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Alternating iterative algorithms for the split equality problem without prior knowledge of operator norms

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Abstract

In this paper, we study the alternating CQ algorithm for solving the split equality problem in Hilbert spaces. It is, however, not easy to implement since its selection of the stepsize requires prior information on the norms of bounded linear operators. To avoid this difficulty, we propose several modified algorithms in which the selection of the stepsize is independent of the norms. In particular, we consider the case whenever the convex sets involved are level sets of given convex functions.

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Keywords: Split equality problem; Alternating CQ algorithm; Variable stepsize

1 Introduction

The split feasibility problem (SFP) requires finding a point $x \in H_1$ satisfying the property

$$x \in C \quad \text{and} \quad Ax \in Q, \tag{1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, and C and Q are two nonempty closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. The SFP was first introduced by Censor and Elfving [5] and has a very broad range of applications in many disciplines including signal processing, image reconstruction problem, and radiation therapy; see [2, 3, 11]. Various iterative algorithms have been constructed to solve SFP (1); see [10, 21–24, 26]. An iterative algorithm for solving the SFP, called the CQ algorithm, has the following iterative step:

$$x_{n+1} = P_C\big(x_n - \gamma A^*(I - P_Q)Ax_n\big), \tag{2}$$

where $\gamma \in (0, \frac{2}{\|A\|^2})$, I denotes the identity operator, A^* denotes the adjoint of A, and P_C and P_Q are projections onto C and Q, respectively. The SFP can be also solved by a different method [19, 29]:

$$x_{n+1} = x_n - \gamma_n\big[(I - P_C)x_n + A^*(I - P_Q)Ax_n\big], \tag{3}$$

where

$$\gamma_n = \frac{\|(I-P_C)x_n\|^2 + \|(I-P_Q)Ax_n\|^2}{\|(I-P_C)x_n + A^*(I-P_Q)Ax_n\|^2}.$$

It is known that both (2) and (3) converge weakly to a solution of the SFP if it is consistent, that is, its solution set is nonempty.

Recently, Moudafi [13] introduced the split equality problem (SEP):

$$\text{Find} \quad x \in C, y \in Q \quad \text{such that} \quad Ax = By. \tag{4}$$

Here, $C \subseteq H_1$, $Q \subseteq H_2$ are two nonempty closed and convex subsets, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, where H_1, H_2, H_3 are Hilbert spaces. In what follows, we always assume that the SEP is consistent, namely

$$S = \{(x,y) \in C \times Q \mid Ax = By\} \neq \emptyset.$$

Many algorithms for solving the SEP have been proposed; see [8, 9, 12, 13, 15, 20]. In particular, Byrne and Moudafi [4, 14] introduced the simultaneous CQ algorithm:

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_n - By_n)), \end{cases} \tag{5}$$

where $\gamma_n \in (\epsilon, 2/(\|A\|^2 + \|B\|^2) - \epsilon)$ for small enough $\epsilon > 0$. To determine stepsize γ_n in (5), one needs first to calculate or estimate the norms $\|A\|$ and $\|B\|$, which is in general difficult or even impossible. To overcome this drawback, many authors have conducted worthwhile works[6–8, 19]. Among these works, Wang [19] suggested a novel variable-step:

$$\tau_n = \frac{\rho_n}{\max(\|A^*(Ax_n - By_n)\|, \|B^*(Ax_n - By_n)\|)}, \tag{6}$$

where $\{\rho_n\}$ is a sequence of positive real numbers such that

$$\sum_{n=0}^{\infty} \rho_n = \infty, \qquad \sum_{n=0}^{\infty} \rho_n^2 < \infty. \tag{7}$$

For solving the SEP, Moudafi [13] also introduced the alternating CQ-algorithm:

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_{n+1} - By_n)), \end{cases} \tag{8}$$

where $\{\gamma_n\}$ is a nondecreasing sequence such that $\gamma_n \in (\epsilon, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}) - \epsilon)$ for small enough $\epsilon > 0$. It is worth noting that in algorithm (8) the choice of the stepsize still depends on the norms $\|A\|$ and $\|B\|$. Thus, a similar question arises: Does there exist a way to select the stepsize in algorithm (8) that does not depend on the operator norms? It is the purpose of this paper to answer this question affirmatively. Motivated by the choice of stepsize (6), we propose three alternating iterative algorithms for the SEP, in which the choice of the stepsize is independent of the norms $\|A\|$ and $\|B\|$. To the best of our knowledge, this is the first work to study alternating iterative algorithms for the SEP without prior knowledge of operator norms.

2 Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and C is a nonempty closed convex subset in H. We denote by I the identity operator on H, and by $\mathrm{Fix}(T)$ the set of the fixed points of an operator T. Given a sequence $\{x_n\}$ in H, $\omega_w(x_n)$ stands for the set of cluster points in the weak topology. The notation $\to$ stands for strong convergence and $\rightharpoonup$ stands for weak convergence.

Definition 2.1 Let $T : H \to H$ be an operator. Then T is

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in H.$$

Definition 2.2 For any $x \in H$, the metric projection onto C is defined as

$$P_C x = \operatorname{argmin}\big\{\|y - x\| \mid y \in C\big\}.$$

The projection P_C has the following well-known properties.

