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Some unity results on entire functions and their difference operators related to 4 CM theorem

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Abstract

This paper is to consider the unity results on entire functions sharing two values with their difference operators and to prove some results related to 4 CM theorem. The main result reads as follows: Let $f(z)$ be a nonconstant entire function of finite order, and let a_1, a_2 be two distinct finite complex constants. If $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 "CM", then $f(z) \equiv \Delta_\eta^n f(z)$, and hence $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 CM.

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1 Introduction and main results

It is well known that a monic polynomial is uniquely determined by its zeros and a rational function by its zeros and poles ignoring a constant factor. But it becomes much more complicated to deal with the transcendental meromorphic function case. In 1929, Nevanlinna proved his famous 5 IM theorem and 4 CM theorem (see e.g. [20, 23]): if meromorphic functions $f(z)$ and $g(z)$ share five (respectively, four) distinct values in the extended complex plane IM (respectively, CM), then $f(z) \equiv g(z)$ ((respectively, $f(z) = T(g(z))$, where T is a Möbius transformation). Here and in what follows, we say that $f(z)$ and $g(z)$ share the finite value a CM(IM) if $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities (ignoring multiplicities), and we say that $f(z)$ and $g(z)$ share the ∞ CM(IM) if $f(z)$ and $g(z)$ have the same poles with the same multiplicities (ignoring multiplicities).

To relax those shared conditions in Nevanlinna's 4 CM theorem, Gundersen provided an example to show that 4 CM shared values cannot be replaced with 4 IM shared values, but with 3 CM shared values and 1 IM shared value in [5]. That is, "4 IM $\neq$ 4 CM" and "3 CM + 1 IM = 4 CM". In addition, he showed that "2 CM + 1 IM = 4 CM" in [6] (see correction in [8]), as well as by Mues in [17]. The problem that "1 CM + 3 IM = 4 CM" is still open. We recall the following result by Mues in [17], which mainly inspired us to write this paper.

Theorem A ([17]) *Let f and g be nonconstant meromorphic functions sharing four distinct values a_j $(j = 1, 2, 3, 4)$ "CM". If $f \not\equiv g$, then f and g share a_j $(j = 1, 2, 3, 4)$ CM.*

In Theorem A, f and g share the value a "CM" means that

$$\overline{N}\left(r,\frac{1}{f-a}\right)-N_E(r,a)=S(r,f),\quad \text{and}\quad \overline{N}\left(r,\frac{1}{g-a}\right)-N_E(r,a)=S(r,g),$$

where $N_E(r,a)$ is defined to be the reduced counting function of common zeros of $f(z)-a$ and $g(z)-a$ with the same multiplicities. Similarly, $N_E^{1)}(r,a)$ used later is defined to be the reduced counting function of common simple zeros of $f(z)-a$ and $g(z)-a$.

Applying Theorem A, one can get (see Theorem 4.8 in [23]) the following.

Theorem B ([23]) *Let f and g be nonconstant meromorphic functions and a_j $(j=1,2,3,4)$ be distinct values. If $f\not\equiv g$ share a_j $(j=1,2,3,4)$ IM and if $\overline{N}(r,\frac{1}{f-a_j})=S(r,f)$ $(j=1,2)$, then f and g share a_j $(j=1,2,3,4)$ CM.*

Remark 1.1 Let $\delta(a,f)$ denote the deficiency of a with respect to $f(z)$, which is defined as

$$\delta(a,f)=\varliminf_{r\to\infty}\frac{m(r,\frac{1}{f-a})}{T(r,f)}=1-\varlimsup_{r\to\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)}.$$

Then we see that the condition $\overline{N}(r,\frac{1}{f-a_j})=S(r,f)$ $(j=1,2)$ in Theorem B means that $\delta(a_j,f)=1$ $(j=1,2)$. And we say that a is a Nevanlinna exceptional value of $f(z)$, provided that $\delta(a,f)>0$.

To reduce the number of shared values, Rubel and Yang appear to be the first to consider the unity of the entire function sharing two values with its first derivative in [21]. They proved that, for a nonconstant entire function f, if f and f' share values a, b CM, then $f\equiv f'$. Mues and Steinmetz [18] improved Rubel and Yang's result by replacing "2 CM" with "2 IM" in 1979, and then by replacing "entire function" with "meromorphic function" in [19] (see also Gundersen [7]). In 2013, Li [15] improved these results by adding some condition on the poles of the meromorphic function f.

This paper is to consider replacing the "derivative" with "difference operator", which is defined as follows:

$$\Delta_\eta f(z)=f(z+\eta)-f(z)\quad \text{and}\quad \Delta_\eta^{n+1}f(z)=\Delta_\eta^n f(z+\eta)-\Delta_\eta^n f(z),\quad n\in\mathbb{N}^+,$$

where η is always a nonzero complex constant. This idea is partly due to the work by Heittokangas et al. in [12]. They were the first to consider a nonconstant meromorphic function $f(z)$ sharing values with its shift $f(z+\eta)$ and to prove the following.

Theorem C ([12]) *Let $f(z)$ be a meromorphic function of finite order, and let $\eta\in\mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share three distinct periodic functions $a_1,a_2,a_3\in\widehat{S}(f)$ with period η CM, then $f(z)=f(z+\eta)$ for all $z\in\mathbb{C}$.*

In Theorem C, $\widehat{S}(f)=S(f)\cup\{\infty\}$, where $S(f)$ is the set containing all meromorphic functions $a(z)$ satisfying

$$T(r,a)=S(r,f),\quad \text{as } r\to\infty, r\notin E,$$

where E is an exceptional set of finite logarithmic measure. Theorem C can be read as a "3 CM" theorem and it has been improved to "2 CM + 1 IM" by Heittokangas et al. [13]. The key theory used in their research consists of the difference counterparts of Nevanlinna theory of meromorphic functions (see e.g. [3, 10, 11]).

