Bakery and Mohamed *Journal of Inequalities and Applications* (2020) 2020:219
https://doi.org/10.1186/s13660-020-02484-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

Eigenvalues of s-type operators on $C(p)$ equipped with a pre-quasi norm

Awad A. Bakery[1,2*] and Elsayed A.E. Mohamed[1,3]

*Correspondence:
aabhassan@uj.edu.sa;
awad_bakery@yahoo.com;
awad_bakry@hotmail.com
[1]University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Cairo, 11566, Abbassia, Egypt
Full list of author information is available at the end of the article

Abstract

We investigate some new topological properties of the multiplication operator on $C(p)$ defined by Lim (Tamkang J. Math. 8(2):213–220, 1977) equipped with the pre-quasi-norm and the pre-quasi-operator ideal formed by this sequence space and s-numbers.

MSC: 46B10; 47B10; 46C05; 46E05; 46E15; 46E30

Keywords: Pre-quasi-norm; Multiplication operator; Fredholm operator; Approximable operator; Simple Banch space

1 Introduction

Throughout the article, we denote the space of all bounded linear operators from a Banach space X into a Banach space Y by $L(X,Y)$ and if $X = Y$, we write $L(X)$, the space of all complex sequences by w, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, $\mathbb{N} = \{0,1,2,\ldots\}$, the space of convergent complex sequences to zero by C_0, the space of bounded complex sequences by ℓ_∞ and all sequences whose its elements are complex by $\mathbb{C}^{\mathbb{N}}$. For a sequence (p_n) with $\inf_n p_n > 0$, Lim (see [1]) defined and studied the sequence space $C(p)$ as follows:

$$C(p) = \{x = (x_n) \in \omega : \rho(\beta x) < \infty \text{ for some } \beta > 0\},$$
$$\text{where } \rho(x) = \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n}\right)^{p_n}.$$

The space $C(p)$ is a Banach space with the Luxemburg norm

$$\|x\| = \inf\left\{\beta > 0 : \rho\left(\frac{x}{\beta}\right) \le 1\right\}.$$

If (p_n) is bounded, we can simply write

$$C(p) = \left\{(x_i) \in \omega : \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n}\right)^{p_n} < \infty\right\}.$$

Remark 1.1 Taking $p_n = p$, for all $n \in \mathbb{N}$, then $C(p)$ is reduced by Lim (see [2]) to ces_p, he defined and determined its dual spaces and characterize some matrix classes.

The multiplication operators and operator ideals theorems give an importance in functional analysis, since it has numerous applications in fixed point theorem, geometry of Banach spaces, spectral theory and eigenvalue distributions theorem etc. For more details see [3–14]). On sequence spaces, Mursaleen and Noman (see [15]) investigated the compact operators on some difference sequence spaces, Komal and Gupta (see [16]) studied the multiplication operators on Orlicz spaces equipped with the Luxemburg norm and Komal et al. (see [17]) examined the multiplication operators on Cesáro sequence spaces equipped with the Luxemburg norm. Some of operator ideals in the class of Hilbert spaces or Banach spaces are generated by numeric sequence spaces. For example the ideal of compact operators is defined by the space of Kolmogorov numbers and C_0. Pietsch (see [18]), examined the quasi-ideals generated by ℓ_p $(0 < p < \infty)$ and the approximation numbers. He proved that the ideals of nuclear operators and of Hilbert–Schmidt operators between Hilbert spaces are defined by ℓ_1 and ℓ_2 respectively. He showed that the class of all finite rank operators are dense in the Banach quasi-ideal and the algebra $L(\ell_p)$, where $(1 \leq p < \infty)$ is simple Banach space. Pietsch (see [19]), showed that the quasi-Banach operator ideal formed by the sequence of approximation numbers is small. Makarov and Faried (see [20]) proved that, for any infinite dimensional Banach spaces X, Y and for any $q > p > 0$, $S^{\mathrm{app}}_{\ell_p}(X,Y)$ is strictly contained in $S^{\mathrm{app}}_{\ell_q}(X,Y)$. Faried and Bakery (see [21]) introduced the concept of pre-quasi-operator ideal which is more general than the usual classes of operator ideal, they studied the operator ideals formed by s-numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M, and showed that the operator ideal formed by approximation numbers and the previous sequence spaces is small under certain conditions. The aim of this article to study the concept of pre-quasi-norm on $C(p)$ which is more general than the usual norm, and give the conditions on $C(p)$ equipped with the pre-quasi-norm to be Banach space. We give the necessity and sufficient conditions on $C(p)$ equipped with the pre-quasi-norm such that the multiplication operator defined on $C(p)$ is a bounded, approximable, invertible, Fredholm and closed range operator. The components of pre-quasi-operator ideal generated by the sequence of s-numbers and $C(p)$ is strictly contained for different powers are determined. Furthermore, we give the sufficient conditions on $C(p)$ equipped with a pre-modular such that the pre-quasi-Banach operator ideal constructed by s-numbers and $C(p)$ is small, simple and its components is closed. Finally the class of all bounded linear operators, whose sequence of eigenvalues belongs to $C(p)$ is the same as the pre-quasi-operator ideal formed by the sequence of s-numbers and $C(p)$.

2 Definitions and preliminaries

The spaces of all finite rank, compact and approximable operators on X are denoted by $F(X)$, $L_c(X)$ and $\Psi(X)$, respectively.

Lemma 2.1 (see [18]) *Let $T \in L(X,Y)$. If T is not approximable, then there are operators $G \in L(X)$ and $B \in L(Y)$ such that $BTGe_k = e_k$, for all $k \in \mathbb{N}$.*

Definition 2.2 (see [18]) A Banach space X is called simple if the algebra $L(X)$ contains one and only one non-trivial closed ideal.

Theorem 2.3 (see [18]) *If X is infinite dimensional Banach space, we have*

$$F(X) \subsetneqq \Psi(X) \subsetneqq L_c(X) \subsetneqq L(X).$$

Definition 2.4 (see [22]) A bounded linear operator $D : X \to X$ is called Fredholm if D has closed range, dim(ker D) and co-dim(range D) are finite.

Definition 2.5 (see [23]) A class of linear sequence spaces X is called a special space of sequences (sss) if

(1) $e_i \in X$ for all $i \in \mathbb{N}$,
(2) if $u = (u_i) \in w$, $v = (v_i) \in X$ and $|u_i| \leq |v_i|$ for every $i \in \mathbb{N}$, then $u \in X$ "i.e. X is solid",
(3) if $(u_i)_{i=0}^{\infty} \in X$, then $(u_{[\frac{i}{2}]})_{i=0}^{\infty} \in X$, wherever $[\frac{i}{2}]$ means the integral part of $\frac{i}{2}$.

Definition 2.6 (see [23]) A subclass of the special space of sequences called a pre-modular (sss) if there is a function $\varrho : X \to [0, \infty[$ satisfying the following conditions:

(i) $\varrho(u) \geq 0$ for each $u \in X$ and $\varrho(u) = 0 \Leftrightarrow u = \theta$, where θ is the zero element of X,
(ii) there exists $L \geq 1$ such that $\varrho(\beta u) \leq L|\beta|\varrho(u)$ for all $u \in X$, and for any scalar β,
(iii) for some $K \geq 1$, $\varrho(u + v) \leq K(\varrho(u) + \varrho(v))$ for every $u, v \in X$,
(iv) if $|u_i| \leq |v_i|$ for all $i \in \mathbb{N}$, then $\varrho((u_i)) \leq \varrho((v_i))$,
(v) for some $K_0 \geq 1$, $\varrho((u_i)) \leq \varrho((u_{[\frac{i}{2}]})) \leq K_0\varrho((u_i))$,
(vi) the set of all finite sequences is ϱ-dense in X. This means that, for each $u = (u_i)_{i=o}^{\infty} \in X$, and for each $\varepsilon > 0$, there exists $s \in \mathbb{N}$ such that $\varrho((u_i)_{i=s}^{\infty}) < \varepsilon$,
(vii) there exists a constant $\xi > 0$ such that $\varrho(\beta, 0, 0, 0, \ldots) \geq \xi|\beta|\varrho(1, 0, 0, 0, \ldots)$ for any $\beta \in \mathbb{R}$.