Lemma 2.3 ([1, 17]) *Let $x, y \in H$ and $z \in C$. Then*

(i) $\langle x - P_C x, z - P_C x\rangle \leq 0$;

(ii) *P_C is firmly nonexpansive*;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$.

Definition 2.4 Let $T : H \to H$ be an operator with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if, for any $\{x_n\}$ in H, the following implication holds:

$$x_n \rightharpoonup x \quad \text{and} \quad (I - T)x_n \to 0 \quad \Longrightarrow \quad x \in \mathrm{Fix}(T).$$

It is well known that $I - T$ is demiclosed at zero if T is nonexpansive [1, 17]. Since P_C is clearly nonexpansive, then $I - P_C$ is demiclosed at zero.

Definition 2.5 Let $\lambda \in (0, 1)$ and $f : H \to (-\infty, +\infty]$ be a proper function.

(i) The function f is convex if

$$f\big(\lambda x + (1 - \lambda)y\big) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in H.$$

(ii) A vector $u \in H$ is a subgradient of f at a point x if

$$f(y) \geq f(x) + \langle u, y - x\rangle, \quad \forall y \in H.$$

(iii) The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the subdifferential of f.

Next, we state the following lemmas which will be used in the sequel.

Lemma 2.6 ([1]) *Let $f : H \to (-\infty, +\infty]$ be a proper convex function. Then f is semicontinuous if and only if it is weakly semicontinuous.*

Lemma 2.7 ([18]) *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.8 ([25]) *Let $\{x_n\}$ be a sequence in H satisfying the properties:*
(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each $x \in C$;*
(ii) $\omega_w(x_n) \subseteq C$.
Then $\{x_n\}$ is weakly convergent to a point in C.

Lemma 2.9 *Assume $\{a_n\}$, $\{\gamma_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\gamma_n)(a_n - b_n + c_n), \quad n \geq 0, \tag{9}$$

where $\sum_{n=0}^{\infty} \gamma_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then
(i) $\lim_{n\to\infty} a_n$ *exists,*
(ii) $\sum_{n=0}^{\infty} b_n < \infty$.

Proof (i) Since $\sum_{n=0}^{\infty} \gamma_n$ is convergent, we have $\prod_{n=0}^{\infty}(1+\gamma_n)$ is convergent. Let $\prod_{n=0}^{\infty}(1+\gamma_n) = \gamma$. It follows from (9) that

$$\begin{aligned}
a_n &\leq (1+\gamma_{n-1})a_{n-1} + (1+\gamma_{n-1})c_{n-1} \\
&\leq (1+\gamma_{n-1})(1+\gamma_{n-2})a_{n-2} + (1+\gamma_{n-1})(1+\gamma_{n-2})c_{n-2} + (1+\gamma_{n-1})c_{n-1} \\
&\leq \cdots \\
&\leq \prod_{k=0}^{n-1}(1+\gamma_k)a_0 + \sum_{j=0}^{n-1}\left(\prod_{k=j}^{n-1}(1+\gamma_k)c_j\right) \\
&\leq \gamma a_0 + \gamma \sum_{j=0}^{n-1} c_j \\
&\leq \gamma a_0 + \gamma c,
\end{aligned}$$

where $c = \sum_{n=0}^{\infty} c_n$. This implies that

$$a_{n+1} \leq (1+\gamma_n)a_n + (1+\gamma_n)c_n$$

$$\leq a_n + \gamma_n(\gamma a_0 + \gamma c) + \gamma c_n.$$

Since $\sum_{n=0}^{\infty} \gamma_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$, by Lemma 2.7, part (i) holds.

(ii) It follows from (9) that

$$\begin{aligned}
a_{n+1} &\leq (1+\gamma_n)a_n - (1+\gamma_n)b_n + (1+\gamma_n)c_n \\
&\leq (1+\gamma_n)(1+\gamma_{n-1})a_{n-1} - (1+\gamma_n)(1+\gamma_{n-1})b_{n-1} - (1+\gamma_n)b_n \\
&\quad + (1+\gamma_n)(1+\gamma_{n-1})c_{n-1} + (1+\gamma_n)c_n \\
&\leq \cdots \\
&\leq \prod_{k=0}^{n}(1+\gamma_k)a_0 - \sum_{j=0}^{n}\left(\prod_{k=j}^{n}(1+\gamma_k)b_j\right) + \sum_{j=0}^{n}\left(\prod_{k=j}^{n}(1+\gamma_k)c_j\right) \\
&\leq \gamma a_0 + \gamma c - \sum_{j=0}^{n} b_j,
\end{aligned}$$

where the last inequality holds since $\prod_{k=j}^{n}(1+\gamma_k) \geq 1$. So we obtain

$$\sum_{j=0}^{n} b_j \leq \gamma a_0 + \gamma c,$$

which means that $\sum_{n=0}^{\infty} b_n < \infty$. □

Lemma 2.10 *Let $\{u_n\}$, $\{\rho_n\}$ be sequences of nonnegative real numbers such that*

(i) $\sum_{k=0}^{\infty} \rho_n u_n < \infty$;

(ii) *there exists some $M > 0$ such that, for all $k \geq 0$, $|u_{n+1} - u_n| \leq M\rho_n$;*

(iii) *$\{\rho_n\}$ satisfies condition (7).*

Then $\lim_{n\to\infty} u_n = 0$.