In 2013, Chen and Yi [2] proved the following Theorem D, which was then extended to Theorem E by Cui and Chen in [4], and to Theorem F by Zhang and Liao in [24].

Theorem D ([2]) *Let $f(z)$ be a transcendental meromorphic function such that its order $\rho(f)$ is not integer or infinite, and let η be a constant such that $f(z+\eta) \not\equiv f(z)$. If $f(z)$ and $\Delta_\eta f(z)$ share three distinct values a, b, ∞ CM, then $f(z) \equiv \Delta_\eta f(z)$.*

Theorem E ([4]) *Let $f(z)$ be a nonconstant meromorphic function of finite order, and let η be a nonzero finite complex constant. Let a, b be two distinct finite complex constants and n be a positive integer. If $f(z)$ and $\Delta_\eta^n f(z)$ share a, b, ∞ CM, then $f(z) \equiv \Delta_\eta^n f(z)$.*

Theorem F ([24]) *Let $f(z)$ be a nonconstant entire function of finite order and η be a nonzero finite complex constant. Let a, b be two distinct finite complex constants. If $f(z)$ and $\Delta_\eta f(z)$ share a, b CM, then $f(z) \equiv \Delta_\eta f(z)$.*

Remark 1.2 We will improve Theorems D–F by the following Theorem 1.1, whose proof is given with a different method from those in [2, 4, 24].

Theorem 1.1 *Let $f(z)$ be a nonconstant entire function of finite order, and let a_1, a_2 be two distinct finite complex constants. If $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 "CM", then $f(z) \equiv \Delta_\eta^n f(z)$, and hence $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 CM.*

Theorem 1.2 *Let $f(z)$ be a nonconstant entire function of finite order, and let a_1, a_2 be two distinct finite complex constants. If $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 IM, and*

$$\overline{N}\left(r, \frac{1}{f-a_1}\right) = S(r,f) \tag{1.1}$$

holds, then $f(z) \equiv \Delta_\eta^n f(z)$, and hence $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 CM.

As a continuation of Theorem B and Theorem 1.2, we prove the following.

Theorem 1.3 *Let $f(z)$ be a nonconstant entire function of finite order, and let a_1, a_2 be two distinct finite complex constants. If $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 IM, and there exists some constant $\lambda > \frac{1}{2}$ such that $\delta(a_1,f) + \delta(a_2,f) \geq \lambda$, then $f(z) \equiv \Delta_\eta^n f(z)$, and hence $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 CM.*

Other basic concepts and fundamental results of the Nevanlinna theory of meromorphic functions (see e.g. [14, 23]) may be used directly in what follows.

2 Lemmas

Now we recall two lemmas which are important in the proofs of our theorems. The first lemma has been used frequently in dealing with value sharing problems related to difference operators.

Lemma 2.1 ([11]) *Let $\eta \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then, for any small periodic function $a(z)$ with period η, with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_\eta^n f}{f-a}\right) = S(r,f),$$

where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.

Use the notation $\overline{N}_{k)}(r, \frac{1}{f-a})$ ($\overline{N}_{(k}(r, \frac{1}{f-a})$) to denote the counting function of the zeros of $f(z) - a$ in the disk $|z| \le r$, whose multiplicities $\le k$ ($\ge k$) and are counted once. Then we have the following.

Lemma 2.2 ([23]) *Let $f(z)$ be a nonconstant meromorphic function, a be an arbitrary complex number, and k be a positive integer. Then*
(i) $\overline{N}(r, \frac{1}{f-a}) \le \frac{k}{k+1}\overline{N}_{k)}(r, \frac{1}{f-a}) + \frac{1}{k+1}N(r, \frac{1}{f-a})$;
(ii) $\overline{N}(r, \frac{1}{f-a}) \le \frac{k}{k+1}\overline{N}_{k)}(r, \frac{1}{f-a}) + \frac{1}{k+1}T(r,f) + O(1)$.

Lemma 2.3 *Suppose that $a_1, a_2 \in \mathbb{C}$ satisfying $a_1 \neq a_2$, $f(z)$ is a nonconstant entire function of finite order sharing a_1 and a_2 "CM" with $\Delta_\eta^n f(z)$. If $f(z) \not\equiv \Delta_\eta^n f(z)$, then*

$$T(r,f) + S(r,f) = \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{f-a_j}\right) = \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_j}\right).$$

What is more, if $f(z) \not\equiv \Delta_\eta^n f(z)$ and (1.1) holds, then
(i) $T(r, \Delta_\eta^n f) = T(r,f) + S(r,f)$;
(ii) $\forall b \in \mathbb{C} \setminus \{a_1, a_2\}$, $\overline{N}(r, \frac{1}{f-b}) = T(r,f) + S(r,f)$, $\overline{N}(r, \frac{1}{\Delta_\eta^n f - b}) = T(r,f) + S(r,f)$;
(iii) $\overline{N}(r, \frac{1}{f'}) = S(r,f)$, $\overline{N}(r, \frac{1}{(\Delta_\eta^n f)'}) = S(r,f)$;
(iv) $\overline{N}^*(r, a_1) + \overline{N}^*(r, a_2) = S(r,f)$, *where $\overline{N}^*(r, a_i)$ is the reduced counting function of the multiple common zeros of $f - a_i$ and $\Delta_\eta^n f - a_i$ ($i = 1, 2$).*

Proof Suppose that $f(z) \not\equiv \Delta_\eta^n f(z)$. Since $f(z)$ and $\Delta_\eta^n f(z)$ share two values a_1 and a_2 "CM", according to the second fundamental theorem and Lemma 2.1, we can easily derive that

$$\begin{aligned}
T(r,f) &\le \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r,f) \\
&= \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_j}\right) + \sum_{j=1}^{2}\left[\overline{N}\left(r, \frac{1}{f-a_j}\right) - N_E(r, a_j)\right] + S(r,f) \\
&\le \sum_{j=1}^{2} \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_j}\right) + S(r,f) \le \overline{N}\left(r, \frac{1}{f - \Delta_\eta^n f}\right) + S(r,f) \\
&\le T\left(r, f - \Delta_\eta^n f\right) + S(r,f) = m\left(r, f - \Delta_\eta^n f\right) + S(r,f) \\
&\le m\left(r, \frac{\Delta_\eta^n f}{f}\right) + m(r,f) + S(r,f) \le T(r,f) + S(r,f).
\end{aligned}$$

Hence we prove the first conclusion.