The concept of pre-quasi-operator ideal which is more general than the usual classes of operator ideal.

Definition 2.7 (see [23]) A function $g : \Omega \to [0, \infty)$ is said to be a pre-quasi-norm on the ideal Ω if the following conditions holds:

(1) For all $T \in \Omega(X, Y)$, $g(T) \geq 0$ and $g(T) = 0$ if and only if $T = 0$,
(2) there exists a constant $M \geq 1$ such that $g(\lambda T) \leq M|\lambda|g(T)$, for all $T \in \Omega(X, Y)$ and $\lambda \in \mathbb{R}$,
(3) there exists a constant $K \geq 1$ such that $g(T_1 + T_2) \leq K[g(T_1) + g(T_2)]$, for all $T_1, T_2 \in \Omega(X, Y)$,
(4) there exists a constant $C \geq 1$ such that if $T \in L(X_0, X)$, $P \in \Omega(X, Y)$ and $R \in L(Y, Y_0)$ then $g(RPT) \leq C\|R\|g(P)\|T\|$, where X_0 and Y_0 are normed spaces.

Definition 2.8 (see [24]) An s-number function is a map defined on $L(X, Y)$ which associate to each operator $T \in L(X, Y)$ a non-negative scalar sequence $(s_n(T))_{n=0}^{\infty}$ assuming that the taking after states are verified:

(a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$, for $T \in L(X, Y)$,
(b) $s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in L(X, Y)$, $m, n \in \mathbb{N}$,
(c) ideal property: $s_n(RVT) \leq \|R\|s_n(V)\|T\|$ for all $T \in L(X_0, X)$, $V \in L(X, Y)$ and $R \in L(Y, Y_0)$, where X_0 and Y_0 are arbitrary Banach spaces,
(d) if $G \in L(X, Y)$ and $\lambda \in \mathbb{R}$, we obtain $s_n(\lambda G) = |\lambda|s_n(G)$,
(e) rank property: if rank$(T) \leq n$, then $s_n(T) = 0$ for each $T \in L(X, Y)$,

(f) norming property: $s_{r\geq n}(I_n) = 0$ or $s_{r<n}(I_n) = 1$, where I_n represents the unit operator on the n-dimensional Hilbert space ℓ_2^n.

There are several examples of s-numbers, we mention the following:

(1) The nth approximation number, denoted by $\alpha_n(T)$, is defined by

$$\alpha_n(T) = \inf\{\|T - B\| : B \in L(X, Y) \text{ and } \operatorname{rank}(B) \leq n\}.$$

(2) The nth Kolmogorov number, denoted by $d_n(T)$, is defined by

$$d_n(T) = \inf_{\dim Y \leq n} \sup_{\|x\|\leq 1} \inf_{y\in Y} \|Tx - y\|.$$

Notations 2.9 (see [21])

$$\begin{aligned}
&S_E := \{S_E(X, Y); X \text{ and } Y \text{ are Banach Spaces}\},\\
&\quad \text{where } S_E(X, Y) := \{T \in L(X, Y) : (s_i(T))_{i=0}^{\infty} \in E\},\\
&S_E^{\mathrm{app}} := \{S_E^{\mathrm{app}}(X, Y); X \text{ and } Y \text{ are Banach Spaces}\},\\
&\quad \text{where } S_E^{\mathrm{app}}(X, Y) := \{T \in L(X, Y) : (\alpha_i(T))_{i=0}^{\infty} \in E\},\\
&S_E^{\mathrm{Kol}} := \{S_E^{\mathrm{Kol}}(X, Y); X \text{ and } Y \text{ are Banach Spaces}\},\\
&\quad \text{where } S_E^{\mathrm{Kol}}(X, Y) := \{T \in L(X, Y) : (d_i(T))_{i=0}^{\infty} \in E\}.
\end{aligned}$$

Theorem 2.10 (see [25]) *Let E_ρ be a pre-modular (sss). Then the linear space $F(X, Y)$ is dense in $S_{E_\rho}(X, Y)$, where $g(T) = \rho(s_n(T)_{n=0}^{\infty})$.*

Theorem 2.11 (see [21]) *The function $g(T) = \varrho(s_i(T))_{i=0}^{\infty}$ is a pre-quasi norm on S_{E_ϱ}, where E_ϱ is a pre-modular (sss).*

During this paper, we define $e_n = \{0, 0, \dots, 1, 0, 0, \dots\}$ where 1 appears at the nth place for all $n \in \mathbb{N}$, the following well-known inequality (see [26]): $|a_i + b_i|^{p_i} \leq H(|a_i|^{p_i} + |b_i|^{p_i})$, where $H = \max\{1, 2^{\sup_i p_i - 1}\}$, $0 \leq p_i \leq \sup_i p_i < \infty$ and $a_i, b_i \in \mathbb{C}$ for every $i \in \mathbb{N}$ are used.

3 Main results

We give here the concept of pre-quasi-norm on $C(p)$ which is more general than the usual norm, and give the conditions on this sequence space equipped with the pre-quasi-norm to be a Banach space.

Definition 3.1 Let X be special space of sequences (sss). Assume there is a function $\varrho : X \to [0, \infty[$ satisfying the following conditions:

(i) $\varrho(x) \geq 0$ for each $x \in X$ and $\varrho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of X,
(ii) there exists $L \geq 1$ such that $\varrho(\lambda x) \leq L|\lambda|\varrho(x)$ for all $x \in X$, and for any scalar λ,
(iii) for some $K \geq 1$, we have $\varrho(x + y) \leq K(\varrho(x) + \varrho(y))$ for every $x, y \in X$.

The space X with ϱ is called pre-quasi-normed (sss) and denoted by X_ϱ, which gives a class more general than the quasi-normed space. If the space X is complete with ϱ, then X_ϱ, is called a pre-quasi-Banach (sss).

We express the following two theorems without verification, since they are clear.

Theorem 3.2 *Every quasi-norm (sss) is pre-quasi-norm (sss).*

Theorem 3.3 *Every pre-modular (sss) is pre-quasi-normed (sss).*

Theorem 3.4 *If (p_n) is an increasing bounded sequence with $p_0 > 1$, then the space $(C(p))_\varrho$ is a pre-modular Banach (sss), where $\varrho(x) = \sum_{n=0}^{\infty}(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n})^{p_n}$, for all $x \in C(p)$.*

Proof (1-i) let $x, y \in C(p)$. Since (p_n) is bounded, we have

$$\begin{aligned}\varrho(x+y) &= \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k+y_k|}{2^n}\right)^{p_n}\\ &\leq H\left(\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|y_k|}{2^n}\right)^{p_n}\right)\\ &= H\big(\varrho(x)+\varrho(y)\big) < \infty,\end{aligned}$$

then $x + y \in C(p)$.

(1-ii) let $\lambda \in \mathbb{C}$, $x \in C(p)$ and since (p_n) is bounded, we have

$$\begin{aligned}\varrho(\lambda x) &= \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|\lambda x_k|}{2^n}\right)^{p_n} \leq \sup_n|\lambda|^{p_n}\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n}\right)^{p_n}\\ &= \sup_n|\lambda|^{p_n}\varrho(x) < \infty.\end{aligned}$$

Then $\lambda x \in C(p)$, from (1-i) and (1-ii) $C(p)$ is a linear space.

