# Discrete fractional order two-point boundary value problem with some relevant physical applications 

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#### Abstract

The results reported in this paper are concerned with the existence and uniqueness of solutions of discrete fractional order two-point boundary value problem. The results are developed by employing the properties of Caputo and Riemann-Liouville fractional difference operators, the contraction mapping principle and the Brouwer fixed point theorem. Furthermore, the conditions for Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the proposed discrete fractional boundary value problem are established. The applicability of the theoretical findings has been demonstrated with relevant practical examples. The analysis of the considered mathematical models is illustrated by figures and presented in tabular forms. The results are compared and the occurrence of overlapping/non-overlapping has been discussed.


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## 1 Introduction

Partial differential equations are invariably important in almost all fields of applied mathematics and science [1-3]. Particularly, one can observe that partial differential equations have been utilized in few places to help in the use of ordinary differential equations such as in the study of waves in liquids, propagation of sound, gravitational attraction and vibrations of strings [4]. On the other hand, partial fractional differential equations have presented adequate interpretations for many physical problems in areas such as fluid mechanics, biological populations, viscoelasticity, advection-diffusion, nuclear science and signals processing [5, 6]. In many cases, like the heat equation, wave equation, Poisson equation and Laplace equation, the problems remained unsolved due to the nonlinearity property of these equations. Owing to this limitation, various techniques like numerical methods for approximating solutions are used to problems modeled by nonlinear partial differential equations involving initial and boundary conditions [7-12].

Very recently, fractional differential equations (FDEs) have become intensively rich theory and found applications in various fields. These equations, which involve derivatives or

[^0]integrals of fractional order, have resulted in a great interest for many researchers due to their effective applications in physics, chemistry, chaotic dynamical systems and random walks with memory in different fields of applied mathematics and engineering. Particularly emphasis has been put to the topics on existence, uniqueness and stability of solutions of differential equations of fractional order; see [13-25] and the references cited therein. The corresponding discrete counter part, fractional order difference equations (FODEs), have appeared as a new research area for mathematicians and scientists. The study of discrete fractional calculus was initiated by Miller and Ross [26] and then developed by several other researchers [27-41]. In the meantime, researchers have adopted the fact that dealing with FODEs provides a more accurate description than FDEs and the use of FODEs facilitates applications that require computational and simulation analysis.

The organization of the remaining part of the paper is outlined as follows: Fundamental definitions and concepts are introduced in Sect. 2. Section 3 is devoted to the discussion on existence and uniqueness results for a discrete FBVP (3.1). The main results of this section are obtained by using the contraction mapping principle and the Brouwer fixed point theorem. In Sect. 4, we develop conditions for Hyers-Ulam and Hyers-Ulam-Rassias stability of the discrete FBVP. The applications are discussed in Sect. 5 which is followed by our conclusion.

## 2 Auxiliary preliminaries

Now we present some fundamental definitions and essential lemmas of discrete fractional calculus that are to be used throughout this paper.

Definition 2.1 (see [31, 32]) Let $\alpha>0$. The $\alpha$ th fractional sum of a function $\Psi$ is defined as

$$
\Delta^{-\alpha} \Psi(x)=\frac{1}{\Gamma(\alpha)} \sum_{\ell=a}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell),
$$

for all $x \in\{a+\alpha, a+\alpha+1, \ldots\}:=\mathbb{N}_{a+\alpha}$ and $x^{(\alpha)}:=\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}$.
Definition 2.2 (see [30-32]) Let $\alpha>0$ and set $\mu=n-\alpha$. The $\alpha$ th fractional Caputo difference operator is defined as

$$
{ }_{0}^{C} \Delta_{x}^{\alpha} \Psi(x)=\Delta^{-\mu}\left(\Delta^{n} \Psi(x)\right)=\frac{1}{\Gamma(\mu)} \sum_{\ell=a}^{x-\mu}(x-\ell-1)^{(\mu-1)} \Delta^{n} \Psi(\ell),
$$

for all $x \in \mathbb{N}_{a+\mu}$ and $n-1<\alpha \leq n$, where $n=\lceil\alpha\rceil$ and $\lceil$.$\rceil is the ceiling of a number.$
Lemma 2.3 (see $[28,39]$ ) Let $x$ and $\alpha$ be any numbers for which $x^{(\alpha)}$ and $x^{(\alpha-1)}$ are defined. Then $\Delta x^{(\alpha)}=\alpha x^{(\alpha-1)}$.

Lemma 2.4 (see $[38,39]$ ) Let $0 \leq N-1<\alpha \leq N$. Then

$$
\Delta_{0}^{-\alpha R L} \Delta_{x}^{\alpha} \Psi(x)=\Psi(x)-\sum_{j=0}^{N-1} \frac{x^{(\alpha-N+j)}}{\Gamma(\alpha+j-1)} \Delta^{j}\left[\Delta^{\alpha-N} \Psi(a)\right],
$$

$$
=\Psi(x)+B_{1} x^{(\alpha-1)}+B_{2} x^{(\alpha-2)}+\cdots+B_{N} x^{(\alpha-N)}
$$

for $B_{i} \in \mathbb{R}$, where $i=1,2, \ldots, N$.

Lemma 2.5 (see $[27,31,32])$ Suppose that $\alpha>0$ and $\Psi$ is defined on $\mathbb{N}_{a}$. Then

$$
\begin{aligned}
\Delta_{0}^{-\alpha C} \Delta_{x}^{\alpha} \Psi(x) & =\Psi(x)-\sum_{j=0}^{n-1} \frac{(x-a)^{(j)}}{j!} \Delta^{j} \Psi(a) \\
& =\Psi(x)+C_{0}+C_{1} x+\cdots+C_{n-1} x^{(n-1)},
\end{aligned}
$$

for $C_{i} \in \mathbb{R}$, where $i=0,1,2, \ldots, n-1$.