Suppose that $f(z) \not\equiv \Delta_\eta^n f(z)$ and (1.1) holds, and we prove conclusions (i)–(iv) step by step.

Step 1. Notice that $f(z)$ and $\Delta_\eta^n f(z)$ share the value a_1 "CM" and (1.1) imply that

$$\overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) = \overline{N}\left(r, \frac{1}{f - a_1}\right) + \left[\overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_j}\right) - N_E(r, a_j)\right] = S(r,f). \tag{2.1}$$

Then, applying the second fundamental theorem again, we have

$$\begin{aligned} T\left(r, \Delta_\eta^n f\right) &\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_2}\right) + S\left(r, \Delta_\eta^n f\right) \\ &\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_2}\right) + S(r,f) \leq T\left(r, \Delta_\eta^n f\right) + S(r,f). \end{aligned}$$

From this and the second equality in the first conclusion, we can see that

$$T\left(r, \Delta_\eta^n f\right) = T(r,f) + S(r,f).$$

Step 2. For all $b \in \mathbb{C} \setminus \{a_1, a_2\}$, from the second fundamental theorem, the second equality in the first conclusion, and conclusion (i), we can derive that

$$\begin{aligned} 2T(r,f) + S(r,f) &= 2T\left(r, \Delta_\eta^n f\right) \\ &\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_2}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - b}\right) + S\left(r, \Delta_\eta^n f\right) \\ &\leq T(r,f) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - b}\right) + S(r,f) \\ &\leq T(r,f) + T\left(r, \Delta_\eta^n f\right) + S(r,f) = 2T(r,f) + S(r,f), \end{aligned}$$

which leads to

$$\overline{N}\left(r, \frac{1}{\Delta_\eta^n f - b}\right) = T(r,f) + S(r,f).$$

Similarly, we can prove that the following equality holds:

$$\overline{N}\left(r, \frac{1}{f - b}\right) = T(r,f) + S(r,f).$$

Step 3. Set

$$h(z) = \frac{(\Delta_\eta^n f)'}{\Delta_\eta^n f - a_1}. \tag{2.2}$$

Then we get from (2.1) and the lemma of logarithmic derivatives that

$$\begin{aligned} T(r,h) &= m(r,h) + N(r,h) \\ &= m\left(r, \frac{(\Delta_\eta^n f)'}{\Delta_\eta^n f - a_1}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) = S(r,f). \end{aligned} \tag{2.3}$$

It is obvious that $h(z) \not\equiv 0$ since $\Delta_\eta^n f(z)$ is not a constant. Hence from (2.1)–(2.3) we can deduce that

$$\begin{aligned}\overline{N}\left(r, \frac{1}{(\Delta_\eta^n f)'}\right) &\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) + \overline{N}\left(r, \frac{1}{h}\right)\\ &\leq \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) + T(r,h) = S(r,f).\end{aligned}$$

Similarly, we can prove that $\overline{N}(r, \frac{1}{f'}) = S(r,f)$.

Step 4. Consider the following function:

$$g(z) = \frac{f'(f - \Delta_\eta^n f)}{(f - a_1)(f - a_2)}. \tag{2.4}$$

The condition that $f(z)$ and $\Delta_\eta^n f(z)$ share two values a_1 and a_2 "CM" ensures that $g(z)$ is a meromorphic function such that all poles of $g(z)$ consist of zeros of $f(z) - a_1$ and $f(z) - a_2$. We obtain from Lemma 2.1 and the lemma of logarithmic derivatives that

$$\begin{aligned}T(r,g) &= m(r,g) + N(r,g) = m\left(r, \frac{f'(f - \Delta_\eta^n f)}{(f - a_1)(f - a_2)}\right) + N\left(r, \frac{f'(f - \Delta_\eta^n f)}{(f - a_1)(f - a_2)}\right)\\ &\leq m\left(r, \frac{ff'}{(f - a_1)(f - a_2)}\right) + m\left(r, \frac{f - \Delta_\eta^n f}{f}\right)\\ &\quad + \sum_{j=1}^{2}\left[\overline{N}\left(r, \frac{1}{f - a_j}\right) - N_E(r,a_j)\right]\\ &\leq m\left(r, \frac{a_1}{a_1 - a_2} \cdot \frac{f'}{f - a_1}\right) + m\left(r, \frac{a_2}{a_1 - a_2} \cdot \frac{f'}{f - a_2}\right) + S(r,f)\\ &= S(r,f).\end{aligned} \tag{2.5}$$

Let z_{ij} $(j = 1,2,\ldots)$ be the multiple common zeros of $f - a_i$ and $\Delta_\eta^n f - a_i$ $(i = 1,2)$, and let m_{ij} and n_{ij} be the multiplicities of the zero z_{ij} of $f - a_i$ and $\Delta_\eta^n f - a_i$, respectively. Note that $m_{ij}, n_{ij} \geq 2$. It follows from expression (2.4) of $g(z)$ that z_{ij} $(j = 1,2,\ldots)$ are zeros of $g(z)$ with multiplicity at least $\min\{m_{ij}, n_{ij}\} - 1 \geq 1$. This and (2.5) show that

$$\overline{N}^*(r,a_1) + \overline{N}^*(r,a_2) \leq \overline{N}\left(r, \frac{1}{g}\right) \leq T(r,g) = S(r,f).$$ □

Remark 2.1 Checking the proof of Lemma 2.3 carefully, we can see that all the conclusions of it still hold when 2 "CM" is replaced with 2 IM.

