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# Jensen–Mercer inequality for GA-convex functions and some related inequalities

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**Abstract**

In this paper, firstly, we prove a Jensen–Mercer inequality for GA-convex functions. After that, we establish weighted Hermite–Hadamard's inequalities for GA-convex functions using the new Jensen–Mercer inequality, and we establish some new inequalities connected with Hermite–Hadamard–Mercer type inequalities for differentiable mappings whose derivatives in absolute value are GA-convex.

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**Keywords:** GA-convex function; Jensen–Mercer inequality; Hadamard fractional integrals; Hermite–Hadamard inequality

## 1 Introduction

Let the real function $\phi$ be defined on some nonempty interval $I$ of the real line $\mathbb{R}$. The function $\phi$ is said to be convex on $I$ if the inequality

$$\phi\bigl(\theta u + (1-\theta)v\bigr) \le \theta\phi(u) + (1-\theta)\phi(v) \tag{1}$$

holds for all $u, v \in I$ and $\theta \in [0,1]$.

The following inequality is well known in the literature as Hermite–Hadamard's inequality.

**Theorem 1.1** *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $u, v \in I$ with $u < v$. The following double inequality holds*:

$$\phi\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v \phi(x)\,dx \le \frac{\phi(u)+\phi(v)}{2}. \tag{2}$$

**Definition 1.1** ([6, 7]) A function $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$\phi\bigl(u^{\theta}v^{1-\theta}\bigr) \le \theta\phi(u) + (1-\theta)\phi(v) \tag{3}$$

for all $u, v \in I$ and $\theta \in [0,1]$.

Since condition (3) can be written as

$$(\phi \circ \exp)\bigl(\theta \ln u + (1-\theta)\ln v\bigr) \le \theta(\phi \circ \exp)(\ln u) + (1-\theta)(\phi \circ \exp)(\ln v),$$

we observe that $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is GA-convex on $I$ if and only if $\phi \circ \exp$ is convex on $\ln I := \{\ln x : x \in I\}$. If $I = [u,v]$, then $\ln I = [\ln u, \ln v]$. By using the useful property, we easily say that if $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ is GA-convex on $I$ and $u, v \in I$ with $u < v$, then

$$\begin{aligned}\phi(\sqrt{uv}) &\le \frac{1}{\ln v - \ln u}\int_{\ln u}^{\ln v} (\phi \circ \exp)(x)\,dx = \frac{1}{\ln v - \ln u}\int_u^v \frac{\phi(x)}{x}\,dx \\ &\le \frac{\phi(u)+\phi(v)}{2}.\end{aligned}$$

**Theorem 1.2** *A real-valued function $\phi$ defined on an interval $I$ is convex if and only if, for all $u_1, u_2, \ldots, u_n$ in $I$ and all scalars $\lambda_i \in [0,1]$ $(i = \overline{1,n})$ with $\sum_{i=1}^n \lambda_i = 1$, we have*

$$\phi\left(\sum_{i=1}^n \lambda_i u_i\right) \le \sum_{i=1}^n \lambda_i \phi(u_i). \tag{4}$$

This inequality is the well-known Jensen inequality in literature [8].

*Remark* 1.1 Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function on $I$, then $\phi \circ \exp$ is convex on $\ln I := \{\ln x : x \in I\}$, and by (4) we get

$$\begin{aligned}(\phi \circ \exp)\left(\sum_{i=1}^n \lambda_i \ln u_i\right) &= \phi\left(\prod_{i=1}^n u_i^{\lambda_i}\right) \\ &\le \sum_{i=1}^n \lambda_i (\phi \circ \exp)(\ln u_i) \\ &= \sum_{i=1}^n \lambda_i \phi(u_i).\end{aligned}$$

Thus, we get Jensen's inequality for GA-convex functions as follows:

$$\phi\left(\prod_{i=1}^n u_i^{\lambda_i}\right) \le \sum_{i=1}^n \lambda_i \phi(u_i). \tag{5}$$

In [5], Mercer proved the following variant of Jensen's inequality known as the Jensen–Mercer inequality.