Lemma 2.6 (see [30,37]) If $\alpha$ and $x$ are any numbers, then
$1 \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}=\frac{\Gamma(x+1)}{\alpha \Gamma(x-\alpha+1)}$.
$2 \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)}=\frac{1}{\alpha} \frac{\Gamma(\alpha+L+1)}{\Gamma(L+1)}$.
Lemma 2.7 (see [29]) Let $\mu \in \mathbb{R} \backslash\{\ldots,-2,-1\}$. Then

$$
\Delta^{-\alpha} x^{(\mu)}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} x^{(\alpha+\mu)} .
$$

## 3 Existence and uniqueness of solutions

In this section, we will discuss the existence and uniqueness of solutions to a discrete fractional boundary value problem (FBVP) of the form

$$
\left\{\begin{array}{l}
{ }_{0}^{C} \Delta_{x}^{\alpha} w(x)=\Psi(x+\alpha-1, w(x+\alpha-1)), \quad 1<\alpha \leq 2  \tag{3.1}\\
\Delta w(\alpha-2)=A, \quad w(\alpha+L)=B
\end{array}\right.
$$

for $x \in[0, L]_{\mathbb{N}_{0}}=\{0,1, \ldots, L\}, A, B$ are some real constants, $\Psi:[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function, ${ }_{0}^{C} \Delta_{x}^{\alpha}$ is the Caputo fractional difference operator (CFDO) and $L \in \mathbb{N}_{1}$. Now, we state and prove an important theorem which will be helpful to obtain a form of the solution of (3.1), provided that the solution exists.

Theorem 3.1 Let $1<\alpha \leq 2$ and $\Psi:[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}} \rightarrow \mathbb{R}$ be given. Then a function $w$ is a solution to the discrete FBVP

$$
\left\{\begin{array}{l}
{ }_{0}^{C} \Delta_{x}^{\alpha} w(x)=\Psi(x+\alpha-1), \quad x \in[0, L]_{\mathbb{N}_{0}},  \tag{3.2}\\
\Delta w(\alpha-2)=A, \quad w(\alpha+L)=B,
\end{array}\right.
$$

if and only if $w(x)$, for $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$ is a solution to the following fractional Taylor's difference formula:

$$
\begin{align*}
w(x)= & p(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1) \\
& -\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1), \tag{3.3}
\end{align*}
$$

where $p$ is the unique solution to the discrete FBVP

$$
\left\{\begin{array}{l}
{ }_{0}^{C} \Delta_{x}^{\alpha} p(x)=0,  \tag{3.4}\\
\Delta p(\alpha-2)=A, \quad p(\alpha+L)=B .
\end{array}\right.
$$

Proof Suppose that $p$ is a solution to (3.4). Using Definition 2.1 together with Lemma 2.5 shows that

$$
\begin{equation*}
p(x)=C_{0}+C_{1} x, \quad x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}, \tag{3.5}
\end{equation*}
$$

where $C_{0}, C_{1} \in \mathbb{R}$. Applying the operator $\Delta$ to both sides in (3.5), we get

$$
\begin{equation*}
\Delta p(x)=C_{1} . \tag{3.6}
\end{equation*}
$$

Using the boundary conditions $\Delta p(\alpha-2)=A$ and $p(\alpha+L)=B$ in (3.5) and (3.6), then it turns out that $C_{0}=B-A(\alpha+L)$ and $C_{1}=A$. Using $C_{0}$ and $C_{1}$ in $p(x)$, we are left with

$$
\begin{equation*}
p(x)=A[x-(\alpha+L)]+B . \tag{3.7}
\end{equation*}
$$

Let $w(x)$ be a solution to (3.2). In view of Lemma 2.5, we obtain a general solution to (3.2) in the form

$$
w(x)=\Delta^{-\alpha} \Psi(x+\alpha-1)+C_{2}+C_{3} x, x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}},
$$

where $C_{2}, C_{3} \in \mathbb{R}$. Whence, by Definition 2.1, we have

$$
\begin{equation*}
w(x)=\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1)+C_{2}+C_{3} x . \tag{3.8}
\end{equation*}
$$

Applying the operator $\Delta$ on both sides in (3.8), we get

$$
\begin{equation*}
\Delta w(x)=\frac{1}{\Gamma(\alpha-1)} \sum_{\ell=0}^{x-\alpha+1}(x-\ell-1)^{(\alpha-2)} \Psi(\ell+\alpha-1)+C_{3} . \tag{3.9}
\end{equation*}
$$

By the boundary conditions $\Delta w(\alpha-2)=A$ and $w(\alpha+L)=B$, we obtain $C_{2}=B-A(\alpha+L)-$ $\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1)$ and $C_{3}=A$. Using the values of $C_{2}, C_{3}$ and $p(x)$ in $w(x)$, we deduce that

$$
\begin{align*}
w(x)= & p(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1) \\
& -\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1) . \tag{3.10}
\end{align*}
$$

Conversely, it is easy to show that the solution (3.10) satisfies the discrete FBVP (3.2). The proof of the theorem is complete.

For applications using the contraction mapping principle and the Brouwer fixed point theorems, the following operator is defined:

$$
\begin{align*}
(T w)(x)= & p(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, w(\ell+\alpha-1)) \\
& -\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, w(\ell+\alpha-1)) \tag{3.11}
\end{align*}
$$

for $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$. Obviously, $w(x)$ is a solution to (3.1) if it is a fixed point of the operator $T$. For our convenience, we consider the Banach space $\mathbb{E}$ with norm $\|w\|=\max |w(x)|$ for $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$.