Lemma 2.4 *Let $f(z)$ be a nonconstant entire function of finite order, and let a_1, a_2 be two distinct finite complex constants. If $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 IM and (1.1) holds, then $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 and a_2 "CM".*

Proof It is easy to find that $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 "CM", since $f(z)$ and $\Delta_\eta^n f(z)$ share the value a_1 IM and (1.1) holds.

From the second fundamental theorem and (1.1), we have

$$\begin{aligned} T(r,f) &\le \overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f) \\ &= \overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f). \end{aligned} \tag{2.6}$$

Let $k=1$. Then (ii) in Lemma 2.2 can be rewritten as

$$\overline{N}\left(r,\frac{1}{f-a_2}\right) \le \frac{1}{2}\overline{N}_{1)}\left(r,\frac{1}{f-a_2}\right)+\frac{1}{2}T(r,f)+O(1). \tag{2.7}$$

(2.6) and (2.7) give

$$T(r,f) \le \overline{N}_{1)}\left(r,\frac{1}{f-a_2}\right)+S(r,f).$$

And due to

$$\overline{N}_{1)}\left(r,\frac{1}{f-a_2}\right) \le \overline{N}\left(r,\frac{1}{f-a_2}\right) \le T(r,f)+S(r,f),$$

the above inequality implies

$$T(r,f) = \overline{N}_{1)}\left(r,\frac{1}{f-a_2}\right)+S(r,f) = \overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f), \tag{2.8}$$

and thus

$$\overline{N}_{(2}\left(r,\frac{1}{f-a_2}\right) = S(r,f).$$

Similarly, the following equality holds:

$$T\left(r,\Delta_\eta^n f\right) = \overline{N}_{1)}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right)+S\left(r,\Delta_\eta^n f\right) = \overline{N}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right)+S\left(r,\Delta_\eta^n f\right).$$

Then, by (i) in Lemma 2.3, we can derive that

$$T(r,f) = \overline{N}_{1)}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right)+S(r,f) = \overline{N}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right)+S(r,f), \tag{2.9}$$

and thus

$$\overline{N}_{(2}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right) = S(r,f). \tag{2.10}$$

By (iv) in Lemma 2.3, one can easily see that

$$N_E(r,a_2)-N_E^{1)}(r,a_2) \le \overline{N}^*(r,a_2) = S(r,f). \tag{2.11}$$

It follows from (2.8), (2.10), and (2.11) that

$$\begin{aligned}
&\overline{N}\left(r, \frac{1}{f-a_2}\right) - N_E(r,a_2) \\
&= \overline{N}\left(r, \frac{1}{f-a_2}\right) - N_E^{1)}(r,a_2) + S(r,f) \\
&\le \overline{N}\left(r, \frac{1}{f-a_2}\right) - \left(\overline{N}_{1)}\left(r, \frac{1}{f-a_2}\right) - \overline{N}_{(2}\left(r, \frac{1}{\Delta_\eta^n f-a_2}\right)\right) + S(r,f) \\
&= S(r,f).
\end{aligned} \tag{2.12}$$

Since $f(z)$ and $\Delta_\eta^n f(z)$ share a_2 IM, from (2.12) we obtain that

$$\overline{N}\left(r, \frac{1}{\Delta_\eta^n f-a_2}\right) - N_E(r,a_2) = S(r,f). \tag{2.13}$$

Thus, $f(z)$ and $\Delta_\eta^n f(z)$ share the value a_2 "CM". □

Remark 2.2 We can find that (2.8), (2.12), and (2.13) used in the proof of Theorem 1.1 still hold when 2 IM is replaced with 2 "CM".

Lemma 2.5 ([3]) *Let $f(z)$ be a meromorphic function of finite order ρ, ε be a positive constant, η_1 and η_2 be two distinct nonzero complex constants. Then there exists a subset $E \subset (1,+\infty)$ of finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E$ and as $r \to \infty$ sufficiently large,*

$$\exp\{-r^{\rho-1+\varepsilon}\} \le \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \le \exp\{r^{\rho-1+\varepsilon}\}.$$

Lemma 2.6 ([1, 9]) *Let $f(z)$ be a meromorphic function with finite order ρ. Then, for any given $\varepsilon > 0$, there exists a set $E \subset (1,+\infty)$ of finite linear measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E$ and r sufficiently large,*

$$\exp\{-r^{\rho+\varepsilon}\} \le |f(z)| \le \exp\{r^{\rho+\varepsilon}\}.$$

Lemma 2.7 ([23]) *Suppose that $f(z)$ is a nonconstant meromorphic function in $|z| < R$ and a_j $(j = 1,2,\dots,q)$ are q distinct finite complex numbers. Then*

$$m\left(r, \sum_{j=1}^{q} \frac{1}{f-a_j}\right) = \sum_{j=1}^{q} m\left(r, \frac{1}{f-a_j}\right) + O(1)$$

holds for $0 < r < R$.

3 Proof of Theorem 1.1

Suppose that $f(z) \not\equiv \Delta_\eta^n f(z)$. Since $f(z)$ is a nonconstant entire function sharing a_1 and a_2 "CM" with $\Delta_\eta^n f(z)$,

$$\frac{\Delta_\eta^n f(z) - a_1}{f(z) - a_1} = p_1(z) e^{q_1(z)} \tag{3.1}$$

and

$$\frac{\Delta_\eta^n f(z) - a_2}{f(z) - a_2} = p_2(z)e^{q_2(z)}, \tag{3.2}$$

where $p_j(z)$ are meromorphic functions such that $\rho(p_j) < \rho(f)$ $(j = 1,2)$, and $q_1(z)$, $q_2(z)$ are polynomials such that $\deg p_1(z) \le \rho(f)$, $\deg q_2(z) \le \rho(f)$.

