Ding *Journal of Inequalities and Applications* (2020) 2020:215
https://doi.org/10.1186/s13660-020-02481-y

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Blow-up analysis for parabolic p-Laplacian equations with a gradient source term

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Abstract

In this work, we deal with the blow-up solutions of the following parabolic p-Laplacian equations with a gradient source term:

$$\begin{cases} (b(u))_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u) + f(x,u,|\nabla u|^2,t) & \text{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where $p > 2$, the spatial domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is bounded, and the boundary $\partial\Omega$ is smooth. Our research relies on the creation of some suitable auxiliary functions and the use of the differential inequality techniques and parabolic maximum principles. We give sufficient conditions to ensure that the solution blows up at a finite time t^*. The upper bounds of the blow-up time t^* and the upper estimates of the blow-up rate are also obtained.

MSC: 35K65; 35B40

Keywords: Blow-up solution; Parabolic p-Laplacian equation; Gradient source term

1 Introduction

The blow-up solutions of parabolic p-Laplician equations have been studied by many authors (see, for instance, [1–10]). In this work, we research the blow-up solutions of the following parabolic p-Laplacian equations with a gradient source term:

$$\begin{cases} (b(u))_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u) + f(x,u,|\nabla u|^2,t) & \text{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}. \end{cases} \tag{1.1}$$

In (1.1), $p > 2$, the spatial domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is bounded, the boundary $\partial\Omega$ is smooth, and t^* is blow-up time, $b(s)$ is a $C^3(\overline{\mathbb{R}_+})$ function with $b'(s) > 0$, $s \in \overline{\mathbb{R}_+}$, $f(x,s,r,t)$ is a nonnegative $C^1(\Omega \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \times \mathbb{R}_+)$ function, and $u_0(x)$ is a nonnegative $C^2(\overline{\Omega})$ function satisfying $\frac{\partial u_0(x)}{\partial n} = 0$, $x \in \partial\Omega$.

As far as I know, there are many papers on the blow-up problems of parabolic equations with a gradient term (see, for instance, [11–23]). For the sake of research (1.1), we mainly focus on the papers [12, 22]. In [22], Zhang et al. researched the following problems:

$$\begin{cases} (b(u))_t = \nabla \cdot (\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + h(x)k(t)f(u) & \text{in } \Omega \times (0,t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}. \end{cases} \tag{1.2}$$

In (1.2), $p \geq 2$, the spatial domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is bounded, and the boundary $\partial\Omega$ is smooth. They derived some conditions which ensure the solution of (1.2) blows up in a finite t^* or exists globally. Moreover, an upper bound and a lower bound of the blow-up time were also specified when the blow-up occurs. Their research relied on using differential inequality techniques. In [12], Ding et al. studied the following problems:

$$\begin{cases} (b(u))_t = \Delta u + f(x,u,|\nabla u|^2,t) & \text{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} = r(u) & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\Omega}. \end{cases} \tag{1.3}$$

In (1.3), the spatial domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is bounded, and the boundary $\partial\Omega$ is smooth. By combining the parabolic maximum principles and the differential inequality techniques, they obtained sufficient conditions for the existence of the blow-up solution and the global solution of (1.3). In addition, the upper bounds of the blow-up time t^* and the upper estimates of the blow-up rate were also given.

Inspired by the above two research works, in this paper we study the blow-up solutions of (1.1). Since the source term of the equation in (1.1) contains a gradient term, and the source term of the equation in (1.2) does not, the research methods in [22] cannot be used to research (1.1). In this paper, we use the method in [12] to study (1.1). In other words, we combine the parabolic maximum principles with the differential inequality techniques to study (1.1). The difficulty in using this research method is that some suitable auxiliary functions need to be constructed. We note that although the source terms in equations of (1.1) and (1.3) are the same, the boundary conditions are different. Therefore, the auxiliary functions in the paper [12] are not suitable for researching (1.1). In this paper, we need to construct some auxiliary functions that are completely different from those in [12] to study the blow-up solution of (1.1). We give sufficient conditions to guarantee that the solution of (1.1) blows up at a finite time t^*. The upper bounds of the blow-up time t^* and the upper estimates of the blow-up rate are also obtained.

