A\| \sigma_n + \delta_n \\
&\quad = \|x_{n+1} - x_n\| + \gamma \|A\| \sigma_n + \delta_n,
\end{aligned} \tag{3.36}$$

where

$$\sigma_n = \left| \frac{r_{n+1} - r_n}{r_{n+1}} \right| \left\| T_{r_{n+1}}^{F_2} A x_{n+1} - A x_n \right\|$$

and

$$\delta_n = \left| \frac{r_{n+1} - r_n}{r_{n+1}} \right| \left\| T_{r_{n+1}}^{F_1}\left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) - \left(x_n + \gamma A^*\left(T_{r_n}^{F_2} - I\right) A x_n\right) \right\|.$$

Hence, using (3.34), (3.35) and (3.36) to (3.32) with the conditions (i) and (ii), we get

$$\limsup_{n\to+\infty}\big(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\big)\le 0.$$

Thus by Lemma 3.4, we conclude that $\lim_{n\to+\infty}\|y_n-x_n\|=0$, which implies that

$$\lim_{n\to+\infty}\|x_{n+1}-x_n\|=0.$$

Since $T_{r_n}^{F_1}x^*=x^*$ and $T_{r_n}^{F_1}$ is firmly nonexpansive, we get

$$\begin{aligned}
\|u_n-x^*\|^2 &= \big\|T_{r_n}^{F_1}\big(x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big)-x^*\big\|^2\\
&= \big\|T_{r_n}^{F_1}\big(x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big)-T_{r_n}^{F_1}x^*\big\|^2\\
&\le \big\langle u_n-x^*, x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big\rangle\\
&= \frac{1}{2}\Big\{\big\|u_n-x^*\big\|^2+\big\|x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big\|^2\\
&\quad -\big\|\big(u_n-x^*\big)-\big[x_n+\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n-x^*\big]\big\|^2\Big\}\\
&= \frac{1}{2}\Big\{\big\|u_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2+\gamma(L\gamma-1)\big\|A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|^2\\
&\quad -\big\|u_n-x_n-\gamma A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|^2\Big\}\\
&= \frac{1}{2}\Big\{\big\|u_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2-\big[\|u_n-x_n\|^2\\
&\quad +\gamma^2\big\|A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|^2-2\gamma\big\langle u_n-x_n, A^*\big(T_{r_n}^{F_2}-I\big)Ax_n\big\rangle\big]\Big\}\\
&= \frac{1}{2}\Big\{\big\|u_n-x^*\big\|^2+\big\|x_n-x^*\big\|^2-\|u_n-x_n\|^2\\
&\quad +2\gamma\big\|A(u_n-x_n)\big\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|\Big\}.
\end{aligned}$$

Hence, we obtain

$$\big\|u_n-x^*\big\|^2\le\big\|x_n-x^*\big\|^2-\|u_n-x_n\|^2+2\gamma\big\|A(u_n-x_n)\big\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|. \tag{3.37}$$

Again,

$$\begin{aligned}
&\big\|x_{n+1}-x^*\big\|^2\\
&\quad= \big\|(1-\alpha_n)\big(x_n-x^*\big)+\alpha_n\big[\tau_n\big(Sx_n-x^*\big)+(1-\tau_n)\big(T_2^nT_1^nu_n-x^*\big)\big]\big\|^2\\
&\quad= \big\|(1-\alpha_n)\big(x_n-x^*\big)+\alpha_n\big(T_2^nT_1^nu_n-x^*\big)+\alpha_n\tau_n\big(Sx_n-T_2^nT_1^nu_n\big)\big\|^2\\
&\quad\le \big\|(1-\alpha_n)\big(x_n-x^*\big)+\alpha_n\big(T_2^nT_1^nu_n-x^*\big)\big\|^2\\
&\qquad +2\tau_n\big\langle Sx_n-T_2^nT_1^nu_n, x_{n+1}-x^*\big\rangle\\
&\quad\le (1-\alpha_n)\big\|x_n-x^*\big\|^2+\alpha_n\big\|T_2^nT_1^nu_n-x^*\big\|^2\\
&\qquad +2\tau_n\big\|Sx_n-T_2^nT_1^nu_n\big\|\big\|x_{n+1}-x^*\big\|
\end{aligned}$$