If $p_1(z)e^{q_1(z)} \equiv p_2(z)e^{q_2(z)}$, then we get from (3.1) and (3.2) that $f(z) \equiv \Delta_\eta^n f(z)$. Next, we suppose that $p_1(z)e^{q_1(z)} \not\equiv p_2(z)e^{q_2(z)}$. From (3.1) and (3.2), we have

$$f(z) - a_1 = \frac{(a_2 - a_1)(1 - p_2(z)e^{q_2(z)})}{p_1(z)e^{q_1(z)} - p_2(z)e^{q_2(z)}}. \tag{3.3}$$

Since $\rho(p_j) < \rho(f)$ $(j = 1,2)$, we can deduce from (3.3) that almost all (except at most $S(r,f)$) zeros of $f(z) - a_1$ are zeros of $g(z) := 1 - p_2(z)e^{q_2(z)}$. Hence

$$\overline{N}\left(r, \frac{1}{f - a_1}\right) \le \overline{N}\left(r, \frac{1}{1 - p_2 e^{q_2}}\right) + S(r,f) \le T\left(r, e^{q_2}\right) + S(r,f). \tag{3.4}$$

Next, we discuss two cases.

Case 1: $\deg q_2(z) < \rho(f)$. It follows from (3.4) that

$$\overline{N}\left(r, \frac{1}{f - a_1}\right) = S(r,f).$$

Therefore, Lemma 2.3 is valid now. Let us consider the following two functions:

$$F(z) = \frac{\Delta_\eta^n f - a_1}{\Delta_\eta^n f - a_2}, \qquad G(z) = \frac{f - a_1}{f - a_2}. \tag{3.5}$$

Notice that $f(z)$ and $\Delta_\eta^n f(z)$ share two values a_1 and a_2 "CM", and we see that $F(z)$ and $G(z)$ are meromorphic functions sharing 0 and ∞ "CM". By (3.5), (ii) in Lemma 2.3, and the Valiron–Mokhon'ko theorem (see e.g. [16, 22]), we have

$$\begin{aligned} &T(r,F) = T\left(r, \Delta_\eta^n f\right) + S\left(r, \Delta_\eta^n f\right) = T(r,f) + S(r,f), \\ &T(r,G) = T(r,f) + S(r,f). \end{aligned} \tag{3.6}$$

Let

$$\varphi(z) = \frac{F''}{F'} - \frac{G''}{G'}, \tag{3.7}$$

and we get, by applying the lemma of logarithmic derivatives,

$$\begin{aligned} m(r,\varphi) &\le m\left(r, \frac{F''}{F'}\right) + m\left(r, \frac{G''}{G'}\right) + O(1) \\ &= S\left(r, F'\right) + S\left(r, G'\right) = S(r,f). \end{aligned} \tag{3.8}$$

Clearly, (3.7) shows that the poles of $\varphi(z)$ are simple, and they can only come from the zeros of $F'(z)$ and $G'(z)$ as well as the poles of $F(z)$ and $G(z)$.

In the following, suppose that z_2 is a pole of $F(z)$ and $G(z)$ with the same multiplicity k, which comes from the zero z_2 of $f-a_2$ and $\Delta_\eta^n f - a_2$ with the same multiplicity k. And suppose that the following two expansions hold in the neighborhood of $z-z_2$:

$$\begin{aligned} F(z) &= \frac{A_{-k}}{(z-z_2)^k} + \frac{A_{-k+1}}{(z-z_2)^{k-1}} + \cdots, \\ G(z) &= \frac{B_{-k}}{(z-z_2)^k} + \frac{B_{-k+1}}{(z-z_2)^{k-1}} + \cdots, \end{aligned}$$

a simple computation shows that

$$\begin{aligned} \varphi &= \frac{F''}{F'} - \frac{G''}{G'} \\ &= \left(-\frac{k+1}{z-z_2} + \frac{(k-1)A_{-k+1}}{kA_{-k}} + O(z-z_2)\right) \\ &\quad - \left(-\frac{k+1}{z-z_2} + \frac{(k-1)B_{-k+1}}{kB_{-k}} + O(z-z_2)\right) \\ &= \frac{k-1}{k}\left(\frac{A_{-k+1}}{A_{-k}} - \frac{B_{-k+1}}{B_{-k}}\right) + O(z-z_2), \end{aligned} \tag{3.9}$$

which implies that z_2 is not the pole of $\varphi(z)$.

To consider the zeros of $F'(z)$ and $G'(z)$, we derive from (3.5) that

$$F' = \frac{(a_1-a_2)(\Delta_\eta^n f)'}{(\Delta_\eta^n f - a_2)^2}, \qquad G' = \frac{(a_1-a_2)f'}{(f-a_2)^2}. \tag{3.10}$$

Now (iv) in Lemma 2.3 and (3.10) imply that

$$\overline{N}\left(r,\frac{1}{F'}\right) \le \overline{N}\left(r,\frac{1}{(\Delta_\eta^n f)'}\right) = S(r,f), \qquad \overline{N}\left(r,\frac{1}{G'}\right) \le \overline{N}\left(r,\frac{1}{f'}\right) = S(r,f). \tag{3.11}$$

Then, by (2.12),(2.13),(3.5),(3.7), and (3.11), we can deduce that

$$\begin{aligned} &N(r,\varphi) \\ &\quad \le \overline{N}\left(r,\frac{1}{F'}\right) + \overline{N}\left(r,\frac{1}{G'}\right) + \overline{N}(r,F) + \overline{N}(r,G) - 2N_E(r,a_2) + S(r,f) \\ &\quad = \overline{N}\left(r,\frac{1}{\Delta_\eta^n f - a_2}\right) + \overline{N}\left(r,\frac{1}{f-a_2}\right) - 2N_E(r,a_2) + S(r,f) = S(r,f). \end{aligned} \tag{3.12}$$