For convenience, in this paper we use a comma to represent partial derivative and adopt the summation convention, for example,

$$u_{,i}u_{,j}u_{,ij} = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

2 The main result and its proof

Due to the need to study the blow-up solution of (1.1), we define the following two constants:

$$\alpha = \min_{x\in\overline{\Omega}} \frac{\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0) + f(x,u_0,|\nabla u_0|^2,0)}{\mathrm{e}^{u_0}}, \tag{2.1}$$

$$\beta = \inf_{(x,s,t)\in\Omega\times\overline{\mathbb{R}_+}\times\mathbb{R}_+} \frac{f(x,s,0,t)}{\mathrm{e}^{s}}, \tag{2.2}$$

where u_0 is the initial value of (1.1). For the same purpose, we also construct two auxiliary functions as follows:

$$G(x,t) = b'(u)u_t - \alpha\mathrm{e}^u, \quad (x,t)\in\overline{\Omega}\times[0,t^*), \tag{2.3}$$

$$H(s) = \int_s^{+\infty} \frac{b'(\tau)}{\mathrm{e}^{\tau}}\,\mathrm{d}\tau, \quad s\in\overline{\mathbb{R}_+}, \tag{2.4}$$

where $u(x,t)$ is a nonnegative $C^3(\Omega\times(0,t^*))\cap C^2(\overline{\Omega}\times[0,t^*))$ solution of (1.1). Now we have

$$H'(s) = -\frac{b'(s)}{\mathrm{e}^s} < 0, \quad s\in\mathbb{R}_+,$$

which implies that the function H has an inverse function H^{-1}. The following Theorem 2.1 is the main result of the blow-up solution to (1.1).

Theorem 2.1 *Let u be a nonnegative $C^3(\Omega\times(0,t^*))\cap C^2(\overline{\Omega}\times[0,t^*))$ solution of* (1.1). *Assume the following four assumptions are true*:

(i)

$$\beta \geq \alpha > 0. \tag{2.5}$$

(ii)

$$\int_{M_0}^{+\infty} \frac{b'(\tau)}{\mathrm{e}^{\tau}}\,\mathrm{d}\tau < +\infty, \quad M_0 = \max_{x\in\overline{\Omega}} u_0(x). \tag{2.6}$$

(iii) *For $s\in\overline{\mathbb{R}_+}$*,

$$(p-1)\left(\frac{1}{b'(s)}\right)' + (p-2)\frac{1}{b'(s)} \geq 0, \qquad \left(\frac{1}{b'(s)}\right)'' + 2\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)} \geq 0. \tag{2.7}$$

(iv) *For $(x,s,r,t)\in\Omega\times\overline{\mathbb{R}_+}\times\overline{\mathbb{R}_+}\times\mathbb{R}_+$*,

$$\begin{aligned} &f_r(x,s,r,t)\geq 0, \qquad f_t(x,s,r,t)\geq 0, \\ &\frac{f_s(x,s,r,t)}{b'(s)} - (p-1)f(x,s,r,t)\left[\left(\frac{1}{b'(s)}\right)' + \frac{1}{b'(s)}\right] \geq 0. \end{aligned} \tag{2.8}$$

Then, $u(x,t)$ must blow up at a finite time t^* and

$$t^* \le \frac{1}{\alpha}\int_{M_0}^{+\infty} \frac{b'(\tau)}{\mathrm{e}^{\tau}}\,\mathrm{d}\tau,$$

as well as

$$u(x,t) \le H^{-1}\bigl(\alpha\bigl(t^*-t\bigr)\bigr), \quad (x,t)\in \overline{\Omega}\times[0,t^*).$$

Proof By directly calculating the auxiliary function $G(x,t)$ defined in (2.3), we have

$$G_{,i} = b'' u_t u_{,i} + b' u_{t,i} - \alpha \mathrm{e}^u u_{,i} \tag{2.9}$$

and

$$G_{,ij} = b''' u_t u_{,i} u_{,j} + b'' u_{t,j} u_{,i} + b'' u_{t,i} u_{,j} + b'' u_t u_{,ij} + b' u_{t,ij} - \alpha \mathrm{e}^u u_{,i} u_{,j} - \alpha \mathrm{e}^u u_{,ij}. \tag{2.10}$$

By (2.10), we get

$$\Delta G = G_{ii} = b'''|\nabla u|^2 u_t + 2b''(\nabla u\cdot\nabla u_t) + b'' u_t \Delta u + b'\Delta u_t - \alpha \mathrm{e}^u|\nabla u|^2 - \alpha \mathrm{e}^u \Delta u. \tag{2.11}$$