**Theorem 1.3** *Let $\phi : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on $[a,b]$, then*

$$\phi\left(u + v - \sum_{i=1}^n \lambda_i x_i\right) \le \phi(u) + \phi(v) - \sum_{i=1}^n \lambda_i \phi(x_i) \tag{6}$$

*for each $x_i \in [u,v]$ and $\lambda_i \in [0,1]$ $(i = \overline{1,n})$ with $\sum_{i=1}^n \lambda_i = 1$.*

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

**Definition 1.2** Let $\phi \in L[u,v]$. The left-sided and right-sided Hadamard fractional integrals $J^{\alpha}_{u+}\phi$ and $J^{\alpha}_{v-}\phi$ of order $\alpha > 0$ with $v > u \geq 0$ are defined by

$$J^{\alpha}_{u+}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_u^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\phi(t)\frac{dt}{t}, \quad u < x < v,$$

and

$$J^{\alpha}_{v-}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^v \left(\ln\frac{t}{x}\right)^{\alpha-1}\phi(t)\frac{dt}{t}, \quad u < x < v,$$

respectively, where $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,dt$ (see [2]).

In this paper, firstly, the Jensen–Mercer inequality is proved for GA-convex functions. After that we prove weighted Hermite–Hadamard's inequalities for GA-convex functions using the new Jensen–Mercer inequality, and we establish some new fractional inequalities connected with the right sides of Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute value are GA-convex.

## 2 Weighted Hermite–Hadamard–Mercer inequalities for GA-convex functions

**Lemma 2.1** *Let $\phi : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$, then*

$$\phi\left(\frac{ab}{x}\right) \leq \phi(a) + \phi(b) - \phi(x) \tag{7}$$

*for each $x \in [a,b]$.*

*Proof* Let $x \in [a,b]$ be an arbitrary point. Then there exists $\mu \in [0,1]$ such that we can write $x = a^{\lambda}b^{1-\lambda}$ and $ab/x = a^{1-\lambda}b^{\lambda}$. By using the GA-convexity of $\phi$, we obtain

$$\begin{aligned}\phi\left(\frac{ab}{x}\right) &\leq (1-\lambda)\phi(a) + \lambda\phi(b)\\ &= \phi(a) + \phi(b) - \left[\lambda\phi(a) + (1-\lambda)\phi(b)\right]\\ &\leq \phi(a) + \phi(b) - \phi\left(a^{\lambda}b^{1-\lambda}\right)\\ &= \phi(a) + \phi(b) - \phi(x).\end{aligned}$$

□

Hermite–Hadamard–Fejer inequalities can be represented for GA-convex functions using a Jensen–Mercer type inequality as follows.

**Theorem 2.1** *Let $\phi : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function on $[a,b]$, then*

$$\phi\left(\frac{ab}{\prod_{i=1}^n x_i^{\lambda_i}}\right) \leq \phi(a) + \phi(b) - \sum_{i=1}^n \lambda_i\phi(x_i) \tag{8}$$

*for each $x_i \in [a,b]$ and $\lambda_i \in [0,1]$ $(i = \overline{1,n})$ with $\sum_{i=1}^n \lambda_i = 1$.*

*Proof* First method: By using inequality (5) and Lemma 2.1, we can write

$$\begin{aligned}\phi\left(\frac{ab}{\prod_{i=1}^{n} x_i^{\lambda_i}}\right) &= \phi\left(\prod_{i=1}^{n}\left(\frac{ab}{x_i}\right)^{\lambda_i}\right)\\ &\le \sum_{i=1}^{n}\lambda_i\phi\left(\frac{ab}{x_i}\right)\\ &\le \phi(a)+\phi(b)-\sum_{i=1}^{n}\lambda_i\phi(x_i).\end{aligned}$$

Second method: Since $\phi$ is a GA-convex function on $[a,b]$, $\phi\circ\exp$ is convex on $[\ln a,\ln b]$. From Theorem 1.3, we get

$$\begin{aligned}&(\phi\circ\exp)\left(\ln a+\ln b-\sum_{i=1}^{n}\lambda_i\ln x_i\right)\\ &\quad\le(\phi\circ\exp)(\ln a)+(\phi\circ\exp)(\ln b)-\sum_{i=1}^{n}\lambda_i(\phi\circ\exp)(\ln x_i)\end{aligned}$$

for each $x_i\in[a,b]$ and $\lambda_i\in[0,1]$ $(i=\overline{1,n})$ with $\sum_{i=1}^{n}\lambda_i=1$. This last inequality gives us the desired result. $\square$