Proof It follows from (i) and the assumption $\sum_{k=0}^{\infty} \rho_n = \infty$ that

$$\liminf_{n\to\infty} u_n = 0.$$

On the other hand, observe that

$$\begin{aligned}
u_{n+1}^2 &= u_n^2 + 2u_n(u_{n+1} - u_n) + (u_{n+1} - u_n)^2 \\
&\leq u_n^2 + 2M\rho_n u_n + M^2\rho_n^2. \qquad (10)
\end{aligned}$$

It is clear that $\sum_{n=0}^{\infty}(2M\rho_n u_n + M^2\rho_n^2) < \infty$ due to (i) and (7). Applying Lemma 2.7 to (10), we obtain that $\lim_{n\to\infty} u_n$ exists. Hence $\lim_{n\to\infty} u_n = 0$, since we have shown that $\liminf_{n\to\infty} u_n = 0$. □

3 Alternating iterative algorithm I

As shown in the introduction, one needs first to calculate (or at least estimate) the norms $\|A\|$ and $\|B\|$ when algorithm (8) is implemented. But this is difficult or even impossible. To overcome this difficulty, we aim to introduce the following alternating iterative algorithm which does not depend on the norms. In this and next sections, we mainly consider the case whenever the projections P_C and P_Q have closed-form expressions; for example, half spaces and closed balls.

Algorithm 3.1 Let $(x_0, y_0) \in H_1 \times H_2$ be arbitrary, $\delta > 0$ be a constant, and $\{\rho_n\}$ be a sequence of positive real numbers. Given (x_n, y_n), construct (x_{n+1}, y_{n+1}) via the formula

$$\begin{cases} x_{n+1} = P_C(x_n - \tau_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \tau_n B^*(Ax_{n+1} - By_n)), \end{cases} \tag{11}$$

where $\tau_n = \rho_n(\max\{\|A^*(Ax_n - By_n)\|, \|B^*(Ax_n - By_n)\|, \delta\})^{-1}$.

Theorem 3.2 *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 3.1. If $\{\rho_n\}$ satisfies condition (7), then $\{(x_n, y_n)\}$ converges weakly to a solution of SEP (4).*

Proof Let $(x^*, y^*) \in S$ be arbitrarily chosen. Then $x^* \in C$, $y^* \in Q$ and $Ax^* = By^*$. Let

$$z_n = P_Q\bigl(y_n + \tau_n B^*(Ax_n - By_n)\bigr).$$

From Lemma 2.3 and the obvious fact that $x_n \in C$, it follows that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \bigl\|P_C\bigl(x_n - \tau_n A^*(Ax_n - By_n)\bigr) - P_C x_n\bigr\| \\ &\le \tau_n \bigl\|A^*(Ax_n - By_n)\bigr\| \le \rho_n, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|y_{n+1} - z_n\| &= \bigl\|P_Q\bigl(y_n + \tau_n B^*(Ax_{n+1} - By_n)\bigr) - P_Q\bigl(y_n + \tau_n B^*(Ax_n - By_n)\bigr)\bigr\| \\ &\le \bigl\|\tau_n B^*(Ax_{n+1} - Ax_n)\bigr\| \\ &\le \tau_n \|B\| \|A\| \|x_{n+1} - x_n\| \\ &\le \rho_n^2 \frac{1}{\delta} \|B\| \|A\|. \end{aligned} \tag{13}$$

On the other hand, in view of (11) and Lemma 2.3, we have

$$\begin{aligned} \bigl\|x_{n+1} - x^*\bigr\|^2 &= \bigl\|P_C\bigl(x_n - \tau_n A^*(Ax_n - By_n)\bigr) - x^*\bigr\|^2 \\ &\le \bigl\|x_n - x^*\bigr\|^2 - 2\tau_n\bigl\langle A^*(Ax_n - By_n), x_n - x^*\bigr\rangle + \tau_n^2\bigl\|A^*(Ax_n - By_n)\bigr\|^2 \\ &\le \bigl\|x_n - x^*\bigr\|^2 - 2\tau_n\bigl\langle Ax_n - By_n, Ax_n - Ax^*\bigr\rangle + \rho_n^2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \bigl\|z_n - y^*\bigr\|^2 &= \bigl\|P_Q\bigl(y_n + \tau_n B^*(Ax_n - By_n)\bigr) - y^*\bigr\|^2 \\ &\le \bigl\|y_n - y^*\bigr\|^2 + 2\tau_n\bigl\langle B^*(Ax_n - By_n), y_n - y^*\bigr\rangle + \tau_n^2\bigl\|B^*(Ax_n - By_n)\bigr\|^2 \end{aligned}$$

$$\leq \|y_n - y^*\|^2 + 2\tau_n\langle Ax_n - By_n, By_n - By^*\rangle + \rho_n^2.$$