Since (p_n) is an increasing bounded sequence with $p_0 > 1$, for all $m \in \mathbb{N}$ we have

$$\varrho(e_m) = \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}(e_m)_k}{2^n}\right)^{p_n} = \left(\frac{1}{2^{n_0}}\right)^{p_{n_0}} < \infty,$$

where $n_0 \in \mathbb{N}$ is such that $2^{n_0} - 1 \leq m \leq 2^{n_0+1} - 2$. Hence $e_m \in C(p)$, for all $m \in \mathbb{N}$.

(2) Let $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$ and $y \in C(p)$. Hence

$$\varrho(x) = \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n}\right)^{p_n} \leq \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|y_k|}{2^n}\right)^{p_n} = \varrho(y) < \infty,$$

we get $x \in C(p)$.

(3) Let $(x_n) \in C(p)$, we have

$$\begin{aligned}\varrho(x_{[\frac{n}{2}]}) &= \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_{[\frac{k}{2}]}|}{2^n}\right)^{p_n}\\ &= \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^{2n}-1}^{2^{2n+1}-2}|x_{[\frac{k}{2}]}|}{2^{2n}}\right)^{p_{2n}}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^{2n+1}-1}^{2^{2n+2}-2}|x_{[\frac{k}{2}]}|}{2^{2n+1}}\right)^{p_{2n+1}}\\ &\leq \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^{2n}-1}^{2^{2n+1}-2}|x_{[\frac{k}{2}]}|}{2^{2n}}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^{2n+1}-1}^{2^{2n+2}-2}|x_{[\frac{k}{2}]}|}{2^{2n+1}}\right)^{p_n}\end{aligned}$$

$$\begin{aligned}&\leq \sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^{2n}-1}^{2^{2n+1}-2}|x_{[\frac{k}{2}]}|}{2^n}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^{2n+1}-1}^{2^{2n+2}-2}|x_{[\frac{k}{2}]}|}{2^n}\right)^{p_n}\\&\leq \left(2H^2+3H\right)\sum_{n=0}^{\infty}\left(\frac{\sum_{k=2^n-1}^{2^{n+1}-2}|x_k|}{2^n}\right)^{p_n},\end{aligned}$$

then $(x_{[\frac{n}{2}]}) \in C(p)$.

(i) Clearly, $\varrho(x) \geq 0$ and $\varrho(x) = 0 \Leftrightarrow x = \theta$.

(ii) There is a number $L = \max\{1, \sup_n |\lambda|^{p_n-1}\} \geq 1$ with $\varrho(\lambda x) \leq L|\lambda|\varrho(x)$ for all $x \in C(p)$ and $\lambda \in \mathbb{C}$.

(iii) We have the inequality $\varrho(x + y) \leq H(\varrho(x) + \varrho(y))$ for all $x, y \in C(p)$.

(iv) It is clear from (2) that $(C(p))_\varrho$ is solid.

(v) It is clear from (3) that $K_0 \geq (2H^2 + 3H) \geq 1$.

(vi) It is clear that $\overline{F} = C(p)$.

(vii) There exists a constant $0 < \xi \leq |\lambda|^{p_0-1}$ such that $\varrho(\lambda, 0, 0, 0, \ldots) \geq \xi|\lambda|\varrho(1, 0, 0, 0, \ldots)$ for any $\lambda \neq 0$ and $\xi > 0$, when $\lambda = 0$.

Hence the space $(C(p))_\varrho$ is pre-modular (sss). To prove that $(C(p))_\varrho$ is a pre-modular Banach (sss), suppose $x^n = (x^n_k)_{k=0}^{\infty}$ is a Cauchy sequence in $(C(p))_\varrho$, then, for every $\varepsilon \in (0, 1)$, there exists a number $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, one has

$$\varrho\left(x^n - x^m\right) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x^n_j - x^m_j|}{2^i}\right)^{p_i} < \varepsilon^{\sup_i p_i}.$$

Hence, for $n, m \geq n_0$ and $i \in \mathbb{N}$, we get

$$\left|x^n_i - x^m_i\right| < \varepsilon.$$

So (x^m_i) is a Cauchy sequence in $\mathbb{R}$ for fixed $i \in \mathbb{N}$, this gives $\lim_{m\to\infty} x^m_i = x^0_i$ for fixed $i \in \mathbb{N}$. Hence $\varrho(x^n - x^0) < \varepsilon^{\sup_i p_i}$, for all $n \geq n_0$. Finally to prove that $x^0 \in C(p)$, we have

$$\varrho\left(x^0\right) = \varrho\left(x^0 - x^n + x^n\right) \leq H\left(\varrho\left(x^n - x^0\right) + \varrho\left(x^n\right)\right) < \infty,$$

so $x^0 \in C(p)$. This means that $(C(p))_\varrho$ is a pre-modular Banach (sss). □

Corollary 3.5 *If* $1 < p < \infty$, *then* $(\mathrm{ces}_p)_\varrho$ *is a pre-modular Banach (sss), where* $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^p$ *for all* $x \in \mathrm{ces}_p$.

4 Multiplication operator on pre-quasi-normed (sss)

We define in this section a multiplication operator on $C(p)$ with a pre-quasi-norm and give the necessity and sufficient conditions on the multiplication operator to be bounded, approximable, invertible, Fredholm and closed range operator.

Definition 4.1 Let $\beta \in \mathbb{C}^{\mathbb{N}}$ be a bounded sequence and E_ϱ be a pre-quasi-normed (sss), the multiplication operator is defined as $T_\beta : E \to E$, where $T_\beta x = \beta x = (\beta_k x_k)_{k=0}^{\infty}$, for all $x \in E$. If T_β is continuous, we call it a multiplication operator induced by β.

Theorem 4.2 *If $\beta \in \mathbb{C}^{\mathbb{N}}$ and (p_n) is a bounded sequence, then $\beta \in \ell_\infty$ if and only if $T_\beta \in L((C(p))_\varrho)$, where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in (C(p))_\varrho$.*

Proof Let $\beta \in \ell_\infty$. Then there exists $C > 0$ such that $|\beta_n| \leq C$, for all $n \in \mathbb{N}$. For $x \in (C(p))_\varrho$, since (p_n) is a bounded sequence, we have

$$\begin{aligned}\varrho(T_\beta x) &= \varrho(\beta x) = \varrho\bigl((\beta_k x_k)_{k=0}^{\infty}\bigr) = \varrho\bigl(\bigl(|\beta_k||x_k|\bigr)_{k=0}^{\infty}\bigr)\\ &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j||x_j|}{2^i}\right)^{p_i} \leq \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}C|x_j|}{2^i}\right)^{p_i} \leq A\varrho(x),\end{aligned}$$

where $A = \sup_n C^{p_n}$, this implies that $T_\beta \in L((C(p))_\varrho)$.

Conversely, let $T_\beta \in L((C(p))_\varrho)$. To show that $\beta \in \ell_\infty$. For, if $\beta \notin \ell_\infty$, then, for every $n \in \mathbb{N}$, there exists some $i_n \in \mathbb{N}$ such that $\beta_{i_n} > n$. Now

$$\varrho(T_\beta e_{i_n}) = \varrho(\beta e_{i_n}) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j|(e_{i_n})_j}{2^i}\right)^{p_i} = \left(\frac{|\beta_{i_n}|}{2^{i_0}}\right)^{p_{i_0}} > \left(\frac{n}{2^{i_0}}\right)^{p_{i_0}} = n^{p_{i_0}}\varrho(e_{i_n}),$$

where $i_0 \in \mathbb{N}$ be such that $2^{i_0} - 1 \leq i_n \leq 2^{i_0+1} - 2$. This shows that T_β is not a bounded operator. So, β must be a bounded sequence. □

Theorem 4.3 *Let $\beta \in \mathbb{C}^{\mathbb{N}}$ and $(C(p))_\varrho$ be a pre-quasi-normed (sss), where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$, for all $x \in (C(p))_\varrho$. $|\beta_n| = 1$, for all $n \in \mathbb{N}$ if and only if T_β is an isometry.*

Proof Let $|\beta_n| = 1$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\varrho(T_\beta x) &= \varrho(\beta x) = \varrho\bigl((\beta_k x_k)_{k=0}^{\infty}\bigr) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j||x_j|}{2^i}\right)^{p_i}\\ &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i}\right)^{p_i} = \varrho(x),\end{aligned}$$

for all $x \in (C(p))_\varrho$. Hence T_β is an isometry.