Making use of the first equation of (1.1), we derive

$$\begin{aligned}
G_t &= \bigl(b'(u)u_t\bigr)_t - \alpha \mathrm{e}^u u_t = \bigl[\bigl(b(u)\bigr)_t\bigr]_t - \alpha \mathrm{e}^u u_t \\
&= \bigl[\nabla\cdot\bigl(|\nabla u|^{p-2}\nabla u\bigr) + f(x,u,q,t)\bigr]_t - \alpha \mathrm{e}^u u_t \\
&= \bigl[|\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4} u_{,i} u_{,j} u_{,ij} + f(x,u,q,t)\bigr]_t - \alpha \mathrm{e}^u u_t \\
&= (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_t)\Delta u + |\nabla u|^{p-2}\Delta u_t \\
&\quad + (p-2)(p-4)|\nabla u|^{p-6}(\nabla u\cdot\nabla u_t)u_{,i} u_{,j} u_{,ij} + 2(p-2)|\nabla u|^{p-4} u_{t,i} u_{,j} u_{,ij} \\
&\quad + (p-2)|\nabla u|^{p-4} u_{,i} u_{,j} u_{t,ij} + f_u u_t + 2f_q(\nabla u\cdot\nabla u_t) + f_t - \alpha \mathrm{e}^u u_t,
\end{aligned} \tag{2.12}$$

where $q = |\nabla u|^2$. It follows from (2.10)–(2.12) that

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{b'}\Delta G + (p-2)\frac{|\nabla u|^{p-4}}{b'} u_{,i} u_{,j} G_{,ij} - G_t \\
&\quad = (p-1)\frac{b'''}{b'}|\nabla u|^p u_t + 2(p-1)\frac{b''}{b'}|\nabla u|^{p-2}(\nabla u\cdot\nabla u_t) + \frac{b''}{b'}|\nabla u|^{p-2} u_t \Delta u \\
&\qquad - \alpha(p-1)\frac{\mathrm{e}^u}{b'}|\nabla u|^p - \alpha\frac{\mathrm{e}^u}{b'}|\nabla u|^{p-2}\Delta u + (p-2)\frac{b''}{b'}|\nabla u|^{p-4} u_t u_{,i} u_{,j} u_{,ij} \\
&\qquad - \alpha(p-2)\frac{\mathrm{e}^u}{b'}|\nabla u|^{p-4} u_{,i} u_{,j} u_{,ij} - (p-2)|\nabla u|^{p-4}(\nabla u\cdot\nabla u_t)\Delta u \\
&\qquad - (p-2)(p-4)|\nabla u|^{p-6}(\nabla u\cdot\nabla u_t)u_{,i} u_{,j} u_{,ij} - 2(p-2)|\nabla u|^{p-4} u_{t,i} u_{,j} u_{,ij} \\
&\qquad + \bigl(\alpha \mathrm{e}^u - f_u\bigr)u_t - 2f_q(\nabla u\cdot\nabla u_t) - f_t.
\end{aligned} \tag{2.13}$$

With (2.9), we obtain

$$u_{t,i} = \frac{1}{b'}G_{,i} - \frac{b''}{b'}u_t u_{,i} + \alpha\frac{\mathrm{e}^u}{b'}u_{,i} \tag{2.14}$$

and

$$\nabla u_t = \frac{1}{b'}\nabla G - \frac{b''}{b'}u_t\nabla u + \alpha\frac{e^u}{b'}\nabla u. \tag{2.15}$$

We insert (2.14) and (2.15) into (2.13) to derive

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{b'}\Delta G + (p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,j}G_{,ij}\\
&\quad+\frac{1}{b'}\bigg((p-2)(p-4)|\nabla u|^{p-6}u_{,i}u_{,j}u_{,ij} + (p-2)|\nabla u|^{p-4}\Delta u - 2(p-1)\frac{b''}{b'}|\nabla u|^{p-2}\\
&\quad+2f_q\bigg)(\nabla u\cdot\nabla G) + 2(p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,ij}G_{,i} - G_t\\
&=\bigg((p-1)\frac{b'''}{b'} - 2(p-1)\frac{(b'')^2}{(b')^2}\bigg)|\nabla u|^p u_t + \bigg(2\alpha(p-1)\frac{b''e^u}{(b')^2} - \alpha(p-1)\frac{e^u}{b'}\bigg)|\nabla u|^p\\
&\quad+(p-1)\frac{b''}{b'}|\nabla u|^{p-2}u_t\Delta u - \alpha(p-1)\frac{e^u}{b'}|\nabla u|^{p-2}\Delta u\\
&\quad+(p-1)(p-2)\frac{b''}{b'}|\nabla u|^{p-4}u_t u_{,i}u_{,j}u_{,ij} - \alpha(p-1)(p-2)\frac{e^u}{b'}|\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij}\\
&\quad+\big(\alpha e^u - f_u\big)u_t + 2\frac{f_q b''}{b'}|\nabla u|^2 u_t - 2\alpha\frac{f_q e^u}{b'}|\nabla u|^2 - f_t.
\end{aligned}\tag{2.16}$$