$$\begin{aligned} &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|u_n - x^*\|^2 \\ &\quad + 2\tau_n \|Sx_n - T_2^n T_1^n u_n\| \|x_{n+1} - x^*\|. \end{aligned} \tag{3.38}$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq{}& (1-\alpha_n)\|x_n - x^*\|^2 \\ &+ \alpha_n \big(\|x_n - x^*\|^2 + \gamma(L\gamma - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2\big) \\ &+ 2\tau_n \|Sx_n - T_2^n T_1^n u_n\| \|x_{n+1} - x^*\|, \end{aligned} \tag{3.39}$$

which gives

$$\begin{aligned} \alpha_n(1 - L\gamma)\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \leq{}& \|x_{n+1} - x_n\|\big(\|x_n - p\| + \|x_{n+1} - x^*\|\big) \\ &+ 2\tau_n \|Sx_n - T_2^n T_1^n u_n\| \|x_{n+1} - x^*\|. \end{aligned} \tag{3.40}$$

Using the condition (iii) in (3.40), we get

$$\lim_{n\to\infty} \|(T_{r_n}^{F_2} - I)Ax_n\| = 0. \tag{3.41}$$

Again,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq{}& (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \|u_n - x^*\|^2 \\ &+ 2\tau_n \|Sx_n - T_2^n T_1^n u_n\| \|x_{n+1} - x^*\|. \end{aligned} \tag{3.42}$$

So, using (3.37) we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq{}& (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\big(\|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\ &+ 2\gamma \|A(u_n - x_n)\| \|(T_{r_n}^{F_2} - I)Ax_n\|\big) \\ &+ 2\tau_n \|Sx_n - T_2^n T_1^n u_n\| \|x_{n+1} - x^*\|, \end{aligned} \tag{3.43}$$

which gives

$$\begin{aligned} \alpha_n \|u_n - x_n\|^2 \leq{}& \|x_{n+1} - x_n\|\big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big) \\ &+ 2\tau_n \|Sx_n - T_2^n T_1^n u_n\| \|x_{n+1} - x^*\| \\ &+ 2\gamma \|A(u_n - x_n)\| \|(T_{r_n}^{F_2} - I)Ax_n\|. \end{aligned} \tag{3.44}$$

Using the condition (iii) to (3.44), we get

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.45}$$

Using (3.45) we get

$$\|x_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \|x_n - u_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.46}$$

Now

$$
\begin{aligned}
\|T_2^nT_1^nu_n-u_n\| &\le \|T_2^nT_1^nu_n-y_n\| + \|y_n-u_n\| \\
&\le \tau_n\|T_2^nT_1^nu_n-Sx_n\| + \|y_n-u_n\| \\
&\le \tau_n\|x_n-Sx_n\| + \tau_n\|T_2^nT_1^nu_n-x_n\| + \|y_n-u_n\| \\
&\le \tau_n\|x_n-Sx_n\| + \tau_n\|T_2^nT_1^nu_n-u_n\| + \tau_n\|u_n-x_n\| \\
&\quad + \|y_n-u_n\|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|T_2^nT_1^nu_n-u_n\| &\le \frac{\tau_n}{1-\tau_n}\|x_n-Sx_n\| + \frac{\tau_n}{1-\tau_n}\|u_n-x_n\| \\
&\quad + \frac{1}{1-\tau_n}\|y_n-u_n\|. \qquad (3.47)
\end{aligned}
$$

Since $\alpha_n\|x_n-y_n\| = \|x_{n+1}-x_n\|$, $\|x_n-y_n\| \to 0$ as $n\to\infty$. So,

$$
\|y_n-u_n\| \le \|y_n-x_n\| + \|x_n-u_n\| \to 0 \quad \text{as } n\to\infty. \qquad (3.48)
$$