**Theorem 2.2** *Let $\phi: I\subseteq(0,\infty)\to\mathbb{R}$ be a function such that $\phi\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\phi$ is a GA-convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is nonnegative and integrable, then the following inequalities hold*:

$$\begin{aligned}&\phi\left(\frac{ab}{\sqrt{xy}}\right)\int_{\frac{ab}{y}}^{\frac{ab}{x}}\frac{g(u)}{u}\,du\\ &\quad\le\frac{1}{2}\left\{\int_{\frac{ab}{y}}^{\frac{ab}{x}}\phi(u)g(u)\frac{du}{u}+\int_{\frac{ab}{y}}^{\frac{ab}{x}}\phi(u)g\left(\frac{(ab)^2}{xyu}\right)\frac{du}{u}\right\}\\ &\quad\le\frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right]\int_{\frac{ab}{y}}^{\frac{ab}{x}}\frac{g(u)}{u}\,du\\ &\quad\le\left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]\int_{\frac{ab}{y}}^{\frac{ab}{x}}\frac{g(u)}{u}\,du\end{aligned}\tag{9}$$

*for all $x,y\in[a,b]$.*

*Proof* Since $\phi$ is a GA-convex function on $[a,b]$, we have

$$\begin{aligned}\phi\left(\frac{ab}{\sqrt{xy}}\right)&=\phi\left(\sqrt{\left[\left(\frac{ab}{x}\right)^{t}\left(\frac{ab}{y}\right)^{1-t}\right]\left[\left(\frac{ab}{x}\right)^{1-t}\left(\frac{ab}{y}\right)^{t}\right]}\right)\\ &\le\frac{\phi((\frac{ab}{x})^{t}(\frac{ab}{y})^{1-t})+\phi((\frac{ab}{x})^{1-t}(\frac{ab}{y})^{t})}{2}\end{aligned}\tag{10}$$

for all $x,y\in[a,b]$ and $t\in[0,1]$.

Multiplying both sides of (10) by $g((\frac{ab}{x})^t(\frac{ab}{y})^{1-t})$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\phi\left(\frac{ab}{\sqrt{xy}}\right)\int_{\frac{ab}{y}}^{\frac{ab}{x}} g(u)\frac{du}{u} \le \frac{1}{2}\left\{\int_{\frac{ab}{y}}^{\frac{ab}{x}} \phi(u)g(u)\frac{du}{u} + \int_{\frac{ab}{y}}^{\frac{ab}{x}} \phi(u)g\left(\frac{(ab)^2}{xyu}\right)\frac{du}{u}\right\}$$

and the first inequality is proved.

For the proof of the second inequality in (9), by the GA-convexity of $\phi$, we have

$$\phi\left(\left(\frac{ab}{x}\right)^t\left(\frac{ab}{y}\right)^{1-t}\right) \le t\phi\left(\frac{ab}{x}\right) + (1-t)\phi\left(\frac{ab}{y}\right)$$

and

$$\phi\left(\left(\frac{ab}{x}\right)^{1-t}\left(\frac{ab}{y}\right)^{t}\right) \le t\phi\left(\frac{ab}{y}\right) + (1-t)\phi\left(\frac{ab}{x}\right).$$

By adding these inequalities, we have

$$\phi\left(\left(\frac{ab}{x}\right)^t\left(\frac{ab}{y}\right)^{1-t}\right) + \phi\left(\left(\frac{ab}{x}\right)^{1-t}\left(\frac{ab}{y}\right)^{t}\right) \le \phi\left(\frac{ab}{x}\right) + \phi\left(\frac{ab}{y}\right). \tag{11}$$

Then multiplying both sides of (11) by $\frac{1}{2}g((\frac{ab}{x})^t(\frac{ab}{y})^{1-t})$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\frac{1}{2}\left\{\int_{\frac{ab}{y}}^{\frac{ab}{x}} \phi(u)g(u)\frac{du}{u} + \int_{\frac{ab}{y}}^{\frac{ab}{x}} \phi(u)g\left(\frac{(ab)^2}{xyu}\right)\frac{du}{u}\right\} \le \frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right]\int_{\frac{ab}{y}}^{\frac{ab}{x}} g(u)\frac{du}{u}.$$

For the proof of the third inequality in (9), by inequality (7), we have

$$\frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right]\int_{\frac{ab}{y}}^{\frac{ab}{x}} g(u)\frac{du}{u} \le \left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]\int_{\frac{ab}{y}}^{\frac{ab}{x}} g(u)\frac{du}{u}.$$

The proof is completed. □

If we take $x = a$ and $y = b$ in Theorem 2.2, then we can derive the following weighted Hermite–Hadamard inequalities for GA-convex functions.