By the first equation of (1.1), we have

$$|\nabla u|^{p-2}\Delta u = b'u_t - (p-2)|\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij} - f. \tag{2.17}$$

We insert (2.17) into (2.16) to get

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{b'}\Delta G + (p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,j}G_{,ij}\\
&\quad+\frac{1}{b'}\bigg((p-2)(p-4)|\nabla u|^{p-6}u_{,i}u_{,j}u_{,ij} + (p-2)|\nabla u|^{p-4}\Delta u - 2(p-1)\frac{b''}{b'}|\nabla u|^{p-2}\\
&\quad+2f_q\bigg)(\nabla u\cdot\nabla G) + 2(p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,ij}G_{,i} - G_t\\
&=\bigg((p-1)\frac{b'''}{b'} - 2(p-1)\frac{(b'')^2}{(b')^2}\bigg)|\nabla u|^p u_t + \bigg(2\alpha(p-1)\frac{b''e^u}{(b')^2} - \alpha(p-1)\frac{e^u}{b'}\bigg)|\nabla u|^p\\
&\quad+(p-1)b''(u_t)^2 - \bigg(\alpha(p-2)e^u + f_u + (p-1)\frac{fb''}{b'}\bigg)u_t\\
&\quad+\alpha(p-1)\frac{fe^u}{b'} + 2\frac{f_d b''}{b'}|\nabla u|^2 u_t - 2\alpha\frac{f_d e^u}{b'}|\nabla u|^2 - f_t.
\end{aligned}\tag{2.18}$$

It follows from (2.3) that

$$u_t = \frac{1}{b'}G + \alpha\frac{e^u}{b'}. \tag{2.19}$$

We insert (2.19) into (2.18) to obtain

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{b'}\Delta G+(p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,j}G_{,ij}\\
&\quad+\frac{1}{b'}\biggl((p-2)(p-4)|\nabla u|^{p-6}u_{,i}u_{,j}u_{,ij}+(p-2)|\nabla u|^{p-4}\Delta u-2(p-1)\frac{b''}{b'}|\nabla u|^{p-2}\\
&\quad+2f_q\biggr)(\nabla u\cdot\nabla G)+2(p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,ij}G_{,i}\\
&\quad+\biggl\{\frac{1}{(b')^2}\biggl[\biggl(2(p-1)\frac{(b'')^2}{b'}-(p-1)b'''\biggr)|\nabla u|^p-2f_qb''|\nabla u|^2\biggr]\\
&\quad+\frac{1}{b'}\biggl[\alpha e^u\biggl(p-2-2(p-1)\frac{b''}{b'}\biggr)+f_u+(p-1)\frac{fb''}{b'}-(p-1)\frac{b''}{b'}G\biggr]\biggr\}G-G_t\\
&=-\alpha(p-1)e^u\biggl[\biggl(\frac{1}{b'}\biggr)''+2\biggl(\frac{1}{b'}\biggr)'+\frac{1}{b'}\biggr]|\nabla u|^p-\alpha^2e^{2u}\biggl[(p-1)\biggl(\frac{1}{b'}\biggr)'+(p-2)\frac{1}{b'}\biggr]\\
&\quad-\alpha e^u\biggl\{\frac{f_u}{b'}-(p-1)f\biggl[\biggl(\frac{1}{b'}\biggr)'+\frac{1}{b'}\biggr]\biggr\}-2\alpha e^uf_q\biggl[\biggl(\frac{1}{b'}\biggr)'+\frac{1}{b'}\biggr]|\nabla u|^2-f_t. \qquad (2.20)
\end{aligned}$$