By adding the last two inequalities and using the fact that $Ax^* = By^*$, we finally obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|z_n - y^*\|^2 \\ &\quad\leq \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - 2\tau_n\|Ax_n - By_n\|^2 + 2\rho_n^2.\end{aligned} \tag{14}$$

It then follows from Young's inequality, (13), and (14) that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ &\quad= \|x_{n+1} - x^*\|^2 + \|y_{n+1} - z_n + z_n - y^*\|^2 \\ &\quad\leq \|x_{n+1} - x^*\|^2 + (1 + \rho_n^2)\|z_n - y^*\|^2 + \left(1 + \frac{1}{\rho_n^2}\right)\|y_{n+1} - z_n\|^2 \\ &\quad\leq (1 + \rho_n^2)\left(\|x_{n+1} - x^*\|^2 + \|z_n - y^*\|^2 + \rho_n^2\|A\|^2\|B\|^2\frac{1}{\delta^2}\right) \\ &\quad\leq (1 + \rho_n^2)\left(\|x_n - x^*\|^2 + \|y_n - y^*\|^2 - 2\tau_n\|Ax_n - By_n\|^2 + \rho_n^2\left(2 + \|A\|^2\|B\|^2\frac{1}{\delta^2}\right)\right).\end{aligned}$$

Now, by setting $a_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$, $\gamma_n = \rho_n^2$, $b_n = 2\tau_n\|Ax_n - By_n\|^2$, and $c_n = \rho_n^2(2 + \|A\|^2\|B\|^2\frac{1}{\delta^2})$, we obtain

$$a_{n+1} \leq (1 + \gamma_n)(a_n - b_n + c_n). \tag{15}$$

Applying Lemma 2.9 to (15), we conclude that there exists $a^* \geq 0$ such that $\lim_{n\to\infty} a_n = a^*$ and

$$\sum_{n=0}^{\infty} \tau_n\|Ax_n - By_n\|^2 < \infty.$$

Hence $\{a_n\}$ is bounded, and so are $\{x_n\}$ and $\{y_n\}$. Since A and B are bounded linear operators, there exists $M_1 > 0$ such that $\|A^*(Ax_n - By_n)\| \leq M_1$, $\|B^*(Ax_n - By_n)\| \leq M_1$, and $\|Ax_{n+1} - By_{n+1}\| + \|Ax_n - By_n\| \leq M_1$. So we have

$$\sum_{n=0}^{\infty} \frac{\rho_n}{\max(M_1, \delta)}\|Ax_n - By_n\|^2 \leq \sum_{n=0}^{\infty} \tau_n\|Ax_n - By_n\|^2 < \infty.$$

This implies that

$$\sum_{n=0}^{\infty} \rho_n\|Ax_n - By_n\|^2 < \infty.$$

We next show $\lim_{n\to\infty}\|Ax_n - By_n\|^2 = 0$. From (13) and the obvious fact that $y_n \in Q$, it follows that

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq \|y_{n+1} - z_n\| + \|z_n - y_n\| \\ &= \|y_{n+1} - z_n\| + \|P_Q(y_n + \tau_n B^*(Ax_n - By_n)) - P_Q y_n\|\end{aligned}$$

$$\le \rho_n^2 \frac{1}{\delta}\|B\|\|A\| + \tau_n \big\|B^*(Ax_n - By_n)\big\|$$
$$\le \rho_n^2 \frac{1}{\delta}\|B\|\|A\| + \rho_n.$$

Let $u_n = \|Ax_n - By_n\|^2$. By the last inequality and (12), we have

$$\begin{aligned}|u_{n+1} - u_n| &= \big|\|Ax_{n+1} - By_{n+1}\| - \|Ax_n - By_n\|\big|\big(\|Ax_{n+1} - By_{n+1}\| + \|Ax_n - By_n\|\big)\\ &\le \big\|(Ax_{n+1} - By_{n+1}) - (Ax_n - By_n)\big\|M_1\\ &\le \big(\|A\|\|x_{n+1} - x_n\| + \|B\|\|y_{n+1} - y_n\|\big)M_1\\ &\le \rho_n\left(\|A\| + \|B\| + \rho_n \frac{1}{\delta}\|B\|^2\|A\|\right)M_1\\ &\le \rho_n M_2,\end{aligned}$$

where M_2 is a positive number such that $(\|A\| + \|B\| + \rho_n \frac{1}{\delta}\|B\|^2\|A\|)M_1 \le M_2$ for all $n \ge 0$. Therefore, by Lemma 2.10, we have $\lim_{n\to\infty} \|Ax_n - By_n\|^2 = 0$.

Now we turn to prove that $\omega_w(x_n, y_n) \subseteq S$. Let $\bar{x}$ and $\bar{y}$ be weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. We assume that $x_{n_k} \rightharpoonup \bar{x}$ and $y_{n_k} \rightharpoonup \bar{y}$, where $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are subsequences of $\{x_n\}$ and $\{y_n\}$, respectively. Since $\{x_{n_k}\} \subseteq C$, $\{y_{n_k}\} \subseteq Q$ and C and Q are closed and convex, we have $\bar{x} \in C$ and $\bar{y} \in Q$. Furthermore, the weak convergence of $\{Ax_{n_k} - By_{n_k}\}$ to $A\bar{x} - B\bar{y}$ and the weakly lower semicontinuity of the squared norm imply

$$\|A\bar{x} - B\bar{y}\|^2 \le \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\|^2 = 0.$$

Hence $(\bar{x}, \bar{y}) \in S$.