Conversely, let T_β be an isometry and $|\beta_n| < 1$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}\varrho(T_\beta x) &= \varrho(\beta x) = \varrho\bigl((\beta_k x_k)_{k=0}^{\infty}\bigr) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j||x_j|}{2^i}\right)^{p_i}\\ &< \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i}\right)^{p_i} = \varrho(x).\end{aligned}$$

Also if $|\beta_n| > 1$, then we get $\varrho(\beta x) > \varrho(x)$. In both cases, we get a contradiction. Hence, $|\beta_n| = 1$, for all $n \in \mathbb{N}$. □

Theorem 4.4 *Let $\beta \in \mathbb{C}^{\mathbb{N}}$, (p_n) be a bounded sequence and $T_\beta \in L((C(p))_\varrho)$, where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in (C(p))_\varrho$. Then $T_\beta \in \Psi((C(p))_\varrho)$ if and only if $(\beta_n)_{n=0}^{\infty} \in C_0$.*

Proof Let T_β be an approximable operator. Hence T_β is compact operator. We have to show that $(\beta_n)_{n=0}^\infty \in C_0$. For if this is not true, then there exists $\delta > 0$ such that $B_\delta = \{r \in \mathbb{N} : |\beta_r| \geq \delta\}$ is an infinite set. Let $d_1, d_2, \dots, d_n, \dots$ be in B_δ. Then $\{e_{d_n} : d_n \in B_\delta\}$ is an infinite bounded set in $(C(p))_\varrho$. We have

$$\begin{aligned}
\varrho(T_\beta e_{d_n} - T_\beta e_{d_m}) = \varrho(\beta e_{d_n} - \beta e_{d_m}) &= \varrho\bigl(\bigl(\beta_k\bigl((e_{d_n})_k - (e_{p_m})_k\bigr)\bigr)_{k=0}^\infty\bigr) \\
&= \sum_{i=0}^\infty \biggl(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |\beta_j((e_{d_n})_j - (e_{p_m})_j)|}{2^i}\biggr)^{p_i} \\
&\geq \sum_{i=0}^\infty \biggl(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |\delta((e_{d_n})_j - (e_{p_m})_j)|}{2^i}\biggr)^{p_i} \\
&\geq \Bigl(\inf_n \delta^{p_n}\Bigr) \sum_{i=0}^\infty \biggl(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |((e_{d_n})_j - (e_{p_m})_j)|}{2^i}\biggr)^{p_i} \\
&= \Bigl(\inf_n \delta^{p_n}\Bigr) \varrho(e_{d_n} - e_{d_m}),
\end{aligned}$$

for all $d_n, d_m \in B_\delta$. This proves $\{e_{d_n} : d_n \in B_\delta\}$ is a bounded sequence, which cannot have a convergent subsequence under T_β. This shows that T_β cannot be a compact, hence is not approximable operator, which is a contradiction. Hence, $\lim_{n\to\infty} \beta_n = 0$.

Conversely, suppose $\lim_{n\to\infty} \beta_n = 0$. Then, for every $\delta > 0$, the set $B_\delta = \{n \in \mathbb{N} : |\beta_n| \geq \delta\}$ is finite. Then

$$\bigl((C(p))_\varrho\bigr)_{B_\delta} = \bigl\{x = (x_n) \in \bigl(C(p)\bigr)_\varrho : n \in B_\delta\bigr\}$$

is a finite dimensional space for each $\delta > 0$. Therefore, $T_\beta|((C(p))_\varrho)_{B_\delta}$ is a finite rank operator. For each $n \in \mathbb{N}$, define $\beta_n : \mathbb{N} \to \mathbb{C}$ by

$$(\beta_n)_m = \begin{cases} \beta_m, & m \in B_{\frac{1}{n}}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, T_{β_n} is a finite rank operator as the space $((C(p))_\varrho)_{B_{\frac{1}{n}}}$ is finite dimensional for each $n \in \mathbb{N}$. Now, by using (p_n) is a bounded sequence, we have

$$\begin{aligned}
\varrho\bigl((T_\beta - T_{\beta_n})x\bigr) &= \varrho\bigl(\bigl(\bigl(\beta_m - (\beta_n)_m\bigr)x_m\bigr)_{m=0}^\infty\bigr) \\
&= \sum_{i=0, i\in B_{\frac{1}{n}}}^\infty \biggl(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |(\beta_j - (\beta_n)_j)x_j|}{2^i}\biggr)^{p_i} \\
&\quad + \sum_{i=0, i\notin B_{\frac{1}{n}}}^\infty \biggl(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |(\beta_j - (\beta_n)_j)x_j|}{2^i}\biggr)^{p_i}
\end{aligned}$$

$$= \sum_{i=0, i \in B_{\frac{1}{n}}}^{\infty} \left(\frac{\sum_{j=2^i-1, j \notin B_{\frac{1}{n}}}^{2^{i+1}-2} |\beta_j x_j|}{2^i} \right)^{p_i} + \sum_{i=0, i \notin B_{\frac{1}{n}}}^{\infty} \left(\frac{\sum_{j=2^i-1, j \notin B_{\frac{1}{n}}}^{2^{i+1}-2} |\beta_j x_j|}{2^i} \right)^{p_i}$$

$$< \left(\frac{1}{n}\right)^{\inf_n p_n} \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |x_j|}{2^i} \right)^{p_i} = \left(\frac{1}{n}\right)^{\inf_n p_n} \varrho(x).$$

This proves that $\|T_\beta - T_{\beta_n}\| \le (\frac{1}{n})^{\inf_n p_n}$ and that T_β is a limit of finite rank operators and hence, T_β is an approximable operator. □

Theorem 4.5 *Let $\beta \in \mathbb{C}^{\mathbb{N}}$, (p_n) be a bounded sequence and $T_\beta \in L((C(p))_\varrho)$, where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in (C(p))_\varrho$. Then $T_\beta \in L_c((C(p))_\varrho)$ if and only if $(\beta_n)_{n=0}^{\infty} \in C_0$.*

Proof It is easy so omitted. □

Corollary 4.6 *Let (p_n) be a bounded sequence, we have*

$$L_c\big(\big(C(p)\big)_\varrho\big) \subsetneqq L\big(\big(C(p)\big)_\varrho\big),$$

where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in C(p)$.