## Theorem 3.2 Assume the following.

$\left(H_{1}\right)$ There exists a constant $K>0$ such that $\left|\Psi(x, w)-\Psi\left(x, w_{1}\right)\right| \leq K\left|w-w_{1}\right|$ for each $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$ and all $w, w_{1} \in \mathbb{E}$.
Then the discrete FBVP (3.1) has a unique solution on $\mathbb{E}$ provided that

$$
\begin{equation*}
\frac{2 \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)}<\frac{1}{K} . \tag{3.12}
\end{equation*}
$$

Proof Let $w, w_{1} \in \mathbb{E}$. Then, for each $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$, we have

$$
\begin{aligned}
& \left|(T w)(x)-\left(T w_{1}\right)(x)\right| \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \\
& \quad \times\left|\Psi(\ell+\alpha-1, w(\ell+\alpha-1))-\Psi\left(\ell+\alpha-1, w_{1}(\ell+\alpha-1)\right)\right| \\
& +\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \\
& \quad \times\left|\Psi(\ell+\alpha-1, w(\ell+\alpha-1))-\Psi\left(\ell+\alpha-1, w_{1}(\ell+\alpha-1)\right)\right| \\
& \leq \frac{K}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}\left|w(\ell+\alpha-1)-w_{1}(\ell+\alpha-1)\right| \\
& \quad+\frac{K}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)}\left|w(\ell+\alpha-1)-w_{1}(\ell+\alpha-1)\right| .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left\|T w-T w_{1}\right\| \\
& \leq \frac{K\left\|w-w_{1}\right\|}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}+\frac{K\left\|w-w_{1}\right\|}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \\
& \leq \frac{K\left\|w-w_{1}\right\|}{\Gamma(\alpha)} \cdot \frac{\Gamma(x+1)}{\alpha \Gamma(x-\alpha+1)}+\frac{K\left\|w-w_{1}\right\|}{\Gamma(\alpha)} \frac{\Gamma(\alpha+L+1)}{\alpha \Gamma(L+1)} \\
& \leq\left[\frac{2 \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)}\right] K\left\|w-w_{1}\right\|,
\end{aligned}
$$

which implies that $T$ is a contraction. By the contraction mapping principle, $T$ has a unique fixed point which is a unique solution to the discrete FBVP (3.1). The proof is complete.

Theorem 3.3 Assume that $\Psi:[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $M \geq$ $\max _{x \in[\alpha-2, \alpha+L]}|p(x)|$, where $p$ is the unique solution of the discrete FBVP (3.4). Let $Q=$ $\max \left\{|\Psi(x, w)|: x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}, w \in \mathbb{E},|w| \leq 2 M\right\}$. Then the discrete FBVP (3.1) has a solution provided

$$
\begin{equation*}
Q \leq \frac{M \Gamma(\alpha+1) \Gamma(L+1)}{2 \Gamma(\alpha+L+1)} . \tag{3.13}
\end{equation*}
$$

Proof Let $M>0$ and we define the set $S=\{w(x) \in \mathbb{E}:\|w\| \leq 2 M\}$. To prove this theorem, we only need to show that $T$ maps $S$ in $S$. For $w(x) \in S$, we have

$$
\begin{aligned}
&|(T w)(x)|= \left\lvert\, p(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, w(\ell+\alpha-1))\right. \\
& \left.-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, w(\ell+\alpha-1)) \right\rvert\, \\
& \leq|p(x)|+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}|\Psi(\ell+\alpha-1, w(\ell+\alpha-1))| \\
&+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)}|\Psi(\ell+\alpha-1, w(\ell+\alpha-1))| \\
& \leq M+\frac{Q}{\Gamma(\alpha)}\left[\sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}+\sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)}\right] \\
& \leq M+\frac{Q}{\Gamma(\alpha+1)}\left[\frac{\Gamma(x+1)}{\Gamma(x+1-\alpha)}+\frac{\Gamma(\alpha+L+1)}{\Gamma(L+1)}\right], \\
&\|T w\| \leq M+2 Q \frac{\Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)} .
\end{aligned}
$$

From (3.13), we have $\|T w\| \leq 2 M$, which implies that $T$ maps $S$ in $S$. Thus, $T$ has at least one fixed point which is a solution to the BVP (3.1) according to the Brouwer fixed point theorem.

## 4 Stability analysis

In this section, the stability analysis is presented for the following discrete FBVP:

$$
\left\{\begin{array}{l}
{ }_{0}^{R L} \Delta_{x}^{\alpha} w(x)=\Psi(x+\alpha-1, w(x+\alpha-1)), \quad 1<\alpha \leq 2,  \tag{4.1}\\
\Delta w(\alpha-2)=A, \quad w(\alpha+L)=B,
\end{array}\right.
$$

for $x \in[0, L]_{\mathbb{N}_{0}}$, where ${ }_{0}^{R L} \Delta_{x}^{\alpha}$ is the Riemann-Liouville fractional difference operator (RLFDO). We now investigate the solution of (4.1), provided that the solution exists.

Theorem 4.1 Let $1<\alpha \leq 2$ and $\Psi:[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}} \rightarrow \mathbb{R}$ be given. A solution to the discrete FBVP

$$
\left\{\begin{array}{l}
{ }_{0}^{R L} \Delta_{x}^{\alpha} w(x)=\Psi(x+\alpha-1), \quad x \in[0, L]_{\mathbb{N}_{0}}  \tag{4.2}\\
\Delta w(\alpha-2)=A, \quad w(\alpha+L)=B,
\end{array}\right.
$$

has the form

$$
\begin{align*}
w(x)= & q(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1) \\
& +\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1) \tag{4.3}
\end{align*}
$$

where $u(x)=\frac{\Gamma(L+2)}{\Gamma(\alpha+L+1)}\left[\beta \Gamma(\alpha)\left(x^{(\alpha-2)}-\frac{x^{(\alpha-1)}}{L+2}\right)-x^{(\alpha-1)}\right]$ such that $\beta=\frac{L+2}{(\alpha-2)(L+2) \Gamma(\alpha-1)-\Gamma(\alpha)}$ and $q$ is the unique solution to the discrete FBVP

$$
\left\{\begin{array}{l}
{ }_{0}^{R L} \Delta_{x}^{\alpha} q(x)=0,  \tag{4.4}\\
\Delta q(\alpha-2)=A, \quad q(\alpha+L)=B
\end{array}\right.
$$

where ${ }_{0}^{R L} \Delta_{x}^{\alpha}$ is the RLFDO.