It is readily seen that $\lim_{n\to\infty} \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ exists for each $(x^*, y^*) \in S$ and $\omega_w(x_n, y_n) \subseteq S$. Therefore, it follows from Lemma 2.8 that $\{(x_n, y_n)\}$ converges weakly to a solution of SEP (4). □

Remark 3.3 It is clear that our choice of the stepsize in Algorithm 3.1 does not need any information on the values of $\|A\|$ and $\|B\|$.

4 Alternating iterative algorithm II

In this section, we propose another alternating iterative algorithm for problem (4), in which the choice of the stepsize does not need any prior information of operator norms.

Algorithm 4.1 Let $(x_0, y_0) \in H_1 \times H_2$ be arbitrary, $\delta > 0$ be a constant, and $\{\rho_n\}$ be a sequence of positive real numbers. Given (x_n, y_n), construct (x_{n+1}, y_{n+1}) via the formula

$$\begin{cases} x_{n+1} = x_n - \tau_n[(I - P_C)x_n + A^*(Ax_n - By_n)],\\ y_{n+1} = y_n - \tau_n[(I - P_Q)y_n - B^*(Ax_{n+1} - By_n)],\end{cases} \tag{16}$$

where $\tau_n = \rho_n(\max\{\|(I - P_C)x_n + A^*(Ax_n - By_n)\|, \|(I - P_Q)y_n - B^*(Ax_n - By_n)\|, \delta\})^{-1}$.

Theorem 4.2 *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 4.1. If $\{\rho_n\}$ satisfies condition (7), then $\{(x_n, y_n)\}$ converges weakly to a solution of SEP (4).*

Proof Let $(x^*, y^*) \in S$ be arbitrarily chosen. Then $x^* \in C$, $y^* \in Q$, and $Ax^* = By^*$. Let

$$z_n = y_n - \tau_n\big[(I - P_Q)y_n - B^*(Ax_n - By_n)\big].$$

Then we have

$$\begin{aligned} \|y_{n+1} - z_n\|^2 &= \big\|\tau_n B^*(Ax_{n+1} - Ax_n)\big\|^2 \\ &\le \tau_n^2\|B\|^2\|A\|^2\|x_{n+1} - x_n\|^2 \\ &\le \tau_n^2\|B\|^2\|A\|^2\rho_n^2 \\ &\le \rho_n^4\frac{1}{\delta^2}\|B\|^2\|A\|^2. \end{aligned} \tag{17}$$

It follows from (16) that

$$\begin{aligned} \big\|x_{n+1} - x^*\big\|^2 &= \big\|x_n - x^*\big\|^2 - 2\tau_n\big\langle (I - P_C)x_n + A^*(Ax_n - By_n), x_n - x^*\big\rangle \\ &\quad + \tau_n^2\big\|(I - P_C)x_n + A^*(Ax_n - By_n)\big\|^2 \\ &\le \big\|x_n - x^*\big\|^2 - 2\tau_n\big\langle (I - P_C)x_n, x_n - x^*\big\rangle - 2\tau_n\big\langle Ax_n - By_n, Ax_n - Ax^*\big\rangle \\ &\quad + \tau_n^2\big\|(I - P_C)x_n + A^*(Ax_n - By_n)\big\|^2 \\ &\le \big\|x_n - x^*\big\|^2 - 2\tau_n\big\|(I - P_C)x_n\big\|^2 - 2\tau_n\big\langle Ax_n - By_n, Ax_n - Ax^*\big\rangle + \rho_n^2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \big\|z_n - y^*\big\|^2 &= \big\|y_n - y^*\big\|^2 - 2\tau_n\big\langle (I - P_Q)y_n - B^*(Ax_n - By_n), y_n - y^*\big\rangle \\ &\quad + \tau_n^2\big\|(I - P_Q)y_n - B^*(Ax_n - By_n)\big\|^2 \\ &\le \big\|y_n - y^*\big\|^2 - 2\tau_n\big\langle (I - P_Q)y_n, y_n - y^*\big\rangle + 2\tau_n\big\langle Ax_n - By_n, By_n - By^*\big\rangle \\ &\quad + \tau_n^2\big\|(I - P_Q)y_n - B^*(Ax_n - By_n)\big\|^2 \\ &\le \big\|y_n - y^*\big\|^2 - 2\tau_n\big\|(I - P_Q)y_n\big\|^2 + 2\tau_n\big\langle Ax_n - By_n, By_n - By^*\big\rangle + \rho_n^2. \end{aligned}$$