Thus (3.8) and (3.12) give immediately

$$T(r,\varphi) = m(r,\varphi) + N(r,\varphi) = S(r,f). \tag{3.13}$$

If $\varphi(z) \not\equiv 0$, suppose that z_2^* is a simple common pole of $F(z)$ and $G(z)$, which comes from the simple common zero z_2 of $f-a_2$ and $\Delta_\eta^n f - a_2$.Then (3.9) implies that z_2^* is a zero of $\varphi(z)$ with the multiplicity at least 1, which means that

$$N_E^{1)}(r,a_2) \le N\left(r,\frac{1}{\varphi}\right) \le T(r,\varphi) = S(r,f). \tag{3.14}$$

Combining (2.8), (2.11), and (2.12) shows that

$$\begin{aligned} N_E^{1)}(r,a_2) &= N_E(r,a_2) + S(r,f) \\ &= \overline{N}\left(r,\frac{1}{f-a_2}\right) + S(r,f) = T(r,f) + S(r,f). \end{aligned} \tag{3.15}$$

Thus clearly $T(r,f) \leq S(r,f)$ follows immediately from (3.14) and (3.15). That is impossible.

Now, we have proved that $\varphi(z) \equiv 0$, that is,

$$\frac{F''}{F'} \equiv \frac{G''}{G'}.$$

Taking integration of this identity twice, we can derive that

$$F \equiv \alpha G + \beta, \tag{3.16}$$

where α ($\neq 0$) and β are constants.

Next, we discuss two subcases.

Case 1.1: a_1 is not a Nevanlinna exceptional value of $f(z)$. The condition that $f(z)$ and $\Delta_\eta^n f(z)$ share the value a_1 "CM" ensures that there exists z_3 such that $f(z_3) = \Delta_\eta^n f(z_3) = a_1$, and then from (3.5), $F(z_3) = G(z_3) = 0$. Therefore $\beta = 0$, and (3.16) becomes

$$F \equiv \alpha G. \tag{3.17}$$

Since $f(z) \not\equiv \Delta_\eta^n f(z)$, we see that $\alpha \neq 1$. Thus, 1 must be a Picard exceptional value of $F(z)$ and $G(z)$, we can deduce easily from (3.17) that 1, α are Picard exceptional values of $F(z)$, and 1, $\frac{1}{\alpha}$ are Picard exceptional values of $G(z)$. This fact and (3.5) show that

$$G = \frac{f-a_1}{f-a_2} \neq \frac{1}{\alpha}.$$

That is,

$$f \neq \frac{a_1\alpha - a_2}{\alpha - 1}, \tag{3.18}$$

which means that $\frac{a_1\alpha-a_2}{\alpha-1}$ is a Picard exceptional value of $f(z)$. Obviously, $\frac{a_1\alpha-a_2}{\alpha-1} \neq a_1, a_2$.

On the other hand, by (ii) in Lemma 2.3, the following equality holds:

$$\overline{N}\left(r,\frac{1}{f-\frac{a_1\alpha-a_2}{\alpha-1}}\right) = T(r,f) + S(r,f).$$

This contradicts (3.18).

Case 1.2: a_1 is a Nevanlinna exceptional value of $f(z)$. Since $f(z)$ and $\Delta_\eta^n f(z)$ share a_1 "CM", a_1 is also a Nevanlinna exceptional value of $\Delta_\eta^n f$, and thus 0 is a Nevanlinna exceptional value of $F(z)$ and $G(z)$. From (3.5) and (3.16), we see that 0, 1, β, $\alpha + \beta$ and 0, 1, $-\frac{\beta}{\alpha}$,

$\frac{1-\beta}{\alpha}$ are Nevanlinna exceptional values of $F(z)$ and $G(z)$, respectively. As $f(z) \not\equiv \Delta_\eta^n f(z)$, we get $\alpha \neq 1$. Hence $\beta = 1$, $\alpha + \beta = 0$. And now (3.16) is of the form

$$F \equiv -G + 1. \tag{3.19}$$

From (3.19), we know that $F(z)$ and $G(z)$ share $\frac{1}{2}$ CM, which implies that $f(z)$ and $\Delta_\eta^n f(z)$ share another value $2a_1 - a_2$ ($\neq a_1, a_2$) CM. Then

$$\begin{aligned}
&\overline{N}\left(r, \frac{1}{f-a_2}\right) + \overline{N}\left(r, \frac{1}{f-(2a_1-a_2)}\right)\\
&\quad \le \overline{N}\left(r, \frac{1}{f-\Delta_\eta^n f}\right)\\
&\quad \le T\left(r, f-\Delta_\eta^n f\right) = m\left(r, f-\Delta_\eta^n f\right)\\
&\quad \le m\left(r, \frac{\Delta_\eta^n f}{f}\right) + m(r,f) \le T(r,f) + S(r,f).
\end{aligned}$$

This and (3.15) yield $\overline{N}(r, \frac{1}{f-(2a_1-a_2)}) = S(r,f)$. On the other hand, by (i) in Theorem 2.3, the following equality holds:

$$\overline{N}\left(r, \frac{1}{f-(2a_1-a_2)}\right) = T(r,f) + S(r,f).$$

This is a contradiction.

Case 2: $\deg q_2(z) = \rho(f)$. If $\deg q_1(z) < \rho(f)$, then we can deduce similar contradictions as in Case 1. Thus, $\deg q_1(z) = \rho(f)$. Suppose that $\deg q_1(z) = \deg q_2(z) = \rho(f) = d$. Obviously, $d \ge 1$. Otherwise, we get a contradiction from (3.3 that

$$\rho(f) \le \max\left\{\rho(p_1), \rho(p_2)\right\}.$$

Set

$$q_1(z) = A_d z^d + A_{d-1} z^{d-1} + \cdots + A_0$$

and

$$q_2(z) = B_d z^d + B_{d-1} z^{d-1} + \cdots + B_0,$$

then $A_d B_d \neq 0$. Denote $A_d = r_1 e^{i\theta_1}$, $B_d = r_2 e^{i\theta_2}$, $A_d + B_d = r_3 e^{i\theta_3}$, where $\theta_j \in [-\pi, \pi)$, $j = 1, 2, 3$.