Proof Let $q$ be a solution to (4.4) defined on $[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$. Using Definition 2.1 and Lemma 2.4, we get

$$
\begin{equation*}
q(x)=C_{4} x^{(\alpha-1)}+C_{5} x^{(\alpha-2)}, \tag{4.5}
\end{equation*}
$$

for some $C_{4}, C_{5} \in \mathbb{R}$. Applying the operator $\Delta$ on both sides in (4.5), we get

$$
\Delta u(x)=C_{4}(\alpha-1) x^{(\alpha-2)}+C_{5}(\alpha-2) x^{(\alpha-3)} .
$$

Using the boundary conditions $\Delta q(\alpha-2)=A$ and $q(\alpha+L)=B$, we deduce that

$$
C_{4}=\frac{B \Gamma(L+2)}{\Gamma(\alpha+L+1)}\left[1+\frac{\beta \Gamma(\alpha)}{L+2}\right]-\frac{A \beta}{L+2}, \quad \text { and } \quad C_{5}=\beta A-\frac{\beta B \Gamma(\alpha) \Gamma(L+2)}{\Gamma(\alpha+L+1)} .
$$

Substituting the values of $C_{4}$ and $C_{5}$ in $q$, we obtain

$$
\begin{equation*}
q(x)=\frac{B \Gamma(L+2)}{\Gamma(\alpha+L+1)} x^{(\alpha-1)}+\beta\left(A-\frac{\Gamma(\alpha) \Gamma(L+2)}{\Gamma(\alpha+L+1)} B\right)\left[x^{(\alpha-2)}-\frac{x^{(\alpha-1)}}{L+2}\right] . \tag{4.6}
\end{equation*}
$$

Assume that $w$ is a solution to (4.2). From Lemma 2.4, we obtain a general solution for (4.2) as

$$
\begin{equation*}
w(x)=\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1)+C_{6} x^{(\alpha-1)}+C_{7} x^{(\alpha-2)} \tag{4.7}
\end{equation*}
$$

for some $C_{6}, C_{7} \in \mathbb{R}$. Applying the operator $\Delta$ on both sides in (4.7), we get

$$
\begin{aligned}
\Delta w(x)= & \frac{1}{\Gamma(\alpha-1)} \sum_{\ell=0}^{x-\alpha+1}(x-\ell-1)^{(\alpha-2)} \Psi(\ell+\alpha-1) \\
& +C_{6}(\alpha-1) x^{(\alpha-2)}+C_{7}(\alpha-2) x^{(\alpha-3)} .
\end{aligned}
$$

In view of $\Delta w(\alpha-2)=A$ and $w(\alpha+L)=B$, we get the value of $C_{6}$ and $C_{7}$ as follows:

$$
\begin{aligned}
C_{6}= & \frac{\Gamma(L+2)}{\Gamma(\alpha+L+1)}\left[1+\frac{\beta \Gamma(\alpha)}{L+2}\right]\left(B-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1)\right) \\
& -\frac{A \beta}{L+2}
\end{aligned}
$$

and

$$
C_{7}=\beta A-\frac{\beta \Gamma(\alpha) \Gamma(L+2)}{\Gamma(\alpha+L+1)}\left[B-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1)\right] .
$$

Substituting the values of $C_{6}, C_{7}$ and $q$ into (4.7), we obtain $w$ in the form

$$
\begin{align*}
w(x)= & q(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1)  \tag{4.8}\\
& +\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1) .
\end{align*}
$$

The definitions of Ulam stability for fractional difference equation are introduced in the sequel on the basis of $[32,37]$.

Definition 4.2 If, for every function $v \in \mathbb{E}$ of

$$
\begin{equation*}
\left|{ }_{0}^{R L} \Delta_{x}^{\alpha} v(x)-\Psi(x+\alpha-1, v(x+\alpha-1))\right| \leq \epsilon, \quad x \in[0, L]_{\mathbb{N}_{0}}, \tag{4.9}
\end{equation*}
$$

where $\epsilon>0$, there exists a solution $w \in \mathbb{E}$ of (4.1) and positive constant $\delta_{1}>0$ such that

$$
\begin{equation*}
|v(x)-w(x)| \leq \delta_{1} \epsilon, \quad x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}, \tag{4.10}
\end{equation*}
$$

then the discrete FBVP (4.1) is said to be Hyers-Ulam stable.

Definition 4.3 If, for every function $v \in \mathbb{E}$ of

$$
\begin{equation*}
\left|{ }_{0}^{R L} \Delta_{x}^{\alpha} v(x)-\Psi(x+\alpha-1, v(x+\alpha-1))\right| \leq \epsilon \Phi(x+\alpha-1), \quad x \in[0, L]_{\mathbb{N}_{0}}, \tag{4.11}
\end{equation*}
$$

where $\epsilon>0$, there is a solution $w \in \mathbb{E}$ of (4.1) and positive constant $\delta_{2}>0$ such that

$$
\begin{equation*}
|v(x)-w(x)| \leq \delta_{2} \epsilon \Phi(x+\alpha-1), \quad x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}, \tag{4.12}
\end{equation*}
$$

then the discrete FBVP (4.1) is said to be Hyers-Ulam-Rassias stable.

Remark 4.4 A function $v \in \mathbb{E}$ is a solution to (4.9) if and only if there exists $g:[\alpha-2, \alpha+$ $L] \rightarrow \mathbb{R}$ satisfying

$$
\begin{aligned}
& \left(H_{2}\right)|g(x+\alpha-1)| \leq \epsilon, x \in[0, b]_{\mathbb{N}_{0}}, \\
& \left(H_{3}\right){ }_{0}^{R L} \Delta_{x}^{\alpha} v(x)=\Psi(x+\alpha-1, v(x+\alpha-1))+g(x+\alpha-1), x \in[0, b]_{\mathbb{N}_{0}} .
\end{aligned}
$$

A similar remark can be formulated for inequality (4.11).