**Corollary 2.1** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a,b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is nonnegative and integrable, then the following inequalities hold*:

$$\begin{aligned}
&\phi(\sqrt{ab})\int_a^b \frac{g(u)}{u}\,du \\
&\quad\le \frac{1}{2}\left\{\int_a^b \phi(u)g(u)\frac{du}{u} + \int_a^b \phi(u)g\left(\frac{ab}{u}\right)\frac{du}{u}\right\} \\
&\quad\le \frac{\phi(a)+\phi(b)}{2}\int_a^b \frac{g(u)}{u}\,du.
\end{aligned} \tag{12}$$

*Remark* 2.1 Specially, if we choose that $g$ is geometrically symmetric to $\sqrt{ab}$ (i.e., $g(ab/u) = g(u)$ for all $u \in [a,b]$) in (12), then we get the following inequality:

$$\phi(\sqrt{ab})\int_a^b \frac{g(u)}{u}\,du \le \int_a^b \phi(u)g(u)\frac{du}{u} \le \frac{\phi(a)+\phi(b)}{2}\int_a^b \frac{g(u)}{u}\,du$$

which coincides with the inequality in [4, Theorem 2.2].

If we choose $g(u) = \frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{ab}{xu})$ or $g(u) = \frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{yu}{ab})$ in Theorem 2.2, then we obtain the following Hermite–Hadamard–Mercer inequalities for GA-convex functions via Hadamard fractional integrals.

**Corollary 2.2** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a,b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$, then the following inequalities for Hadamard fractional integrals hold*:

$$\begin{aligned}
\phi\left(\frac{ab}{\sqrt{xy}}\right) &\le \frac{\Gamma(\alpha+1)}{2\ln^{\alpha}(y/x)}\left\{J^{\alpha}_{\frac{ab}{y}+}\phi\left(\frac{ab}{x}\right) + J^{\alpha}_{\frac{ab}{x}-}\phi\left(\frac{ab}{y}\right)\right\} \\
&\le \frac{1}{2}\left[\phi\left(\frac{ab}{x}\right) + \phi\left(\frac{ab}{y}\right)\right] \\
&\le \phi(a)+\phi(b) - \frac{\phi(x)+\phi(y)}{2}
\end{aligned} \tag{13}$$

*for all $x,y \in [a,b]$ and $\alpha > 0$. Specially, we take $\alpha = 1$ in the above inequalities, then we get*

$$\begin{aligned}
\phi\left(\frac{ab}{\sqrt{xy}}\right) &\le \frac{1}{\ln(y/x)}\int_x^y \phi\left(\frac{ab}{u}\right)\frac{du}{u} \le \frac{1}{2}\left[\phi\left(\frac{ab}{x}\right) + \phi\left(\frac{ab}{y}\right)\right] \\
&\le \phi(a)+\phi(b) - \frac{\phi(x)+\phi(y)}{2}
\end{aligned}$$

*for all $x,y \in [a,b]$ with $x < y$.*

*Remark* 2.2 Specially, if we choose $x = a$ and $y = b$ in (13), then we get the following inequalities:

$$\phi(\sqrt{ab}) \le \frac{\Gamma(\alpha+1)}{2\ln^{\alpha}(b/a)}\left\{J^{\alpha}_{a+}\phi(b) + J^{\alpha}_{b-}\phi(a)\right\} \le \frac{\phi(a)+\phi(b)}{2}$$

which coincide with the inequality in [1, Theorem 2.1].

Let $w : [a,b] \to \mathbb{R}$ be a nonnegative and integrable function. If we choose $g(u) = \frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{ab}{xu})w(u)$ and $g(u) = \frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{yu}{ab})w(u)$ in Theorem 2.2, then we obtain the following weighted Hermite–Hadamard–Mercer inequalities for GA-convex functions via Hadamard fractional integrals.