From (3.1) and (3.2), we have

$$\frac{\Delta_\eta^n f}{f-a_1} = \frac{a_2 p_1 e^{q_1} - a_1 p_2 e^{q_2} + (a_1 - a_2) p_1 p_2 e^{q_1+q_2}}{(a_2-a_1)(1-p_2 e^{q_2})}. \tag{3.20}$$

Notice that

$$\Delta_\eta^n f = \Delta_\eta^n (f - a_j) = \sum_{j=0}^{n} (-1)^j C_n^j \left(f\left(z + (n-j)\eta\right) - a_j\right), \quad j = 1, 2.$$

Set $\varrho = \max\{\rho(p_1), \rho(p_2)\}$ and $\varepsilon = \min\{\frac{d-\varrho}{2}, \frac{1}{2}\}$, then applying Lemma 2.5 we see that there exists a subset $E_1 \subset (1, +\infty)$ of finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_1$ and as $r \to \infty$ sufficiently large,

$$\exp\{-r^{d-1+\varepsilon}\} \le \left|\frac{\Delta_\eta^n f}{f - a_j}\right| \le \exp\{r^{d-1+\varepsilon}\}, \quad j = 1, 2. \tag{3.21}$$

By Lemma 2.6, for ε given above, there exists a set $E_2 \subset (1, +\infty)$ of finite linear measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_2$ and r sufficiently large,

$$\exp\{-r^{\varrho+\varepsilon}\} \le |p_j(z)| \le \exp\{r^{\varrho+\varepsilon}\}, \quad j = 1, 2. \tag{3.22}$$

Case 2.1: $r_1 > \max\{r_2, r_3\} := r_4$. For the point $\varphi_1 = -\theta_1/d \in [-\pi, \pi)$, we see that, for all $z = |z|e^{i\varphi_1} = re^{i\varphi_1}$,

$$A_d z^d = r_1 r^d > r_4 r^d = \max\{r_2 r^d, r_3 r^d\} \ge \max\{\operatorname{Re} A_d z^d, \operatorname{Re}(A_d + B_d) z^d\}. \tag{3.23}$$

Then we deduce from (3.20)–(3.22) that, for all $z = re^{i\varphi_1}$ satisfying $|z| = r \notin [0,1] \cup E_1 \cup E_2$ and r sufficiently large,

$$\begin{aligned}
&|a_2| \exp\{r_1 r^d(1 + o(1)) - r^{\varrho+\varepsilon}\} \\
&\quad < |a_2 p_1 e^{q_1}| = \left|(a_2 - a_1)(1 - p_2 e^{q_2})\frac{\Delta_\eta^n f}{f - a_1} + a_1 p_2 e^{q_2} - (a_1 - a_2) p_1 p_2 e^{q_1 + q_2}\right| \\
&\quad \le \left|(a_2 - a_1)(1 - p_2 e^{q_2})\frac{\Delta_\eta^n f}{f - a_1}\right| + |a_1 p_2 e^{q_2}| + |(a_1 - a_2) p_1 p_2 e^{q_1 + q_2}| \\
&\quad < (|a_1| + |a_2|)(1 + \exp\{r^{\varrho+\varepsilon}\}) \exp\{r^{d-1+\varepsilon}\} + |a_1| \exp\{r_2 r^d(1 + o(1)) + r^{\varrho+\varepsilon}\} \\
&\qquad + (|a_1| + |a_2|) \exp\{r_3 r^d(1 + o(1)) + 2r^{\varrho+\varepsilon}\} \\
&\quad < \exp\{r_4 r^d(1 + o(1))\}.
\end{aligned}$$

However, from (3.23) we see that this is impossible.

Case 2.2: $r_2 > \max\{r_1, r_3\}$. Rewrite (3.20) as the form

$$\frac{\Delta_\eta^n f}{f - a_2} = \frac{a_2 p_1 e^{q_1} - a_1 p_2 e^{q_2} + (a_1 - a_2) p_1 p_2 e^{q_1 + q_2}}{(a_2 - a_1)(1 - p_1 e^{q_1})}.$$

With this and reasoning as in Case 2.1, we can deduce a similar contradiction.

Case 2.3: $r_3 > \max\{r_1, r_2\}$. Reasoning as in Case 2.1, we can deduce a similar contradiction again.

Case 2.4: $r_1 = r_2 = r_3$. Since $A_d = r_1 e^{i\theta_1}$, $B_d = r_2 e^{i\theta_2}$, $A_d + B_d = r_3 e^{i\theta_3}$, θ_1, θ_2, and θ_3 must be distinct and satisfy

$$|\theta_j - \theta_k| \notin \{0, 2\pi\}, \quad 1 \le j < k \le 3.$$

Thus, for $\varphi_1 = -\theta_1/d$ and $z = re^{i\varphi_1}$, we have

$$\begin{aligned}
A_d z^d = r_1 r^d &> \max\{r_1 \cos(\theta_2 + d\varphi_2) r^d, r_1 \cos(\theta_2 + d\varphi_2) r^d\} \\
&= \max\{\operatorname{Re} B_d z^d, \operatorname{Re}(A_d + B_d) z^d\}.
\end{aligned}$$

With this and arguing as in Case 2.1, we can also deduce a similar contradiction.

Thus, we finally prove that $f(z) \equiv \Delta_\eta^n f(z)$.

Remark 3.1 From Lemma 2.4 and Theorem 1.1, we can get Theorem 1.2 immediately. And we omit it.