Lemma 4.5 If $v$ solves (4.9), then

$$
\begin{aligned}
& \left\lvert\, v(x)-q(x)-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1))\right. \\
& \\
& \left.\quad-\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \right\rvert\, \\
& \leq \frac{\epsilon \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)} .
\end{aligned}
$$

Proof If $v$ solves the inequality (4.9), then from Remark 4.4 and Lemma 2.4, the solution to $\left(H_{3}\right)$ satisfies

$$
\begin{aligned}
v(x)= & q(x)+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \\
& +\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \\
& +\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} g(\ell+\alpha-1) .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mid \nu(x) & -q(x)-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \\
& \left.-\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, \nu(\ell+\alpha-1)) \right\rvert\, \\
= & \left|\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} g(\ell+\alpha-1)\right| \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}|g(\ell+\alpha-1)| \\
& \leq \frac{\epsilon}{\Gamma(\alpha)} \frac{\Gamma \Gamma(x+1)}{\alpha \Gamma(x+1-\alpha)} \\
& \leq \frac{\epsilon \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)} .
\end{aligned}
$$

Theorem 4.6 Suppose that the hypothesis $\left(H_{1}\right)$ together with the inequality (4.9) is satisfied. Then the discrete FBVP (4.1) is Hyers-Ulam stable provided that

$$
\begin{equation*}
K<\frac{\Gamma(\alpha+1) \Gamma(L+1)}{2 \Gamma(\alpha+L+1)} . \tag{4.13}
\end{equation*}
$$

Proof With the help of solution (4.8) and Lemma 4.5, for $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$, we have

$$
\begin{aligned}
& \mid v(x)-w(x) \mid \\
& \leq \left\lvert\, v(x)-q(x)-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, w(\ell+\alpha-1))\right. \\
& \left.-\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, w(\ell+\alpha-1)) \right\rvert\, \\
& \leq \left\lvert\, v(x)-q(x)-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1))\right. \\
& \left.-\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \right\rvert\, \\
&+\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \\
& \quad \times|\Psi(\ell+\alpha-1, v(\ell+\alpha-1))-\Psi(\ell+\alpha-1, w(\ell+\alpha-1))| \\
& \quad+\frac{|u(x)|}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \\
& \quad \times|\Psi(\ell+\alpha-1, v(\ell+\alpha-1))-\Psi(\ell+\alpha-1, w(\ell+\alpha-1))| .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|v(x)-w(x)| \leq & \frac{\epsilon \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)} \\
& +\frac{K}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}|v(\ell+\alpha-1)-w(\ell+\alpha-1)| \\
& +\frac{K|u(x)|}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)}|v(\ell+\alpha-1)-w(\ell+\alpha-1)| \\
\leq & \frac{\epsilon \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)}+\frac{K\|v-w\|}{\Gamma(\alpha)}\left[\frac{\Gamma(x+1)}{\alpha \Gamma(x+1-\alpha)}\right] \\
& +\frac{K\|v-w\||u(x)|}{\Gamma(\alpha)}\left[\frac{\Gamma(\alpha+L+1)}{\alpha \Gamma(L+1)}\right] .
\end{aligned}
$$

Therefore, we are left with

$$
\|v-w\| \leq \frac{\epsilon \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)}+\frac{2 K \Gamma(\alpha+L+1)}{\Gamma(\alpha+1)(L+1)}\|v-w\| .
$$

From the above inequalities, we have $\|v-w\| \leq \delta_{1} \epsilon$, where $\delta_{1}=\frac{\Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)-2 K \Gamma(\alpha+L+1)}>0$. Thus, Eq. (4.1) is Hyers-Ulam stable.

We assume the following.
$\left(H_{4}\right)$ Let $\Phi \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}} \rightarrow \mathbb{R}^{+}$be an increasing function, and there exists a constant $\lambda>0$ such that

$$
\frac{\epsilon}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Phi(\ell+\alpha-1) \leq \lambda \epsilon \Phi(x+\alpha-1), \quad x \in[0, L]_{\mathbb{N}_{0}} .
$$

Lemma 4.7 If $v$ solves (4.11), then

$$
\begin{aligned}
\mid v(x) & -q(x)-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \\
& \left.-\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \right\rvert\, \\
\leq & \lambda \in \Phi(x+\alpha-1) .
\end{aligned}
$$

Proof From inequality (4.11), for $x \in[\alpha-2, \alpha+L]_{\mathbb{N}_{\alpha-2}}$, we can find a function ${ }_{0}^{R L} \Delta_{x}^{\alpha} v(x)=$ $\Psi(x+\alpha-1, v(x+\alpha-1))+g(x+\alpha-1)$ and $|g(x+\alpha-1)| \leq \epsilon \Phi(x+\alpha-1)$. It follows that

$$
\begin{aligned}
& \mid v(x)-q(x)-\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \\
& \left.-\frac{u(x)}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)} \Psi(\ell+\alpha-1, v(\ell+\alpha-1)) \right\rvert\, \\
&=\left|\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} g(\ell+\alpha-1)\right| \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}|g(\ell+\alpha-1)| \\
& \leq \frac{\epsilon}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)} \Phi(\ell+\alpha-1) \\
& \leq \lambda \epsilon \Phi(x+\alpha-1) .
\end{aligned}
$$

Theorem 4.8 If the hypotheses $\left(H_{1}\right),\left(H_{4}\right)$ and the inequality (4.13) are satisfied, then a discrete FBVP (4.1) is Hyers-Ulam-Rassias stable.

Proof With the help of Lemma 2.6, Lemma 4.7 and solution (4.8), we obtain

$$
\|v-w\| \leq \delta_{2} \epsilon \Phi(x+\alpha-1)
$$

where $\delta_{2}=\frac{\lambda \Gamma(\alpha+1) \Gamma(L+1)}{\Gamma(\alpha+1) \Gamma(L+1)-2 K \Gamma(\alpha+L+1)}>0$. Thus (4.1) is Hyers-Ulam-Rassias stable.

## 5 Applications

Some suitable examples are presented to validate the theoretical results and numerical solutions to the discrete FBVP (3.1) and (4.1) with different applications by using Caputo and Riemann-Liouville fractional difference operators. Computational aspects regarding numerical values and diagrams are executed with MATLAB.