By adding the last two inequalities and using the fact $Ax^* = By^*$, we finally obtain

$$\begin{aligned} \big\|x_{n+1} - x^*\big\|^2 + \big\|z_n - y^*\big\|^2 &\le \big\|x_n - x^*\big\|^2 + \big\|y_n - y^*\big\|^2 - 2\tau_n\big(\big\|(I - P_C)x_n\big\|^2 \\ &\quad + \big\|(I - P_Q)y_n\big\|^2 + \|Ax_n - By_n\|^2\big) + 2\rho_n^2. \end{aligned} \tag{18}$$

It then follows from Young's inequality, (17), and (18) that

$$\begin{aligned} &\big\|x_{n+1} - x^*\big\|^2 + \big\|y_{n+1} - y^*\big\|^2 \\ &\quad = \big\|x_{n+1} - x^*\big\|^2 + \big\|y_{n+1} - z_n + z_n - y^*\big\|^2 \\ &\quad \le \big\|x_{n+1} - x^*\big\|^2 + \big(1 + \rho_n^2\big)\big\|z_n - y^*\big\|^2 + \left(1 + \frac{1}{\rho_n^2}\right)\|y_{n+1} - z_n\|^2 \\ &\quad \le \big(1 + \rho_n^2\big)\left(\big\|x_{n+1} - x^*\big\|^2 + \big\|z_n - y^*\big\|^2 + \rho_n^2\frac{1}{\delta^2}\|B\|^2\|A\|^2\right) \end{aligned}$$

$$\leq \left(1+\rho_n^2\right)\Bigg[\left\|x_n-x^*\right\|^2+\left\|y_n-y^*\right\|^2-2\tau_n\left(\left\|(I-P_C)x_n\right\|^2 + \left\|(I-P_Q)y_n\right\|^2+\|Ax_n-By_n\|^2\right)+\rho_n^2\left(2+\frac{1}{\delta^2}\|B\|^2\|A\|^2\right)\Bigg].$$

Now, by setting $a_n = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$, $\gamma_n = \rho_n^2$, $c_n = \rho_n^2(2 + \frac{1}{\delta^2}\|B\|^2\|A\|^2)$ and

$$b_n = 2\tau_n\left(\left\|(I-P_C)x_n\right\|^2+\left\|(I-P_Q)y_n\right\|^2+\|Ax_n-By_n\|^2\right),$$

we can immediately obtain

$$a_{n+1} \leq (1+\gamma_n)(a_n - b_n + c_n). \tag{19}$$

Applying Lemma 2.9 to (19), we conclude that $\lim_{n\to\infty} a_n$ exists and

$$\sum_{n=0}^{\infty} 2\tau_n\left(\left\|(I-P_C)x_n\right\|^2+\left\|(I-P_Q)y_n\right\|^2+\|Ax_n-By_n\|^2\right)<\infty.$$

Therefore $\{a_n\}$ is bounded, and so are $\{x_n\}$ and $\{y_n\}$. Observe that

$$\left\|P_Cx_n-x^*\right\|\leq\left\|x_n-x^*\right\|,\qquad \left\|P_Qy_n-y^*\right\|\leq\left\|y_n-y^*\right\|.$$

This implies that $\{(I-P_C)x_n\}$ and $\{(I-P_Q)y_n\}$ are also bounded. Since A and B are bounded linear operators, there exists $M_3 > 0$ such that, for all $n \geq 0$, $\|(I-P_C)x_n + A^*(Ax_n - By_n)\| \leq M_3$, $\|(I-P_Q)y_n - B^*(Ax_n - By_n)\| \leq M_3$, $\|(I-P_C)x_{n+1}\| + \|(I-P_C)x_n\| < M_3$, and $\|(I-P_Q)y_{n+1}\| + \|(I-P_Q)y_n\| < M_3$. Thus, we have

$$\sum_{n=0}^{\infty}\frac{\rho_n}{\max(M_3,\delta)}\left(\left\|(I-P_C)x_n\right\|^2+\left\|(I-P_Q)y_n\right\|^2+\|Ax_n-By_n\|^2\right) \leq \sum_{n=0}^{\infty}\tau_n\left(\left\|(I-P_C)x_n\right\|^2+\left\|(I-P_Q)y_n\right\|^2+\|Ax_n-By_n\|^2\right)<\infty.$$

This implies that

$$\sum_{n=0}^{\infty}\rho_n\left\|(I-P_C)x_n\right\|^2<\infty,\qquad \sum_{n=0}^{\infty}\rho_n\left\|(I-P_Q)y_n\right\|^2<\infty, \tag{20}$$

and

$$\sum_{n=0}^{\infty}\rho_n\|Ax_n-By_n\|^2<\infty.$$

We next prove that $\lim_{n\to\infty}\|(I-P_C)x_n\|^2 = 0$, $\lim_{n\to\infty}\|(I-P_Q)y_n\|^2 = 0$, and $\lim_{n\to\infty}\|Ax_n - By_n\|^2 = 0$. From (17), we get

$$\|y_{n+1}-y_n\|\leq\|y_{n+1}-z_n\|+\|z_n-y_n\|\leq\rho_n^2\frac{1}{\delta}\|B\|\|A\|+\rho_n.$$