Proof Since the identity operator I on $(C(p))_\varrho$ is a multiplication operator induced by the sequence $\beta = (1, 1, \ldots)$, we have $I \notin L_c((C(p))_\varrho)$ and $I \in L((C(p))_\varrho)$. □

Theorem 4.7 *Let $(C(p))_\varrho$ be a pre-quasi-Banach (sss), where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in C(p)$ and $T_\beta \in L((C(p))_\varrho)$. Then β is bounded away from zero on $\mathbb{N} \setminus \ker(\beta) := \ker(\beta)^c$ if and only if T_β has closed range.*

Proof Let β be bounded away from zero on $\ker(\beta)^c$. Then there exists $\epsilon > 0$ such that $|\beta_n| \ge \epsilon$, for all $n \in \ker(\beta)^c$. We have to prove that range (T_β) is closed. Let f be a limit point of range (T_β). Then there exists a sequence $T_\beta x_n$ in $(C(p))_\varrho$, for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} T_\beta x_n = f$. Clearly, the sequence $T_\beta x_n$ is a Cauchy sequence. Now, since ϱ is non-decreasing, we have

$$\begin{aligned}
&\varrho(T_\beta x_n - T_\beta x_m) \\
&\quad = \varrho\big(\big(\beta_j(x_n)_j - \beta_j(x_m)_j\big)_{j=0}^{\infty}\big) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j(x_n)_j - \beta_j(x_m)_j|}{2^i}\right)^{p_i} \\
&\quad = \sum_{i=0, i \in \ker(\beta)^c}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j(x_n)_j - \beta_j(x_m)_j|}{2^i}\right)^{p_i} \\
&\qquad + \sum_{i=0, i \notin \ker(\beta)^c}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j(x_n)_j - \beta_j(x_m)_j|}{2^i}\right)^{p_i}
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=0, i\in \ker(\beta)^c}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j(x_n)_j-\beta_j(x_m)_j|}{2^i}\right)^{p_i} = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j(y_n)_j-\beta_j(y_m)_j|}{2^i}\right)^{p_i}\\
&> \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\epsilon(y_n)_j-\epsilon(y_m)_j|}{2^i}\right)^{p_i} = \varrho\bigl(\epsilon(y_n-y_m)\bigr),
\end{aligned}$$

where

$$(y_n)_k = \begin{cases} (x_n)_k, & k\in \ker(\beta)^c,\\ 0, & k\notin \ker(\beta)^c.\end{cases}$$

This proves that (y_n) is a Cauchy sequence in $(C(p))_\varrho$. But $(C(p))_\varrho$ is complete. Therefore, there exists $x \in (C(p))_\varrho$ such that $\lim_{n\to\infty} y_n = x$. In view of the continuity of T_β, hence $\lim_{n\to\infty} T_\beta y_n = T_\beta x$. But $\lim_{n\to\infty} T_\beta x_n = \lim_{n\to\infty} T_\beta y_n = f$. Therefore, $T_\beta x = f$. Hence $f \in \operatorname{range}(T_\beta)$. This proves that T_β has closed range.

Conversely, suppose that T_β has closed range. Then T_β is bounded away from zero on $((C(p))_\varrho)_{\ker(\beta)^c}$. That is, there exists $\epsilon > 0$ such that $\varrho(T_\beta x) \geq \varrho(\epsilon x)$, for all $x \in ((C(p))_\varrho)_{\ker(\beta)^c}$. Let $D = \{k \in \ker(\beta)^c : |\beta_k| < \epsilon\}$. If $D \neq \phi$, then, for $n_0 \in D$, since ϱ is non-decreasing, we have

$$\begin{aligned}
\varrho(T_\beta e_{n_0}) &= \varrho\bigl(\bigl(\beta_k(e_{n_0})_k\bigr)_{k=0}^{\infty}\bigr) = \varrho\bigl(\bigl(|\beta_k||(e_{n_0})_k|\bigr)_{k=0}^{\infty}\bigr) = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|\beta_j||(e_{n_0})_j|}{2^i}\right)^{p_i}\\
&< \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}\epsilon|(e_{n_0})_j|}{2^i}\right)^{p_i} = \varrho(\epsilon e_{n_0}),
\end{aligned}$$

which is a contradiction. Hence, $D = \phi$ so that $|\beta_k| \geq \epsilon$, for all $k \in \ker(\beta)^c$. This proves the theorem. □

Theorem 4.8 *Let $\beta \in \mathbb{C}^{\mathbb{N}}$ and $(C(p))_\varrho$ be a pre-quasi-Banach (sss), where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in C(p)$. There exist $a > 0$ and $A > 0$ such that $a < \beta_n < A$; for all $n \in \mathbb{N}$ if and only if $T_\beta \in L((C(p))_\varrho)$ is invertible.*

Proof Let the condition be true. Define $\eta \in \mathbb{C}^{\mathbb{N}}$ by $\eta_n = \frac{1}{\beta_n}$. Then by Theorem 4.2, T_β and T_η are bounded linear operators. Also $T_\beta.T_\eta = T_\eta.T_\beta = I$. Hence, T_β is the inverse of T_η. Conversely, let T_β be invertible. Then $\operatorname{range}(T_\beta) = ((C(p))_\varrho)_{\mathbb{N}}$. Therefore, $\operatorname{range}(T_\beta)$ is closed. Hence, by Theorem 4.7, there exists $a > 0$ such that $|\beta_n| \geq a$, for all $n \in \ker(\beta)^c$. Now $\ker(\beta) = \phi$; otherwise $\beta_{n_0} = 0$, for some $n_0 \in \mathbb{N}$, in which case $e_{n_0} \in \ker(T_\beta)$ which is a contradiction, since $\ker(T_\beta)$ is trivial. Hence, $|\beta_n| \geq a$, for all $n \in \mathbb{N}$. Since T_β is bounded, so by Theorem 4.2, there exists $A > 0$ such that $|\beta_n| \leq A$, for all $n \in \mathbb{N}$. Thus, we have proved that $a \leq |\beta_n| \leq A$, for all $n \in \mathbb{N}$. □

Theorem 4.9 *Let $(C(p))_\varrho$ be a pre-quasi-Banach (sss), where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in C(p)$ and $T_\beta \in L((C(p))_\varrho)$. Then T_β is a Fredholm operator if and only if:*

(i) *$\ker(\beta)$ is a finite subset of $\mathbb{N}$.*

(ii) *$|\beta_n| \geq \epsilon$, for all $n \in \ker(\beta)^c$.*

Proof Let T_β be Fredholm. If $\ker(\beta)$ is an infinite subset of $\mathbb{N}$, then $e_n \in \ker(T_\beta)$, for all $n \in \ker(\beta)$. But e_n's are linearly independent, which shows that $\ker(T_\beta)$ is an infinite dimensional which is a contradiction. Hence, $\ker(\beta)$ must be a finite subset of $\mathbb{N}$. The condition (ii) comes from Theorem 4.7. Conversely, If the conditions (i) and (ii) are true, then we prove that T_β is Fredholm. By Theorem 4.7, the condition (ii) implies that T_β has closed range. The condition (i) implies that $\ker(T_\beta)$ and $(\text{range}(T_\beta))^c$ are finite dimensional. This proves that T_β is Fredholm. □

5 Completeness and closedness of the pre-quasi-ideal components

We give here the sufficient conditions on $C(p)$ such that the components of pre-quasi-operator ideal $S_{C(p)}$ are complete.

Theorem 5.1 *If X, Y are Banach spaces and (p_n) is an increasing bounded sequence with $p_0 > 1$, then $(S_{(C(p))_\varrho}, g)$, where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in C(p)$ and $g(T) = \varrho((s_n(T))_{n=0}^{\infty})$ is a pre-quasi-Banach operator ideal.*

Proof Since $(C(p))_\varrho$ is a pre-modular (sss) by Theorem 3.4, then from Theorem 2.11, the function $g(T) = \varrho((s_n(T))_{n=0}^{\infty})$ is a pre-quasi-norm on $S_{(C(p))_\varrho}$. Let (T_i) be a Cauchy sequence in $S_{(C(p))_\varrho}(X,Y)$, since $L(X,Y) \supseteq S_{(C(p))_\varrho}(X,Y)$, we get

$$g(T_i - T_j) = \sum_{n=0}^{\infty}\left(\frac{\sum_{m=2^n-1}^{2^{n+1}-2} s_m(T_i - T_j)}{2^n}\right)^{p_n} \geq \left(\|T_i - T_j\|\right)^{p_0},$$

then $(T_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L(X,Y)$. While the space $L(X,Y)$ is a Banach space, there exists $T \in L(X,Y)$ with $\lim_{i\to\infty}\|T_i - T\| = 0$ and while $(s_n(T_i))_{n=0}^{\infty} \in (C(p))_\varrho$ for each $i \in \mathbb{N}$, using definition (2.6) conditions (iii), (iv) and ϱ being continuous at θ, we get

$$\begin{aligned}
g(T) &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{p_i} = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T - T_m + T_m)}{2^i}\right)^{p_i} \\
&\leq K\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_{[\frac{j}{2}]}(T - T_m)}{2^i}\right)^{p_i} + K\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_{[\frac{j}{2}]}(T_m)}{2^i}\right)^{p_i} \\
&\leq K\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} \|T_m - T\|}{2^i}\right)^{p_i} + KK_0\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T_m)}{2^i}\right)^{p_i} < \varepsilon,
\end{aligned}$$

we have $(s_n(T))_{n=0}^{\infty} \in (C(p))_\varrho$, then $T \in S_{(C(p))_\varrho}(X,Y)$.