From assumptions (2.7) and (2.8) we know that the right-hand side of equality (2.20) is nonpositive. So now we have

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{b'}\Delta G+(p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,j}G_{,ij}\\
&\quad+\frac{1}{b'}\biggl((p-2)(p-4)|\nabla u|^{p-6}u_{,i}u_{,j}u_{,ij}+(p-2)|\nabla u|^{p-4}\Delta u-2(p-1)\frac{b''}{b'}|\nabla u|^{p-2}\\
&\quad+2f_q\biggr)(\nabla u\cdot\nabla G)+2(p-2)\frac{|\nabla u|^{p-4}}{b'}u_{,i}u_{,ij}G_{,i}\\
&\quad+\biggl\{\frac{1}{(b')^2}\biggl[\biggl(2(p-1)\frac{(b'')^2}{b'}-(p-1)b'''\biggr)|\nabla u|^p-2f_qb''|\nabla u|^2\biggr]\\
&\quad+\frac{1}{b'}\biggl[\alpha e^u\biggl(p-2-2(p-1)\frac{b''}{b'}\biggr)+f_u+(p-1)\frac{fb''}{b'}-(p-1)\frac{b''}{b'}G\biggr]\biggr\}G\\
&\quad-G_t\le 0\quad\text{in }\Omega\times\bigl(0,t^*\bigr). \qquad (2.21)
\end{aligned}$$

Combining (2.21) and parabolic maximum principles ([24], Theorems 2.7–2.9, pp. 20–21), it follows that G may take its nonpositive minimum value under the following three possible cases:

(1) for $t=0$; (2) at a point $(\hat{x},\hat{t})\in\Omega\times(0,t^*)$ where $|\nabla u(\hat{x},\hat{t})|=0$; (3) on the boundary $\partial\Omega\times(0,t^*)$.

We first study the first case. By (2.1), we derive

$$\begin{aligned}
&\min_{x\in\overline{\Omega}}G(x,0)\\
&=\min_{x\in\overline{\Omega}}\bigl\{\nabla\cdot\bigl(|\nabla u_0|^{p-2}\nabla u_0\bigr)+f\bigl(x,u_0,|\nabla u_0|^2,0\bigr)-\alpha e^{u_0}\bigr\}\\
&=\min_{x\in\overline{\Omega}}\biggl\{e^{u_0}\biggl(\frac{\nabla\cdot(|\nabla u_0|^{p-2}\nabla u_0)+f(x,u_0,|\nabla u_0|^2,0)}{e^{u_0}}-\alpha\biggr)\biggr\}=0. \qquad (2.22)
\end{aligned}$$

Then, we study the second case. With (2.5), we have

$$
\begin{aligned}
G(\hat{x},\hat{t}) &= \Big[\nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big) + f\big(x,u,|\nabla u|^2,t\big) - \alpha \mathrm{e}^u\Big]\Big|_{(\hat{x},\hat{t})} \\
&= \Big[|\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij} + f\big(x,u,|\nabla u|^2,t\big) - \alpha \mathrm{e}^u\Big]\Big|_{(\hat{x},\hat{t})} \\
&\geq \Big[-|\nabla u|^{p-2}|\Delta u| - (p-2)|\nabla u|^{p-4}|\nabla u||\nabla u||u_{,ij}| + f\big(x,u,|\nabla u|^2,t\big) - \alpha \mathrm{e}^u\Big]\Big|_{(\hat{x},\hat{t})} \\
&= \bigg[-|\nabla u|^{p-2}|\Delta u| - (p-2)|\nabla u|^{p-2}|u_{,ij}| + \mathrm{e}^u\bigg(\frac{f(x,u,|\nabla u|^2,t)}{\mathrm{e}^u} - \alpha\bigg)\bigg]\bigg|_{(\hat{x},\hat{t})} \\
&= \mathrm{e}^{u(\hat{x},\hat{t})}\bigg(\frac{f(\hat{x},u(\hat{x},\hat{t}),0,\hat{t})}{\mathrm{e}^{u(\hat{x},\hat{t})}} - \alpha\bigg) \geq \mathrm{e}^{u(\hat{x},\hat{t})}(\beta-\alpha) \geq 0.
\end{aligned}
\tag{2.23}
$$