Hence,

$$
\|T_2^nT_1^nu_n-u_n\| \to 0 \quad \text{as } n\to\infty. \qquad (3.49)
$$

Now, we show that

$$
\limsup_{n\to\infty}\langle Sp-p, x_n-p\rangle \le 0,
$$

where p is the unique solution of the variational inequality (3.24). Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$ as $j\to\infty$ and

$$
\limsup_{n\to\infty}\langle Sp-p, x_n-p\rangle = \lim_{j\to\infty}\langle Sp-p, x_{n_j}-p\rangle.
$$

Since $\|x_n-u_n\| \to 0$ as $n\to\infty$, $u_{n_j} \rightharpoonup \bar{x}$. Now, following similar steps to Theorem 3.1, we can show that $\bar{x} \in \Gamma\cap\mathcal{S}$. Hence

$$
\lim_{j\to\infty}\langle Sp-p, x_{n_j}-p\rangle = \langle Sp-p, \bar{x}-p\rangle \le 0. \qquad (3.50)
$$

Finally, we show that $x_n \to p$ and $n\to\infty$. From (2.1) and (3.23), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \big\|(1-\alpha_n)(x_n-p) + \alpha_n\big[\tau_nSx_n + (1-\tau_n)T_2^nT_1^nu_n - p\big]\big\|^2 \\
&= \big\|(1-\alpha_n)(x_n-p) + \alpha_n(1-\tau_n)\big(T_2^nT_1^nu_n-p\big) + \alpha_n\tau_n(Sx_n-p)\big\|^2 \\
&\le \big\|(1-\alpha_n)(x_n-p) + \alpha_n(1-\tau_n)\big(T_2^nT_1^nu_n-p\big)\big\|^2 \\
&\quad + 2\alpha_n\tau_n\langle Sx_n-p, x_{n+1}-p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1-\tau_n)^2\left\|T_2^nT_1^nu_n-p\right\|^2\\
&\quad+2\alpha_n\tau_n\langle Sx_n-p,x_{n+1}-p\rangle\\
&\leq \bigl(1-\alpha_n+\alpha_n(1-\tau_n)^2\bigr)\|x_n-p\|^2+2\alpha_n\tau_n\langle Sx_n-Sp,x_{n+1}-p\rangle\\
&\quad+2\alpha_n\tau_n\langle Sp-p,x_{n+1}-p\rangle\\
&\leq \bigl(1-\alpha_n+\alpha_n(1-\tau_n)^2\bigr)\|x_n-p\|^2+2\alpha_n\tau_n\|Sx_n-Sp\|\|x_{n+1}-p\|\\
&\quad+2\alpha_n\tau_n\langle Sp-p,x_{n+1}-p\rangle\\
&\leq \bigl(1-\alpha_n+\alpha_n(1-\tau_n)^2\bigr)\|x_n-p\|^2\\
&\quad+\alpha_n\tau_n\bigl[\|Sx_n-Sp\|^2+\|x_{n+1}-p\|^2\bigr]+2\alpha_n\tau_n\langle Sp-p,x_{n+1}-p\rangle\\
&\leq \bigl(1-\alpha_n+\alpha_n(1-\tau_n)^2+\alpha_n\tau_n\rho\bigr)\|x_n-p\|^2+\alpha_n\tau_n\|x_{n+1}-p\|^2\\
&\quad+2\alpha_n\tau_n\langle Sp-p,x_{n+1}-p\rangle\\
&= \bigl(1-\alpha_n\tau_n-\alpha_n\tau_n(1-\rho-\tau_n)\bigr)\|x_n-p\|^2+\alpha_n\tau_n\|x_{n+1}-p\|^2\\
&\quad+2\alpha_n\tau_n\langle Sp-p,x_{n+1}-p\rangle.
\end{aligned}
$$