We give here the sufficient conditions on $C(p)$ such that the components of pre-quasi-operator ideal $S_{C(p)}$ are closed. □

Theorem 5.2 *If X, Y are normed spaces, (p_n) is an increasing bounded sequence with $p_0 > 1$, then $(S_{(C(p))_\varrho}, g)$ is a pre-quasi-closed operator ideal, where $\varrho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2}|x_j|}{2^i})^{p_i}$ for all $x \in C(p)$ and $g(T) = \varrho((s_n(T))_{n=0}^{\infty})$.*

Proof Since $C(p)$ is a pre-modular (sss) by Theorem 3.4, from Theorem 2.11, the function $g(T) = \varrho((s_n(T))_{n=0}^{\infty})$ is a pre-quasi-norm on $S_{(C(p))_\varrho}$. Let $T_m \in S_{(C(p))_\varrho}(X,Y)$ for all $m \in \mathbb{N}$

and $\lim_{m\to\infty} g(T_m - T) = 0$, since $L(X,Y) \supseteq S_{(C(p))_\varrho}(X,Y)$, we get

$$g(T - T_j) = \sum_{n=0}^{\infty}\left(\frac{\sum_{m=2^n-1}^{2^{n+1}-2} s_m(T - T_j)}{2^n}\right)^{p_n} \geq \left(\|T - T_j\|\right)^{p_0},$$

then $(T_j)_{j\in\mathbb{N}}$ is a convergent sequence in $L(X,Y)$. While $(s_n(T_j))_{n=0}^{\infty} \in (C(p))_\varrho$ for each $j \in \mathbb{N}$, hence using definition (2.6) conditions (iii), (iv) and ϱ is continuous at θ, we obtain

$$\begin{aligned} g(T) &= \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{p_i} = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T - T_m + T_m)}{2^i}\right)^{p_i} \\ &\leq K\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_{[\frac{j}{2}]}(T - T_m)}{2^i}\right)^{p_i} + K\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_{[\frac{j}{2}]}(T_m)}{2^i}\right)^{p_i} \\ &\leq K\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} \|T_m - T\|}{2^i}\right)^{p_i} + KK_0\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T_m)}{2^i}\right)^{p_i} < \varepsilon, \end{aligned}$$

we have $(s_n(T))_{n=0}^{\infty} \in (C(p))_\varrho$, then $T \in S_{(C(p))_\varrho}(X,Y)$. □

6 Smallness and simpleness of the pre-quasi-Banach operator ideal

We give here the sufficient conditions on $C(p)$ such that the pre-quasi-operator ideal formed by the sequence of s-numbers and this sequence space is strictly contained for different powers.

Theorem 6.1 *Let X, Y be infinite dimensional Banach spaces and for any $1 < p_n < q_n$ for all $n \in \mathbb{N}$, it is true that*

$$S_{C(p)}(X,Y) \subsetneqq S_{C(q)}(X,Y) \subsetneqq L(X,Y).$$

Proof Let the conditions be satisfied, if $T \in S_{C(p)}(X,Y)$, then $(s_j(T)) \in C(p)$. We have

$$\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{q_i} < \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{p_i} < \infty,$$

hence $T \in S_{C(q)}(X,Y)$. Next, if we take $(s_j(T))_{j=0}^{\infty}$ such that

$$\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T) = 2^i(i+1)^{-\frac{1}{p_i}},$$

one can find $T \in L(X,Y)$ with

$$\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{p_i} = \sum_{i=0}^{\infty}\frac{1}{i+1} = \infty$$

and

$$\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{q_i}=\sum_{i=0}^{\infty}\left(\frac{1}{i+1}\right)^{\frac{q_i}{p_i}}<\infty.$$

Hence T does not belong to $S_{C(p)}(X,Y)$ and $T\in S_{C(q)}(X,Y)$.

It is easy to verify that $S_{C(q)}(X,Y)\subset L(X,Y)$. Next, if we take $(s_j(T))_{j=0}^{\infty}$ such that $\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)=2^i(i+1)^{-\frac{1}{q_i}}$. One can find $T\in L(X,Y)$ such that T does not belong to $S_{C(q)}(X,Y)$. This completes the proof. □

Corollary 6.2 *For any infinite dimensional Banach spaces X, Y and $1<p<q<\infty$, then*

$$S_{\mathrm{ces}_p}(X,Y)\underset{\neq}{\subsetneq}S_{\mathrm{ces}_q}(X,Y)\underset{\neq}{\subsetneq}L(X,Y).$$

In this part, we give the conditions for which the pre-quasi-Banach operator ideal $S^{\mathrm{app}}_{C(p)}$ is small.

Theorem 6.3 *If (p_n) is an increasing bounded sequence with $p_0>1$, then the pre-quasi-Banach operator ideal $S^{\mathrm{app}}_{C(p)}$ is small.*

Proof Let (p_n) be an increasing bounded sequence with $p_0>1$, take $\epsilon=(\sum_{i=0}^{\infty}2^{-ip_i})^{\frac{1}{\sup_i p_i}}$. By using Theorems 3.3 and 3.4. Then $(S^{\mathrm{app}}_{C(p)},g)$, where

$$g(T)=\frac{1}{\epsilon}\left[\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(T)}{2^i}\right)^{p_i}\right]^{\frac{1}{\sup_i p_i}}$$

is a pre-quasi-Banach operator ideal. Let X and Y be any two Banach spaces. Suppose that $S^{\mathrm{app}}_{C(p)}(X,Y)=L(X,Y)$, then there exists a constant $C>0$ such that $g(T)\le C\|T\|$ for all $T\in L(X,Y)$. Suppose that X and Y be infinite dimensional Banach spaces. Hence by Dvoretzky's theorem (see [27]) for $n\in\mathbb{N}$, we have quotient spaces X/N_n and subspaces M_n of Y which can be mapped onto ℓ_2^n by isomorphisms H_n and A_n such that $\|H_n\|\|H_n^{-1}\|\le 2$ and $\|A_n\|\|A_n^{-1}\|\le 2$. Let I_n be the identity map on ℓ_2^n, Q_n be the quotient map from X onto X/N_n and J_n be the natural embedding map from M_n into Y. Let γ_j be the Bernstein numbers (see [28]) then

$$\begin{aligned}1&=\gamma_j(I_n)=\gamma_j\big(A_nA_n^{-1}I_nH_nH_n^{-1}\big)\le\|A_n\|\gamma_j\big(A_n^{-1}I_nH_n\big)\big\|H_n^{-1}\big\|\\&=\|A_n\|\gamma_j\big(J_nA_n^{-1}I_nH_n\big)\big\|H_n^{-1}\big\|\le\|A_n\|d_j\big(J_nA_n^{-1}I_nH_n\big)\big\|H_n^{-1}\big\|\\&=\|A_n\|d_j\big(J_nA_n^{-1}I_nH_nQ_n\big)\big\|H_n^{-1}\big\|\le\|A_n\|\alpha_j\big(J_nA_n^{-1}I_nH_nQ_n\big)\big\|H_n^{-1}\big\|,\end{aligned}$$

for $0\le i\le n$. Now

$$\begin{aligned}&\sum_{j=2^i-1}^{2^{i+1}-2}2^{-i}\le\sum_{j=2^i-1}^{2^{i+1}-2}2^{-i}\|A_n\|\alpha_j\big(J_nA_n^{-1}I_nH_nQ_n\big)\big\|H_n^{-1}\big\|\\&\quad\Rightarrow\quad 1\le\|A_n\|2^{-i}\sum_{j=2^i-1}^{2^{i+1}-2}\alpha_j\big(J_nA_n^{-1}I_nH_nQ_n\big)\big\|H_n^{-1}\big\|\end{aligned}$$

$$\Rightarrow \quad 1 \le \left(\|A_n\| \left\|H_n^{-1}\right\|\right)^{p_i} \left(2^{-i} \sum_{j=2^i-1}^{2^{i+1}-2} \alpha_j\left(J_n A_n^{-1} I_n H_n Q_n\right)\right)^{p_i}.$$