Finally, we study the third case. Using of the boundary condition of (1.1), we obtain

$$
\frac{\partial G}{\partial n} = b''\frac{\partial u}{\partial n}u_t + b'\frac{\partial u_t}{\partial n} - \alpha \mathrm{e}^u\frac{\partial u}{\partial n} = b'\bigg(\frac{\partial u}{\partial n}\bigg)_t = 0 \quad \text{on } \partial\Omega\times\big(0,t^*\big). \tag{2.24}
$$

It follows from parabolic maximum principles and (2.22)–(2.24) that the minimum value of G in $\overline{\Omega}\times[0,t^*)$ is zero. In fact, if the minimum value of G in $\overline{\Omega}\times[0,t^*)$ is negative, then this minimum value must be taken on $\partial\Omega\times(0,t^*)$. So there is $(\tilde{x},\tilde{t})\in\partial\Omega\times(0,t^*)$ such that $G(\tilde{x},\tilde{t}) = \min_{(x,t)\in\overline{\Omega}\times[0,t^*)} G(x,t) < 0$. The parabolic maximum principle means

$$
\frac{\partial G}{\partial n}\bigg|_{(\tilde{x},\tilde{t})} < 0,
$$

which contradicts (2.24). Hence, the minimum value of G in $\overline{\Omega}\times[0,t^*)$ is zero. In other words, we have

$$
G(x,t)\geq 0 \quad \text{in } \overline{\Omega}\times[0,t^*),
$$

from which we get the following differential inequality:

$$
\frac{b'(u)}{\alpha \mathrm{e}^u}u_t \geq 1 \quad \text{in } \overline{\Omega}\times[0,t^*). \tag{2.25}
$$

At the point $\tilde{x}\in\overline{\Omega}$ where $u_0(\tilde{x}) = M_0$, we integrate (2.25) from 0 to t to get

$$
\frac{1}{\alpha}\int_0^t \frac{b'(u)}{\mathrm{e}^u}u_t\,\mathrm{d}t = \frac{1}{\alpha}\int_{M_0}^{u(\tilde{x},t)}\frac{b'(\tau)}{\mathrm{e}^\tau}\,\mathrm{d}\tau \geq t. \tag{2.26}
$$

It follows from (2.26) that u must blow up at some finite time t^*. In fact, assuming that the solution u does not blow up, we have, for any $t>0$,

$$
\frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{b'(\tau)}{\mathrm{e}^\tau}\,\mathrm{d}\tau > \frac{1}{\alpha}\int_{M_0}^{u(\tilde{x},t)}\frac{b'(\tau)}{\mathrm{e}^\tau}\,\mathrm{d}\tau \geq t. \tag{2.27}
$$

Letting $t \to +\infty$ in (2.27), we derive

$$\frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{b'(\tau)}{\mathrm{e}^{\tau}}\,\mathrm{d}\tau = +\infty,$$

which contradicts (2.6). Hence, u must blow up at some finite time t^*. Furthermore, letting $t \to t^*$ in (2.26), we obtain

$$t^* \le \frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{b'(\tau)}{\mathrm{e}^{\tau}}\,\mathrm{d}\tau.$$

At each fixed point $x \in \overline{\Omega}$, we integrate (2.25) from t to $\check{t}$ ($0 < t < \check{t} < t^*$) and use (2.4) to deduce

$$H\bigl(u(x,t)\bigr) \ge H\bigl(u(x,t)\bigr) - H\bigl(u(x,\check{t})\bigr) = \int_{u(x,t)}^{u(x,\check{t})}\frac{b'(\tau)}{\mathrm{e}^{\tau}}\,\mathrm{d}\tau \ge \alpha(\check{t}-t). \tag{2.28}$$

Letting $\check{t} \to t^*$ in (2.28), we have

$$H\bigl(u(x,t)\bigr) \ge \alpha\bigl(t^*-t\bigr)$$

and

$$u(x,t) \le H^{-1}\bigl(\alpha\bigl(t^*-t\bigr)\bigr).$$

The proof is complete. □

In Theorem 2.1, we have the following conclusions when $b(u) \equiv u$:

Corollary 2.1 *Let u be a nonnegative $C^3(\Omega \times (0,t^*)) \cap C^2(\overline{\Omega} \times [0,t^*))$ solution of the following problem*:

$$\begin{cases} u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u) + f(x,u,|\nabla u|^2,t) & \textit{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} = 0 & \textit{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \ge 0 & \textit{in } \overline{\Omega}, \end{cases}$$

where $p > 2$, the spatial domain $\Omega \subset \mathbb{R}^N$ ($N \ge 2$) is bounded, and the boundary $\partial\Omega$ is smooth. Assume the following two assumptions are true:

(i)

$$\beta \ge \alpha > 0.$$

(ii) *For $(x,s,r,t) \in \Omega \times \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \times \mathbb{R}_+$,*

$$f_r(x,s,r,t) \ge 0, \qquad f_t(x,s,r,t) \ge 0, \qquad f_s(x,s,r,t) - (p-1)f(x,s,r,t) \ge 0.$$

Then $u(x,t)$ must blow up at a finite time t^* and

$$t^* \le \frac{\mathrm{e}^{-M_0}}{\alpha}, \quad M_0 = \max_{x\in\overline{\Omega}} u_0(x),$$

as well as

$$u(x,t) \le \ln \frac{1}{\alpha(t^*-t)}.$$

3 Application

In the following, we give an example to illustrate the application of Theorem 2.1

Example 3.1 Let u be a nonnegative $C^3(\Omega \times (0,t^*)) \cap C^2(\overline{\Omega} \times [0,t^*))$ solution of the following problem:

$$\begin{cases} (\mathrm{e}^{\frac{u}{2}})_t = \nabla \cdot (|\nabla u|\nabla u) + (1 + \sum_{i=1}^3 x_i^2 + |\nabla u|^2 t)\mathrm{e}^u & \text{in } \Omega \times (0,t^*), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = 1 & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega = \{x = (x_1,x_2,x_3) | \sum_{i=1}^3 x_i^2 < 1\}$. We now have

$$p = 3, \qquad b(u) = \mathrm{e}^{\frac{u}{2}}, \qquad f\bigl(x,u,|\nabla u|^2,t\bigr) = \left(1 + \sum_{i=1}^3 x_i^2 + |\nabla u|^2 t\right)\mathrm{e}^u, \qquad u_0(x) = 1.$$

It follows from (2.1) and (2.2) that

$$\alpha = \min_{x\in\overline{\Omega}} \frac{\nabla \cdot (|\nabla u_0|^{p-2}\nabla u_0) + f(x,u_0,|\nabla u_0|^2,0)}{\mathrm{e}^{u_0}} = \min_{x\in\overline{\Omega}} \left(1 + \sum_{i=1}^3 x_i^2\right) = 1$$

and

$$\beta = \inf_{(x,s,t)\in\Omega\times\overline{\mathbb{R}_+}\times\mathbb{R}_+} \frac{f(x,s,0,t)}{\mathrm{e}^s} = \inf_{(x,s,t)\in\Omega\times\overline{\mathbb{R}_+}\times\mathbb{R}_+} \left(1 + \sum_{i=1}^3 x_i^2\right) = 1,$$

from which we know that the assumption (2.5) holds. We also easily check that the assumptions (2.6)–(2.8) hold true. Hence, Theorem 2.1 implies that $u(x,t)$ must blow up at a finite time t^* and

$$t^* \le \frac{1}{\alpha}\int_{M_0}^{+\infty} \frac{b'(\tau)}{\mathrm{e}^\tau}\,\mathrm{d}\tau = \frac{1}{2}\int_1^{+\infty} \mathrm{e}^{-\frac{\tau}{2}}\,\mathrm{d}\tau = \mathrm{e}^{-\frac{1}{2}},$$

as well as

$$u(x,t) \le H^{-1}\bigl(\alpha\bigl(t^*-t\bigr)\bigr) = \ln\frac{1}{(t^*-t)^2}, \quad (x,t) \in \overline{\Omega} \times [0,t^*).$$

4 Conclusions

In this paper, we combine parabolic maximum principles with the differential inequality techniques to study the blow-up solution of problem (1.1). The key to our research is constructing two auxiliary functions (2.3) and (2.4). With their help, we obtain sufficient conditions for the existence of the blow-up solution of problem (1.1). In addition, we also give an upper bound on the blow-up time and an upper estimate of the blow-up rate.

Acknowledgements
The author is sincerely grateful to the editors and anonymous referees for careful reading of the original manuscript and useful comments.

Funding
This work was supported by the National Natural Science Foundation of China (No 61473180).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All results are due to JD. All authors read and approved the final manuscript.

Publisher's Note

Received: 18 February 2020 Accepted: 21 August 2020 Published online: 01 September 2020

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