### 5.1 Steady-state heat equation

Consider the mathematical model of heat flow in a rod made out of a heat-conducting material, subject to an external heat source along its length and some boundary conditions at each end. Let $w(x, t)$ denote the temperature distribution at the real position $x$ varying only with any real time $t$, where $a<x<b$ along some finite length of the rod. The general form of the one dimensional non-homogeneous heat equation with heat generation [4] is given as

$$
\begin{equation*}
w_{t}(x, t)=\left(k(x) w_{x}(x, t)\right)_{x}+\psi(x, t) \tag{5.1}
\end{equation*}
$$

where $k(x)$ is the coefficient of heat conduction, which may vary with $x$, and $\psi(x, t)$ is the heat source (or sink). Equation (5.1) is often called the diffusion equation. The heat equation (5.1) reduces to

$$
\begin{equation*}
w_{t}(x, t)=k w_{x x}(x, t)+\psi(x, t), \tag{5.2}
\end{equation*}
$$

subject to the initial condition

$$
\begin{equation*}
w(x, 0)=w(x), \tag{5.3}
\end{equation*}
$$

and the Neumann-Dirichlet boundary conditions

$$
\begin{equation*}
w_{x}(a, t)=A(t), \quad \text { and } \quad w(b, t)=B(t), \tag{5.4}
\end{equation*}
$$

where the rod is assumed to be insulated at one end and the temperature is specified at the other end. In general, we expect the temperature distribution to change with time. However, if $\psi(x, t), A(t)$ and $B(t)$ are all time independent, Eqs. (5.2), (5.3) and (5.4) are called initial boundary value problems.
Now we are interested in computing the steady-state solution to the above problem and by setting $w_{t}=0$ in (5.2), we obtain the steady-state heat equation in $x$ as the following ordinary differential equation:

$$
\begin{cases}w^{\prime \prime}(x)=\Psi(x), & a<x<b  \tag{5.5}\\ w^{\prime}(a)=A, & w(b)=B\end{cases}
$$

where $\Psi(x)=-\frac{\psi(x)}{k}, a, b, A$ and $B$ are real valued constants. The above steady-state heat equation with Neumann-Dirichlet boundary conditions (5.5) is transformed to the discrete FBVPs (3.2) and (4.2).

Example 5.1 Suppose that $\Psi(x)=x^{(8)}, L=1, A=0$ and $B=0$ where we have different fractional orders $\alpha$ like $\alpha=1.1, \alpha=1.4, \alpha=1.7$ and $\alpha=2$. Then the discrete fractional order steady-state boundary value problems (3.2) and (4.2) become

$$
\left\{\begin{array}{l}
{ }_{0}^{*} \Delta_{x}^{\alpha} w(x)=(x+\alpha-1)^{(8)}, \quad x \in[0,1]_{\mathbb{N}_{0}}  \tag{5.6}\\
\Delta w(\alpha-2)=0, \quad w(\alpha+1)=0
\end{array}\right.
$$

where ${ }_{0}^{*} \Delta_{x}^{\alpha}={ }_{0}^{C} \Delta_{x}^{\alpha}$ or ${ }_{0}^{*} \Delta_{x}^{\alpha}={ }_{0}^{R L} \Delta_{x}^{\alpha}$. The solutions of (5.6) can be formulated for different values of $\alpha$ by using the procedure discussed in the previous sections. Indeed, by using Definition 2.1 and Lemma 2.7 in (3.3) and (4.3) we obtain

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}(\ell+\alpha-1)^{(8)} & =\Delta^{-\alpha}(x+\alpha-1)^{(8)} \\
& =\frac{\Gamma(9)}{\Gamma(9+\alpha)}(x+\alpha-1)^{(\alpha+8)} \\
\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{x-\alpha}(x-\ell-1)^{(\alpha-1)}(\ell+\alpha-1)^{(8)} & =\frac{\Gamma(9)}{\Gamma(9+\alpha)} \cdot \frac{\Gamma(x+\alpha)}{\Gamma(x-8)}
\end{aligned}
$$

Similarly, we find

$$
\frac{1}{\Gamma(\alpha)} \sum_{\ell=0}^{L}(\alpha+L-\ell-1)^{(\alpha-1)}(\ell+\alpha-1)^{(8)}=\frac{\Gamma(9)}{\Gamma(9+\alpha)} \cdot \frac{\Gamma(2 \alpha+1)}{\Gamma(\alpha-7)} .
$$

If ${ }_{0}^{*} \Delta_{x}^{\alpha}={ }_{0}^{C} \Delta_{x}^{\alpha}$ then the analytical solution to (5.6) has the form

$$
w(x)=p(x)+\frac{\Gamma(9)}{\Gamma(9+\alpha)}\left[\frac{\Gamma(x+\alpha)}{\Gamma(x-8)}\right]-\frac{\Gamma(9)}{\Gamma(9+\alpha)}\left[\frac{\Gamma(2 \alpha+1)}{\Gamma(\alpha-7)}\right]
$$

where $p(x)$ is defined in Theorem 3.1. If ${ }_{0}^{*} \Delta_{x}^{\alpha}={ }_{0}^{R L} \Delta_{x}^{\alpha}$ then the analytical solution to (5.6) is

$$
w(x)=q(x)+\frac{\Gamma(9)}{\Gamma(9+\alpha)}\left[\frac{\Gamma(x+\alpha)}{\Gamma(x-8)}\right]+u(x) \frac{\Gamma(9)}{\Gamma(9+\alpha)}\left[\frac{\Gamma(2 \alpha+1)}{\Gamma(\alpha-7)}\right]
$$

where $q$ and $u$ are defined as in Theorem 4.1. For the values of the fractional orders $\alpha \in$ $(1,2]$ and $x \in[0,1]$, the solutions with respect to the Riemann-Liouville operator exhibit better results than the Caputo difference operator, as seen in Fig. 1(a) and Fig. 1(b) and as shown in Table 1 and Table 2. When $\alpha=2$, the graphs of both solutions are provided in Fig. 1(c). Note that both curves are overlapping. In three dimensions the solution surfacing over different values of $\alpha$ and $x$ are shown in Fig. 2.