Therefore,

$$\begin{aligned}
(n+1)^{\frac{1}{\sup_i p_i}} &\le L\|A_n\| \left\|H_n^{-1}\right\| \left[\sum_{i=0}^{n} \left(2^{-i} \sum_{j=2^i-1}^{2^{i+1}-2} \alpha_j\left(J_n A_n^{-1} I_n H_n Q_n\right)\right)^{p_i}\right]^{\frac{1}{\sup_i p_i}} \\
\Rightarrow \quad \frac{1}{\epsilon}(n+1)^{\frac{1}{\sup_i p_i}} &\le L\|A_n\| \left\|H_n^{-1}\right\| \frac{1}{\epsilon}\left[\sum_{i=0}^{n} \left(2^{-i} \sum_{j=2^i-1}^{2^{i+1}-2} \alpha_j\left(J_n A_n^{-1} I_n H_n Q_n\right)\right)^{p_i}\right]^{\frac{1}{\sup_i p_i}} \\
\Rightarrow \quad \frac{1}{\epsilon}(n+1)^{\frac{1}{\sup_i p_i}} &\le L\|A_n\| \left\|H_n^{-1}\right\| g\left(J_n A_n^{-1} I_n H_n Q_n\right) \\
\Rightarrow \quad \frac{1}{\epsilon}(n+1)^{\frac{1}{\sup_i p_i}} &\le LC\|A_n\| \left\|H_n^{-1}\right\| \left\|J_n A_n^{-1} I_n H_n Q_n\right\| \\
\Rightarrow \quad \frac{1}{\epsilon}(n+1)^{\frac{1}{\sup_i p_i}} &\le LC\|A_n\| \left\|H_n^{-1}\right\| \left\|J_n A_n^{-1}\right\| \|I_n\| \|H_n Q_n\| \\
&= LC\|A_n\| \left\|H_n^{-1}\right\| \left\|A_n^{-1}\right\| \|I_n\| \|H_n\| \\
\Rightarrow \quad \frac{1}{\epsilon}(n+1)^{\frac{1}{\sup_i p_i}} &\le 4LC,
\end{aligned}$$

for some $L \ge 1$. Since n is an arbitrary, this gives a contradiction. Thus X and Y both cannot be infinite dimensional when $S^{\mathrm{app}}_{C(p)}(X,Y) = L(X,Y)$. This finishes the proof. □

We express the accompanying theorem without verifications, these can be set up utilizing a standard procedure.

Theorem 6.4 *If (p_n) is an increasing bounded sequence with $p_0 > 1$, then the pre-quasi-Banach operator ideal $S^{\mathrm{Kol}}_{C(p)}$ is small.*

Corollary 6.5 *If $1 < p < \infty$, then the quasi-Banach operator ideal $S^{\mathrm{app}}_{\mathrm{ces}_p}$ is small.*

Corollary 6.6 *If $1 < p < \infty$, then the quasi-Banach operator ideal $S^{\mathrm{Kol}}_{\mathrm{ces}_p}$ is small.*

The following question arises naturally: for which $C(p)$, the pre-quasi-Banach ideal is simple?

Theorem 6.7 *For any infinite dimensional Banach spaces X, Y, and for any bounded sequences (p_n), (q_n) with $1 < p_n < q_n$ for all $n \in \mathbb{N}$,*

$$L(S_{C(q)}, S_{C(p)}) = \Psi(S_{C(q)}, S_{C(p)}).$$

Proof Suppose that there exists $T \in L(S_{C(q)}, S_{C(p)})$ which is not approximable. According to Lemma 2.1, we can find $A \in L(S_{C(q)}, S_{C(q)})$ and $B \in L(S_{C(p)}, S_{C(p)})$ with $BTAI_k = I_k$. Then it follows for all $k \in \mathbb{N}$ that

$$\|I_k\|_{S_{C(p)}} = \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(I_k)}{2^i}\right)^{p_i} \leq \|BTA\| \|I_k\|_{S_{C(q)}} \leq \sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} s_j(I_k)}{2^i}\right)^{q_i}.$$

This contradicts Theorem (6.1). Hence $T \in \Psi(S_{C(q)}, S_{C(p)})$, which finishes the proof. □

Corollary 6.8 *For any infinite dimensional Banach spaces X and Y. If $1 < p < q$, then*

$$L(S_{\mathrm{ces}_q}, S_{\mathrm{ces}_p}) = L_C(S_{\mathrm{ces}_q}, S_{\mathrm{ces}_p}).$$

Theorem 6.9 *If (p_n) is an increasing bounded sequence with $p_0 > 1$, then the pre-quasi-Banach space $S_{C(p)}$ is simple.*

Proof Suppose that the closed ideal $L_C(S_{C(p)})$ contains an operator T which is not approximable. According to Lemma 2.1, we can find $A, B \in L(S_{C(p)})$ with $BTAI_k = I_k$. This means that $I_{S_{C(p)}} \in L_C(S_{C(p)})$. Consequently $L(S_{C(p)}) = L_C(S_{C(p)})$. Therefore $\Psi(S_{C(p)})$ is the only non-trivial closed ideal in $L(S_{C(p)})$.

We give here the sufficient conditions such that the pre-quasi-ideal $S_{(C(p))_\rho}$ is approximable. □

Theorem 6.10 *Take any infinite dimensional Banach spaces X and Y. If (p_n) is an increasing bounded sequence with $p_0 > 1$, then $\overline{F(X, Y)} = S_{(C(p))_\rho}(X, Y)$, where $\rho(x) = \sum_{i=0}^{\infty}(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} |x_j|}{2^i})^{p_i}$ for all $x \in C(p)$ and the converse is not necessarily true.*

Proof Since $(C(p))_\rho$ is a pre-modular (sss), then from Theorem 2.10, we have $\overline{F(X, Y)} = S_{(C(p))_\rho}(X, Y)$. Conversely since $I_4 \in S_{(C(1))_\rho}$ the condition is not satisfied, which is a counter example. This finishes the proof. □

7 Eigenvalues of *s*-type operators

We give here the sufficient conditions on $C(p)$ such that the pre-quasi-operator ideal formed by the sequence of s-numbers and this sequence space is equal the class of all bounded linear operators whose sequence of eigenvalues belongs to this sequence space.