From the above inequality, it follows that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \left(1-\frac{\alpha_n\tau_n(1-\rho-\tau_n)}{1-\alpha_n\tau_n}\right)\|x_n-p\|^2\\
&\quad+2\frac{\alpha_n\tau_n}{1-\alpha_n\tau_n}\langle Sp-p,x_{n+1}-p\rangle.
\end{aligned}
\tag{3.51}
$$

Considering $a_n=\|x_n-p\|^2$, $b_n=\frac{\alpha_n\tau_n(1-\rho-\tau_n)}{1-\alpha_n\tau_n}$ and $c_n=2\frac{\alpha_n\tau_n}{1-\alpha_n\tau_n}\langle Sp-p,x_{n+1}-p\rangle$, we get $a_{n+1}\leq(1-b_n)a_n+c_n$. Hence, from Lemma 3.5, we conclude that $\{x_n\}$ converges strongly to p. □

The following consequence is a strong convergence theorem for computing a common solution of an equilibrium problem and a hierarchical fixed point problem in a real Hilbert space.

Corollary 3.7 *Let H_1 be a Hilbert spaces. Let C be nonempty closed and convex subset of H_1. Let $F_1: C\times C\to\mathbb{R}$ and be a bifunction satisfying Assumption A. Let $S: C\to C$ be a contraction mapping with coefficient $\rho\geq 0$ and $\{T_i\}_{i=1}^N: C\to H_1$ be k_i-strictly pseudocontractive nonself-mappings. Assume that $EP(F,C)\cap\mathcal{S}\neq\emptyset$. Define a sequence $\{x_n\}$ as follows*:

$$
\begin{cases}
x_0\in C,\\
u_n=T_{r_n}^{F_1}(x_n),\quad n\geq 1,\\
x_{n+1}=(1-\alpha_n)x_n+\alpha_n(\tau_nSx_n+(1-\tau_n)T_N^nT_{N-1}^n\cdots T_1^nu_n),
\end{cases}
\tag{3.52}
$$

where $T_i^n=(1-\delta_n^i)I+\delta_n^iP_C(\beta_iI+(1-\beta_i)T_i)$, $0\leq k_i\leq\beta_i<1$, and $\delta_n^i\in(0,1)$ for $i=1,2,\ldots,N$. Let $\{\alpha_n\}$ and $\{\tau_n\}$ be two real sequences in $(0,1)$. Suppose the following conditions (i)–(iv) *of Theorem 3.6 are satisfied. Then the sequence $\{x_n\}$ converges strongly to $p\in EP(F,C)\cap\mathcal{S}$, which is the unique solution of the following variational inequality*:

$$
\langle p-Sp,p-y\rangle\leq 0,\quad\forall y\in EP(F,C)\cap\mathcal{S}.
\tag{3.53}
$$

Proof Taking $A = 0$ in Theorem 3.6, the conclusion of Corollary 3.7 is followed. □

4 Conclusions

In this paper, we have introduced a modified Krasnoselski–Mann type iterative method for approximating a common solution of a split mixed equilibrium problem and a hierarchical fixed point problem of a finite collection of k-strictly pseudocontractive nonself-mappings.

Our main results improve and extend the corresponding results of Moudafi and Mainge [24], Moudafi [22] and Kazmi et al. [13] from single nonexpansive self-mapping to a finite collection of k_i-strictly pseudocontractive nonself-mappings. Also, we have studied our iterative algorithm by giving an explicit formula for selecting the step size so that the implementation of the proposed algorithm does not require any prior information of operator norm. We also have established strong convergence results for a special class of hierarchical fixed point and split mixed equilibrium problem.

In [5], Ceng and Petruşel have introduced a cyclic algorithm for HFPP of a finite collection of nonexpansive nonself-mappings in Banach spaces. Whether we can extend Theorems 3.1 and 3.6 to Banach spaces will be an issue of future research.

Funding
This work was supported by the Basic Science Research Program through the National Research Foundation (NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contribute equally. Both authors read and approved the final version of the manuscript.

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Publisher's Note

Received: 12 May 2020 Accepted: 17 September 2020 Published online: 29 September 2020

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