Example 5.2 Suppose that $\Psi(x)=x^{(6)}, L=4, A=0$ and $B=1$ where we have different fractional orders $\alpha$ like $\alpha=1.1, \alpha=1.4, \alpha=1.7$ and $\alpha=2$. Then the discrete fractional order steady-state boundary value problems (3.2) and (4.2) become

$$
\left\{\begin{array}{l}
{ }_{0}^{*} \Delta_{x}^{\alpha} w(x)=(x+\alpha-1)^{(6)}, \quad x \in[0,4]_{\mathbb{N}_{0}},  \tag{5.7}\\
\Delta x(\alpha-2)=0, \quad x(\alpha+4)=1,
\end{array}\right.
$$



Figure 1 Steady-state solutions curves for $x$ using (a) CFDO, (b) RLFDO and (c) comparison of steady-state solutions at $\alpha=2$ of Example 5.1

Table 1 Steady-state solutions to fractional order $\alpha \in(1,2]$ varying $x \in[0,1]$ using CFDO

| $\alpha$ | $x$ |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :---: | :---: |
|  | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |  |
| 1.1 | -12.6308 | 379.5102 | 398.6517 | 265.5418 | 108.0197 | -12.6308 |  |
| 1.4 | -21.9303 | 175.2079 | 196.3280 | 132.6556 | 47.8625 | -21.9303 |  |
| 1.7 | -11.8009 | 94.3783 | 110.9082 | 78.4466 | 30.3342 | -11.8009 |  |
| 2 | 0 | 60.3317 | 72.2430 | 54.8485 | 26.3561 | 0 |  |

Table 2 Steady-state solutions to fractional order $\alpha \in(1,2]$ varying $x \in[0,1]$ using RLFDO

| $\alpha$ | $x$ |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | ---: | :---: |
|  | 0 | 0.2 | 0.4 | 0.6 | 1 |  |
| 1.1 | -11.6531 | 380.3625 | 399.3875 | 266.1692 | 108.5459 | -12.1991 |
| 1.4 | -17.0644 | 179.7338 | 200.4577 | 136.3639 | 51.1397 | -19.0852 |
| 1.7 | -9.3400 | 96.8351 | 113.2805 | 80.6855 | 32.4082 | -9.9130 |
| 2 | 0 | 60.3317 | 72.2430 | 54.8485 | 26.3561 | 0 |

where ${ }_{0}^{*} \Delta_{x}^{\alpha}$ is either ${ }_{0}^{C} \Delta_{x}^{\alpha}$ or ${ }_{0}^{R L} \Delta_{x}^{\alpha}$. By using a similar method to Example 5.1, we obtain the steady-state solutions to this problem. If ${ }_{0}^{*} \Delta_{x}^{\alpha}={ }_{0}^{C} \Delta_{x}^{\alpha}$ then the analytical solution to (5.7) is

$$
w(x)=p(x)+\frac{\Gamma(7)}{\Gamma(7+\alpha)}\left[\frac{\Gamma(x+\alpha)}{\Gamma(x-6)}\right]-\frac{\Gamma(7)}{\Gamma(7+\alpha)}\left[\frac{\Gamma(2 \alpha+4)}{\Gamma(\alpha-2)}\right]
$$

where $p$ is defined in Theorem 3.1. If ${ }_{0}^{*} \Delta_{x}^{\alpha}={ }_{0}^{R L} \Delta_{x}^{\alpha}$ then the analytical solution to (5.7) is

$$
w(x)=q(x)+\frac{\Gamma(7)}{\Gamma(7+\alpha)}\left[\frac{\Gamma(x+\alpha)}{\Gamma(x-6)}\right]+u(x) \frac{\Gamma(7)}{\Gamma(7+\alpha)}\left[\frac{\Gamma(2 \alpha+4)}{\Gamma(\alpha-2)}\right],
$$



Figure 2 Surface plots of corresponding steady-state solutions to different values of $\alpha$ and $x$ using (a) CFDO and (b) RLFDO of Example 5.1


Figure 3 Steady-state solutions curve varies x using (a) CFDO and (b) RLFDO of Example 5.2

Table 3 Steady-state solutions to fractional order $\alpha \in(1,2]$ varying $x \in[0,4]$ using CFDO

| $\alpha$ | $x$ |  |  |  |  |  |  |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | :---: |
|  | 0 | 0.8 | 1.6 |  |  |  |  |
| 1.1 | 2.8703 | 6.4278 | 0.4364 | 4.4414 | 1.9626 | 2.8703 |  |
| 1.4 | 9.4796 | 11.6840 | 7.7750 | 10.6799 | 8.7377 | 9.4796 |  |
| 1.7 | 13.0258 | 14.4482 | 11.8027 | 13.9575 | 12.4130 | 13.0258 |  |
| 2 | 1.0000 | 1.9489 | 0.1038 | 1.7332 | 0.4891 | 1.0000 |  |

Table 4 Steady-state solutions to fractional order $\alpha \in(1,2]$ varying $x \in[0,4]$ using RLFDO

| $\alpha$ | $x$ |  |  |  |  |  |
| :--- | :--- | :--- | :--- | ---: | ---: | ---: |
|  | 0 | 0.8 | 1.6 | 2.4 | 3.2 |  |
| 1.1 | 2.4241 | 6.0714 | 0.1503 | 4.2122 | 1.7813 | 2.7303 |
| 1.4 | 5.8591 | 8.5303 | 5.1595 | 8.5738 | 7.1033 | 8.2826 |
| 1.7 | 7.8519 | 9.3432 | 7.1890 | 9.9419 | 9.0286 | 10.2804 |
| 2 | 1.0000 | 1.2822 | -1.2295 | -0.2668 | -2.1776 | -2.3333 |

where $q$ and $u$ are defined as in Theorem 4.1. For the values of the fractional orders $\alpha \in$ $(1,2]$ and $x \in[0,4]$, it is realized that the solutions with respect to the Riemann-Liouville operator exhibit better results than the Caputo difference operator, as seen in Fig. 3(a) and Fig. 3(b) and as shown in Table 3 and Table 4. As we see in Fig. 3, the trajectories of both solutions do not overlap for fractional orders $\alpha \in(1,2]$ with $x$ increasing. In three dimensions, the solution surfaces over different values of $\alpha$ and $x$ are shown in Fig. 4.