Observe that

$$\begin{aligned}
&\left|\left\|(I-P_C)x_{n+1}\right\|^2-\left\|(I-P_C)x_n\right\|^2\right| \\
&\quad=\left|\left\|(I-P_C)x_{n+1}\right\|-\left\|(I-P_C)x_n\right\|\right|\left(\left\|(I-P_C)x_{n+1}\right\|+\left\|(I-P_C)x_n\right\|\right) \\
&\quad\le\left\|(I-P_C)x_{n+1}-(I-P_C)x_n\right\|M_3 \\
&\quad\le\|x_{n+1}-x_n\|M_3 \\
&\quad\le M_3\rho_n.
\end{aligned}$$

This along with (20) and Lemma 2.10 implies that $\lim_{n\to\infty}\|(I-P_C)x_n\|^2=0$. Similarly, we have

$$\begin{aligned}
&\left|\left\|(I-P_Q)y_{n+1}\right\|^2-\left\|(I-P_Q)y_n\right\|^2\right| \\
&\quad=\left|\left\|(I-P_Q)y_{n+1}\right\|-\left\|(I-P_Q)y_n\right\|\right|\left(\left\|(I-P_Q)y_{n+1}\right\|+\left\|(I-P_Q)y_n\right\|\right) \\
&\quad\le\left\|(I-P_Q)y_{n+1}-(I-P_Q)y_n\right\|M_3 \\
&\quad\le\|y_{n+1}-y_n\|M_3 \\
&\quad\le\rho_n\left(\rho_n\frac{1}{\delta}\|B\|\|A\|+1\right)M_3 \\
&\quad\le M_4\rho_n,
\end{aligned}$$

where M_4 is a positive number such that $(\rho_n\frac{1}{\delta}\|B\|\|A\|+1)M_3\le M_4$ for all $n\ge 0$. Hence, by Lemma 2.10, we conclude that $\lim_{n\to\infty}\|(I-P_Q)y_n\|^2=0$. Using a similar method in Theorem 3.2, we obtain $\lim_{n\to\infty}\|Ax_n-By_n\|^2=0$.

Now, we show that $\omega_w(x_n,y_n)\subseteq S$. Let $\bar{x}$ and $\bar{y}$ be weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. We assume that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are subsequences of $\{x_n\}$ and $\{y_n\}$ such that $x_{n_k}\rightharpoonup\bar{x}$ and $y_{n_k}\rightharpoonup\bar{y}$, respectively. Since $I-P_C$ and $I-P_Q$ are demiclosed at zero, it follows from $(I-P_C)x_{n_k}\to 0$ and $(I-P_Q)y_{n_k}\to 0$ that $\bar{x}\in C$ and $\bar{y}\in Q$. Furthermore, the weak convergence of $\{Ax_{n_k}-By_{n_k}\}$ to $A\bar{x}-B\bar{y}$ and the weakly lower semicontinuity of the squared norm imply

$$\|A\bar{x}-B\bar{y}\|^2\le\liminf_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|^2=0.$$

Hence $(\bar{x},\bar{y})\in S$. In summary, we have proved that $\lim_{n\to\infty}\|x_n-x^*\|^2+\|y_n-y^*\|^2$ exists for each $(x^*,y^*)\in S$ and $\omega_w(x_n,y_n)\subseteq S$. Thus, we conclude from Lemma 2.8 that $\{(x_n,y_n)\}$ converges weakly to a solution of SEP (4). □

5 A relaxed alternating iterative algorithm

In this section, we consider the case whenever P_C or P_Q fails to have a closed-form expression. Indeed, Moudafi [12] considered one of such cases when C and Q are level sets:

$$C=\{x\in H_1\mid c(x)\le 0\} \tag{21}$$

and

$$Q=\{y\in H_2\mid q(y)\le 0\}, \tag{22}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are two convex lower semicontinuous and subdifferential functions on H_1 and H_2, respectively. Here the subdifferential operators ∂c and ∂q of c and q are assumed to be bounded, i.e., bounded on bounded sets. In this case, it is known that the associated projections are very hard to calculate. To overcome this difficulty, Moudafi [12] presented the relaxed alternating CQ-algorithm (RACQA):

$$\begin{cases} x_{n+1} = P_{C_n}(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}(y_n + \gamma B^*(Ax_{n+1} - By_n)), \end{cases} \tag{23}$$

where $\{C_n\}$ and $\{Q_n\}$ are two sequences of closed convex sets defined by

$$C_n = \{x \in H_1 \mid c(x_n) + \langle \xi_n, x - x_n\rangle \leq 0\}, \quad \xi_n \in \partial c(x_n) \tag{24}$$

and

$$Q_n = \{y \in H_2 \mid q(y_n) + \langle \eta_n, y - y_n\rangle \leq 0\}, \quad \eta_n \in \partial q(y_n). \tag{25}$$

Since C_n and Q_n are clearly half-spaces, the associated projections thus have closed form expressions. This indicates that the implementation of RACQA is very easy. Under suitable conditions, Moudafi [12] proved that the RACQA converges weakly to a solution of (4).

Following the RACQA and our proposed algorithms, we now present a relaxed alternating iterative algorithm in which we just need projections onto half-spaces [16, 27, 28].