4 Proof of Theorem 1.3

We begin our proof by supposing that $f(z) \not\equiv \Delta_\eta^n f(z)$. We get immediately from Lemma 2.1 that

$$T(r, \Delta_\eta^n f) = m(r, \Delta_\eta^n f) \leq m\left(r, \frac{\Delta_\eta^n f}{f}\right) + m(r,f) = T(r,f) + S(r,f),$$

which also gives $S(r, \Delta_\eta^n f) \leq S(r,f)$.

Since $f(z)$ and $\Delta_\eta^n f(z)$ share two values a_1 and a_2 IM, we get by applying the second fundamental theorem that

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r,f) \\ &= \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_1}\right) + \overline{N}\left(r, \frac{1}{\Delta_\eta^n f - a_2}\right) + S(r,f) \leq 2T(r, \Delta_\eta^n f) + S(r,f). \end{aligned}$$

And hence

$$T(r, \Delta_\eta^n f) \geq \frac{1}{2} T(r,f) + S(r,f). \tag{4.1}$$

Using Lemma 2.1 and Lemma 2.7, one can easily prove that

$$\begin{aligned} &m\left(r, \frac{1}{f-a_1}\right) + m\left(r, \frac{1}{f-a_2}\right) \\ &\quad = m\left(r, \frac{1}{f-a_1} + \frac{1}{f-a_2}\right) + O(1) \\ &\quad \leq m\left(r, \frac{1}{\Delta_\eta^n f}\right) + m\left(r, \frac{\Delta_\eta^n f}{f-a_1} + \frac{\Delta_\eta^n f}{f-a_2}\right) + O(1) \leq m\left(r, \frac{1}{\Delta_\eta^n f}\right) + S(r,f). \end{aligned} \tag{4.2}$$

The assumption $\delta(a_1,f) + \delta(a_2,f) \geq \lambda$ means that

$$N\left(r, \frac{1}{f-a_1}\right) + N\left(r, \frac{1}{f-a_2}\right) \leq (2-\lambda)T(r,f) + S(r,f). \tag{4.3}$$

We get by combining (4.2) and (4.3) that

$$\begin{aligned} m\left(r, \frac{1}{\Delta_\eta^n f}\right) &\geq 2T(r,f) - N\left(r, \frac{1}{f-a_1}\right) - N\left(r, \frac{1}{f-a_2}\right) + O(1) \\ &\geq \lambda T(r,f) + S(r,f). \end{aligned} \tag{4.4}$$

On the other hand, we can derive by using Lemma 2.7 and the lemma of logarithmic derivatives that

$$\begin{aligned} & m\left(r,\frac{1}{\Delta_\eta^n f}\right)+m\left(r,\frac{1}{\Delta_\eta^n f-a_1}\right)+m\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right) \\ &\quad = m\left(r,\frac{1}{\Delta_\eta^n f}+\frac{1}{\Delta_\eta^n f-a_1}+\frac{1}{\Delta_\eta^n f-a_2}\right)+O(1) \\ &\quad \le m\left(r,\frac{(\Delta_\eta^n f)'}{\Delta_\eta^n f}+\frac{(\Delta_\eta^n f)'}{\Delta_\eta^n f-a_1}+\frac{(\Delta_\eta^n f)'}{\Delta_\eta^n f-a_2}\right)+m\left(r,\frac{1}{(\Delta_\eta^n f)'}\right)+O(1) \\ &\quad \le m\left(r,\frac{1}{(\Delta_\eta^n f)'}\right)+S(r,f). \end{aligned} \tag{4.5}$$

Noting that $f(z)$ shares two values a_1 and a_2 IM with $\Delta_\eta^n f(z)$, we can derive that

$$\begin{aligned} & \overline{N}\left(r,\frac{1}{\Delta_\eta^n f-a_1}\right)+\overline{N}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right) \\ &\quad \le \overline{N}\left(r,\frac{1}{f-\Delta_\eta^n f}\right) \le T\left(r,f-\Delta_\eta^n f\right)=m\left(r,f-\Delta_\eta^n f\right) \\ &\quad \le m\left(r,\frac{\Delta_\eta^n f}{f}\right)+m(r,f) \le T(r,f)+S(r,f). \end{aligned} \tag{4.6}$$

Furthermore, from (4.6) we know

$$\begin{aligned} & N\left(r,\frac{1}{\Delta_\eta^n f-a_1}\right)+N\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right) \\ &\quad \le \overline{N}\left(r,\frac{1}{\Delta_\eta^n f-a_1}\right)+\overline{N}\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right)+N\left(r,\frac{1}{(\Delta_\eta^n f)'}\right) \\ &\quad \le T(r,f)+N\left(r,\frac{1}{(\Delta_\eta^n f)'}\right)+S(r,f). \end{aligned} \tag{4.7}$$

Hence (4.5) and (4.7) lead to

$$\begin{aligned} & m\left(r,\frac{1}{\Delta_\eta^n f}\right)+2T\left(r,\Delta_\eta^n f\right)+O(1) \\ &\quad = m\left(r,\frac{1}{\Delta_\eta^n f}\right)+T\left(r,\frac{1}{\Delta_\eta^n f-a_1}\right)+T\left(r,\frac{1}{\Delta_\eta^n f-a_2}\right) \\ &\quad \le T(r,f)+T\left(r,\frac{1}{(\Delta_\eta^n f)'}\right)+S(r,f) \\ &\quad \le T(r,f)+T\left(r,\Delta_\eta^n f\right)+S(r,f). \end{aligned}$$

Since $\lambda > \frac{1}{2}$, we can conclude easily from the above inequality and (4.4) that

$$\begin{aligned} T\left(r,\Delta_\eta^n f\right) &\le T(r,f)-m\left(r,\frac{1}{\Delta_\eta^n f}\right)+S(r,f) \\ &\le (1-\lambda)T(r,f)+S(r,f) < \frac{1}{2}T(r,f)+S(r,f), \end{aligned}$$

which contradicts (4.1). Hence $f(z) \equiv \Delta_\eta^n f(z)$.

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