Figure 4 Surface plots of corresponding steady-state solutions to different values of $\alpha$ and $x$ using (a) CFDO and (b) RLFDO of Example 5.2

### 5.2 Simple gravity pendulum

In [4], the following differential equation, which represents the motion of a simple pendulum, is considered:

$$
\begin{equation*}
\frac{d^{2} w}{d x^{2}}+\frac{g}{\gamma} \sin (w(x))=0, \tag{5.8}
\end{equation*}
$$

where $g$ is the acceleration due to the gravitational constant, $\gamma$ is the length of the pendulum and $w$ is the angular displacement.

Example 5.3 Let us consider the parameters $A=0, B=1, L=1$ with fractional order $\alpha=$ 1.3. Then the discrete fractional order boundary value problems (3.1) and (4.1) for the simple pendulum become

$$
\left\{\begin{array}{l}
\left\{_{0}^{*} \Delta_{x}^{1.3} w(x)=\Psi(x+0.3, w(x+0.3)), \quad x \in[0,1]_{\mathbb{N}_{0}},\right.  \tag{5.9}\\
\Delta w(-0.7)=0, \quad w(2.3)=1,
\end{array}\right.
$$

where ${ }_{0}^{*} \Delta_{x}^{1.3}={ }_{0}^{C} \Delta_{x}^{1.3}$ or ${ }_{0}^{*} \Delta_{x}^{1.3}={ }_{0}^{R L} \Delta_{x}^{1.3}$ and $\Psi(x+0.3, w(x+0.3))=-\frac{g}{\gamma} \sin (w(x+0.3))$. For the values of $g=9.8 \mathrm{~m} / \mathrm{s}^{2}, \gamma=60 \mathrm{~m}$, we choose $K=\frac{g}{\gamma}=0.1633$. If ${ }_{0}^{*} \Delta_{x}^{1.3}={ }_{0}^{C} \Delta_{x}^{1.3}$, in this case, inequality (3.12) takes the form

$$
K\left[\frac{2 \Gamma(\alpha+L+1)}{\Gamma(\alpha+1) \Gamma(L+1)}\right] \approx 0.7513<1 .
$$

Therefore, from Theorem 3.2, we conclude that the boundary value problem (5.9) has a unique solution. Furthermore, if ${ }_{0}^{*} \Delta_{x}^{1.3}={ }_{0}^{R L} \Delta_{x}^{1.3}$, we obtain

$$
\frac{\Gamma(\alpha+1) \Gamma(L+1)}{2 \Gamma(\alpha+L+1)} \approx 0.2174
$$

If $K=0.1633<0.2174$ and the inequality

$$
\left|{ }_{0}^{R L} \Delta_{x}^{1.3} v(x)-\Psi(x+0.3, v(x+0.3))\right| \leq \epsilon, \quad x \in[0,1]_{\mathbb{N}_{0}},
$$

holds, then the boundary value problem (5.9) is Hyers-Ulam stable by Theorem 4.6.

### 5.3 Temperature distribution equation

In $[8,10,16]$, the authors considered the following mathematical model, which describes the temperature distribution in lumped system of combined convection-radiation in a slab made of materials with variable thermal conductivity:

$$
\left\{\begin{array}{l}
D^{\alpha} w(x)-\eta w^{4}(x)=0, \quad 1<\alpha \leq 2, x \in[0,1]  \tag{5.10}\\
w^{\prime}(0)=0, \quad w(1)=1
\end{array}\right.
$$

where $w=\frac{\mathbb{T}-\mathbb{T}_{a}}{\mathbb{T}_{i}-\mathbb{T}_{a}}$ and $x=\frac{t}{V \rho c_{a} / \mathbb{S} h}$ are dimensionless temperature and time, respectively. The parameter $\eta=\left(\mathbb{T}-\mathbb{T}_{a}\right) \xi$, where $V, \mathbb{S}, \rho, c, \mathbb{T}_{i}, \mathbb{T}_{a}, c_{a}$ and $h$ are the volume, surface area, density, specific heat, the initial temperature, temperature of the convection environment, specific heat at temperature $\mathbb{T}_{a}$ and heat transfer coefficient of the lumped system, respectively.

Example 5.4 Suppose that $\alpha=1.4, L=1, \eta=4 \times 10^{-10}$ and $M=250$ with $\Psi(x, w)=\eta w^{4}(x)$. Then we obtain the discrete fractional order heat transfer boundary value problem (5.10) and it takes the form

$$
\left\{\begin{array}{l}
{ }_{0}^{C} \Delta_{x}^{1.4} w(x)=\eta w^{4}(x+0.4), \quad x \in[0,1]  \tag{5.11}\\
\Delta w(-0.6)=0, \quad w(2.4)=1
\end{array}\right.
$$

The main result of (5.11) is discussed in Sect. 2. The Banach space is

$$
\mathbb{E}:=\left\{w(x) \mid[-0.6,2.4]_{\mathbb{N}_{-0.6}} \rightarrow \mathbb{R},\|w\| \leq 500\right\} .
$$

We note that

$$
\frac{M \Gamma(\alpha+1) \Gamma(L+1)}{2 \Gamma(\alpha+L+1)}=250(0.2083) \approx 52.0833
$$

It is clear that $|\Psi(x, w)| \leq 25<52.0833$, whenever $w \in[-500,500]$. Therefore by Theorem 3.3, we conclude that the boundary value problem (5.11) has at least one solution.

## 6 Conclusion

This work made a study of a discrete fractional boundary value problem with the Caputo and Riemann-Liouville difference operators. The existence and uniqueness of solutions and various types of Hyers-Ulam stability are discussed for the addressed problem based on the properties of fractional operators, the contraction mapping principle and the Brouwer fixed point theorem. Theoretical results are complemented with suitable examples accompanied by numerical solutions for different values of $\alpha$ and $x$. The discrete FBVP appearing in mathematical models of engineering applications is solved via the Riemann-Liouville and Caputo difference operators. Subsequently, the dynamics exhibited by RLFDO and CFDO for the proposed models is illustrated through diagrams. Furthermore, the occurrence of overlapping or non-overlapping cases is presented in the graphical representations.

Results obtained in the present paper can be considered as a contribution to the developing field of discrete fractional boundary value problems describing mathematical and physical applications.

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## Authors' contributions

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