Notations 7.1

$$S_E^{\lambda} := \{S_E^{\lambda}(X, Y); X \text{ and } Y \text{ are Banach Spaces}\},$$

where

$$S_E^{\lambda}(X, Y) := \Big\{T \in L(X, Y) : \big(\lambda_n(T)\big)_{n=0}^{\infty} \in E \text{ and } \\ \big\|T - \lambda_n(T)I\big\| \text{ is not invertible for all } n \in \mathbb{N}\Big\}.$$

Theorem 7.2 *Take any infinite dimensional Banach spaces X and Y. If (p_n) is an increasing bounded sequence with $p_0 > 1$, then*

$$S_{C(p)}^{\lambda}(X, Y) = S_{C(p)}(X, Y).$$

Proof Let $T \in S_{C(p)}(X,Y)$, then $(s_n(T))_{n=0}^{\infty} \in C(p)$. Since (p_n) is an increasing bounded sequence with $p_n > 1$ for all $n \in \mathbb{N}$, we have

$$\sum_{i=0}^{\infty}\left(\frac{\sum_{j=2^i-1}^{2^{i+1}-2} x_j(T)}{2^i}\right)^{p_i} \geq \sum_{i=0}^{\infty}\left[s_{2^{i+1}-2}(T)\right]^{p_i}.$$

Hence $(s_n(T))_{n=0}^{\infty} \in \ell_{(p_n)}$, so $\lim_{n\to\infty} s_n(T) = 0$. Suppose $\|T - s_n(T)I\|$ is invertible for all $n \in \mathbb{N}$, then $\|T - s_n(T)I\|^{-1}$ exists and is bounded for all $n \in \mathbb{N}$. This shows $\lim_{n\to\infty}\|T - s_n(T)I\|^{-1} = \|T\|^{-1}$ exists and is bounded. Since $(S_{C(p)}, g)$ is a pre-quasi-operator ideal, we have

$$I = TT^{-1} \in S_{C(p)}(X,Y) \quad \Rightarrow \quad \left(s_n(I)\right)_{n=0}^{\infty} \in C(p) \quad \Rightarrow \quad \lim_{n\to\infty} s_n(I) = 0.$$

But $\lim_{n\to\infty} s_n(I) = 1$. This is a contradiction, then $\|T - s_n(T)I\|$ is not invertible for all $n \in \mathbb{N}$. Therefore the sequence $(s_n(T))_{n=0}^{\infty}$ is the eigenvalues of T.

Conversely, let $T \in S_{C(p)}^{\lambda}(X,Y)$, then $(\lambda_n(T))_{n=0}^{\infty} \in C(p)$ and $\|T - \lambda_n(T)I\| = 0$ for all $n \in \mathbb{N}$. This gives $T = \lambda_n(T)I$ for all $n \in \mathbb{N}$, then $s_n(T) = s_n(\lambda_n(T)I) = |\lambda_n(T)|$ for all $n \in \mathbb{N}$. Hence $(s_n(T))_{n=0}^{\infty} \in C(p)$, so $T \in S_{C(p)}(X,Y)$. This finishes the proof. □

Acknowledgements
This work was funded by the University of Jeddah, Saudi Arabia, under grant No. (UJ-02-048-DR). The authors, therefore, acknowledge with thanks the University technical and financial support. Also, the authors thank the anonymous referees for their constructive suggestions and helpful comments which led to significant improvement of the original manuscript of this paper.

Funding
Applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science, Ain Shams University, P.O. Box 1156, Cairo, 11566, Abbassia, Egypt.
[3]Department of Mathematics, Faculty of Education, Alzaeim Alazhari University, P.O. Box 1432, 13311 Khartoum, Sudan.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 February 2020 Accepted: 4 September 2020 Published online: 11 September 2020

References

1. Lim, K.P.: Matrix transformation on certain sequence space. Tamkang J. Math. **8**(2), 213–220 (1977)
2. Lim, K.P.: Matrix transformation in the Cesáro sequence spaces. Kyungpook Math. J. **14**, 221–227 (1979)
3. Singh, R.K., Kumar, A.: Multiplication operators with closed range. Bull. Aust. Math. Soc. **16**, 247–252 (1977)
4. Abrahmse, M.B.: Multiplication Operators. Lecture Notes in Mathematics, vol. 693, pp. 17–36. Springer, Berlin (1978)
5. Raj, K., Sharma, S.K., Kumar, A.: Multiplication operator on Musielak–Orlicz spaces of Bochner type. J. Adv. Stud. Topol. **3**, 1–7 (2012)
6. Sharma, A., Raj, K., Sharma, S.K.: Products of multiplication composition and differentiation operators from H^{∞} to weighted Bloch spaces. Indian J. Math. **54**, 159–179 (2012)
7. Singh, R.K., Manhas, J.S.: Composition Operators on Function Spaces. North-Holland, Amsterdam (1993)
8. Singh, R.K., Manhas, J.S.: Multiplication operators and composition operators with closed ranges. Bull. Aust. Math. Soc. **16**, 247–252 (1977)

9. Takagi, H., Yokouchi, K.: Multiplication and composition operators between two L_p-spaces. Contemp. Math. **232**, 321–338 (1999)
10. Bakery, A.A., Abou Elmatty, A.R.: Some properties of pre-quasi norm on Orlicz sequence space. J. Inequal. Appl. **2020**, Article ID 55 (2020)
11. İlkhan, M., Demiriz, S., Kara, E.E.: Multiplication operators on Cesáro second order function spaces. Positivity **24**, 605–614 (2020)
12. Kara, E.E., Zengin Alp, P.: A new class of operator ideals and approximation numbers. New Trends Math. Sci. **6**(4), 8–15 (2018). https://doi.org/10.20852/ntmsci.2018.310
13. Zengin Alp, P., Kara, E.E.: Some equivalent quasinorms on L_φ, E. Facta Univ. (NIS), Ser. Math. Inform. **33**(5), 739–746 (2018)
14. Bakery, A.A.: The spectrum generated by s-numbers and pre-quasi normed Orlicz–Cesáro mean sequence spaces. Open Math. **18**, 846–857 (2020)
15. Mursaleen, M., Noman, A.K.: Compactness of matrix operators on some new difference sequence spaces. Linear Algebra Appl. **436**, 41–52 (2012)
16. Komal, B.S., Gupta, S.: Multiplication operators between Orlicz spaces. Integral Equ. Oper. Theory **41**(3), 324–330 (2001)
17. Komal, B.S., Pandoh, S., Raj, K.: Multiplication operators on Cesáro sequence spaces. Demonstr. Math. **49**(4), 430–436 (2016)
18. Pietsch, A.: Operator Ideals. North-Holland, Amsterdam (1980)
19. Pietsch, A.: Small ideals of operators. Stud. Math. **51**, 265–267 (1974)
20. Makarov, B.M., Faried, N.: Some properties of operator ideals constructed by s numbers. In: Theory of Operators in Functional Spaces, pp. 206–211. Academy of Science. Siberian Section, Novosibirsk (1977) (in Russian)
21. Faried, N., Bakery, A.A.: Small operator ideals formed by s numbers on generalized Cesáro and Orlicz sequence spaces. J. Inequal. Appl. **2018**(1), Article ID 357 (2018). https://doi.org/10.1186/s13660-018-1945-y
22. Mrowka, T.: A brief introduction to linear analysis: Fredholm operators, geometry of manifolds, Fall 2004 (Massachusetts Institute of Technology: MIT OpenCouseWare) (2004)
23. Bakery, A.A., Mohammed, M.M.: Small pre-quasi Banach operator ideals of type Orlicz–Cesàro mean sequence spaces. J. Funct. Spaces **2019**, Article ID 7265010 (2019)
24. Pietsch, A.: Eigenvalues and s-Numbers. Cambridge University Press, New York (1986)
25. Bakery, A.A., Mohammed, M.M.: Some properties of pre-quasi operator ideal of type generalized Cesáro sequence space defined by weighted means. Open Math. **17**, 1703–1715 (2019). https://doi.org/10.1515/math-2019-0135
26. Altay, B., Başar, F.: Generalization of the sequence space $\ell(p)$ derived by weighted means. J. Math. Anal. Appl. **330**(1), 147–185 (2007)
27. Pietsch, A.: Operator Ideals. VEB, Berlin (1978)
28. Pietsch, A.: s-Numbers of operators in Banach spaces. Stud. Math. **51**, 201–223 (1974)