In what follows, we will treat SEP (4) under the following assumptions.

(A1) The sets C and Q are given in (21) and (22), respectively.

(A2) For any $x \in H_1$ and $y \in H_2$, at least one subgradient $\xi \in \partial c(x)$ and $\eta \in \partial q(y)$ can be calculated.

Remark 5.1 It follows from Lemma 2.6 that both c and q are weakly lower semicontinuous by condition (A2), since $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex.

We are now in a position to present a relaxed alternative iterative algorithm that does not depend on operator norms for solving SEP (4).

Algorithm 5.2 Let (x_0, y_0) be arbitrary, $\delta > 0$ be a constant, and $\{\rho_n\}$ be a sequence of positive real numbers. Given (x_n, y_n), construct (x_{n+1}, y_{n+1}) via the formula

$$\begin{cases} x_{n+1} = x_n - \tau_n[(I - P_{C_n})x_n + A^*(Ax_n - By_n)], \\ y_{n+1} = y_n - \tau_n[(I - P_{Q_n})y_n - B^*(Ax_{n+1} - By_n)], \end{cases} \tag{26}$$

where C_n and Q_n are given as in (24) and (25), respectively, and

$$\tau_n = \rho_n\bigl(\max\bigl\{\bigl\|(I - P_{C_n})x_n + A^*(Ax_n - By_n)\bigr\|, \bigl\|(I - P_{Q_n})y_n - B^*(Ax_n - By_n)\bigr\|, \delta\bigr\}\bigr)^{-1}.$$

Remark 5.3 By the definition of the subgradient, it is obvious that $C \subseteq C_n$ and $Q \subseteq Q_n$ for all $n \geq 0$. Since C_n and Q_n are both half-spaces, the projections onto C_n and Q_n can be directly calculated. Thus Algorithm 5.2 is easily implementable.

Theorem 5.4 *Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 5.2. If $\{\rho_n\}$ satisfies condition (7), then $\{(x_n, y_n)\}$ converges weakly to a solution of SEP (4).*

Proof Take $(x^*, y^*) \in S$, i.e., $x^* \in C$ (and thus $x^* \in C_n$); $y^* \in Q$ (and thus $y^* \in Q_n$), $Ax^* = By^*$. Similarly as in the proof of Theorem 4.2, we can conclude that $\lim_{n\to\infty}(\|x_n - x^*\|^2 + \|y_n - y^*\|^2)$ exists, $\lim_{n\to\infty} \|(I - P_{C_n})x_n\|^2 = 0$, $\lim_{n\to\infty} \|(I - P_{Q_n})y_n\|^2 = 0$, and $\lim_{n\to\infty} \|Ax_n - By_n\|^2 = 0$.

We next show that $\omega_w(x_n, y_n) \subseteq S$. Since $\lim_{n\to\infty}(\|x_n - x^*\|^2 + \|y_n - y^*\|^2)$ exists, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Let $\bar{x}$ and $\bar{y}$ be weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. Without loss of generality, we assume that $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$. Since ∂c is bounded on bounded sets, there is a constant $\sigma_1 > 0$ such that $\|\xi_n\| \le \sigma_1$ for all $n \ge 0$. From (24) and the fact that $P_{C_n}(x_n) \in C_n$, it follows that

$$c(x_n) \le \langle \xi_n, x_n - P_{C_n}x_n\rangle \le \sigma_1 \big\|(I - P_{C_n})x_n\big\|.$$

The weakly lower semicontinuity of c leads to

$$c(\bar{x}) \le \liminf_{n\to+\infty} c(x_n) \le \sigma_1 \liminf_{n\to+\infty} \big\|(I - P_{C_n})x_n\big\| = 0,$$

and therefore $\bar{x} \in C$. Likewise, since ∂q is bounded on bounded sets, there is a constant $\sigma_2 > 0$ such that $\|\eta_n\| \le \sigma_2$ for all $n \ge 0$. From (25) and the fact that $P_{Q_n}(y_n) \in Q_n$, it follows that

$$q(y_n) \le \langle \eta_n, y_n - P_{Q_n}y_n\rangle \le \sigma_2 \big\|(I - P_{Q_n})y_n\big\|.$$

Again, the weakly lower semicontinuity of q leads to

$$q(\bar{y}) \le \liminf_{n\to+\infty} q(y_n) \le \sigma_2 \liminf_{n\to+\infty} \big\|(I - P_{Q_n})y_n\big\| = 0,$$

and therefore $\bar{y} \in Q$. Furthermore, the weak convergence of $\{Ax_n - By_n\}$ to $A\bar{x} - B\bar{y}$ and the weakly lower semicontinuity of the squared norm imply

$$\|A\bar{x} - B\bar{y}\|^2 \le \liminf_{n\to+\infty} \|Ax_n - By_n\|^2 = 0.$$

Hence $(\bar{x}, \bar{y}) \in S$.

Finally, we deduce from Lemma 2.8 that $\{(x_n, y_n)\}$ converges weakly to a solution of SEP (4), since $\lim_{n\to\infty} \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ exists for each $(x^*, y^*) \in S$ and $\omega_w(x_n, y_n) \subseteq S$. □

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