**Corollary 2.3** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a,b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$ and $w : [a,b] \to \mathbb{R}$ is nonnegative and* *integrable, then the following inequalities for Hadamard fractional integrals hold*:

$$\begin{aligned}
&\phi\left(\frac{ab}{\sqrt{xy}}\right)J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)\\
&\quad\le\frac{1}{2}\left\{J^{\alpha}_{\frac{ab}{y}+}\phi w\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi w\left(\frac{ab}{y}\right)\right\}\\
&\quad\le\frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right]J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)\\
&\quad\le\left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)
\end{aligned}$$

*and*

$$\begin{aligned}
&\phi\left(\frac{ab}{\sqrt{xy}}\right)J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)\\
&\quad\le\frac{1}{2}\left\{J^{\alpha}_{\frac{ab}{y}+}\phi w\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi w\left(\frac{ab}{y}\right)\right\}\\
&\quad\le\frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right]J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)\\
&\quad\le\left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)
\end{aligned}$$

*for all* $x,y\in[a,b]$ *with* $x<y$. *Also, we obtain the following inequalities from both inequalities above*:

$$\begin{aligned}
&\phi\left(\frac{ab}{\sqrt{xy}}\right)\left[J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)+J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)\right]\\
&\quad\le J^{\alpha}_{\frac{ab}{y}+}\phi w\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi w\left(\frac{ab}{y}\right)\\
&\quad\le\frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right]\left[J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)+J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)\right]\\
&\quad\le\left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]\left[J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)+J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)\right]
\end{aligned}\tag{14}$$

*for all* $x,y\in[a,b]$ *with* $x<y$.

*Remark* 2.3 Specially, if we choose $x=a$ and $y=b$ in (14), then we get the following inequalities:

$$\begin{aligned}
\phi(\sqrt{ab})\left[J^{\alpha}_{b-}w(a)+J^{\alpha}_{a+}w(b)\right]&\le\left[J^{\alpha}_{b-}\phi w(a)+J^{\alpha}_{a+}\phi w(b)\right]\\
&\le\frac{\phi(a)+\phi(b)}{2}\left[J^{\alpha}_{b-}w(a)+J^{\alpha}_{a+}w(b)\right],
\end{aligned}$$

which coincide with the inequality in [3, Theorem 2.1].

If we choose $g(u)=1$ in Theorem 2.2, then we obtain the following Hermite–Hadamard–Mercer inequalities for GA-convex functions.

**Corollary 2.4** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$, then the following inequalities hold*:

$$\begin{aligned}\phi\left(\frac{ab}{\sqrt{xy}}\right) &\le \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du \\ &\le \frac{1}{2}\left[\phi\left(\frac{ab}{x}\right)+\phi\left(\frac{ab}{y}\right)\right] \\ &\le \left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]\end{aligned} \tag{15}$$

*for all $x, y \in [a,b]$ with $x < y$.*

**Theorem 2.3** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is nonnegative and integrable, then the following inequalities for fractional integrals hold*:

$$\begin{aligned}&\phi\left(\frac{ab}{\sqrt{xy}}\right)\int_{\frac{ab}{y}}^{\frac{ab}{x}} \frac{g(u)}{u}\,du \\ &\quad\le \frac{1}{2}\left\{\int_{\frac{ab}{y}}^{\frac{ab}{x}} \phi(u)g(u)\frac{du}{u} + \int_{\frac{ab}{y}}^{\frac{ab}{x}} \phi(u)g\left(\frac{(ab)^2}{xyu}\right)\frac{du}{u}\right\} \\ &\quad\le \left[\phi(a)+\phi(b)\right]\int_x^y g\left(\frac{ab}{u}\right)\frac{du}{u} - \frac{1}{2}\left[\int_x^y \phi(u)g\left(\frac{ab}{u}\right)\frac{du}{u} + \int_x^y \phi(u)g\left(\frac{abu}{xy}\right)\frac{du}{u}\right] \\ &\quad\le \left[\phi(a)+\phi(b)-\phi(\sqrt{xy})\right]\int_x^y g\left(\frac{ab}{u}\right)\frac{du}{u}\end{aligned} \tag{16}$$

*for all $x, y \in [a,b]$.*

*Proof* The first inequality of (16) was proved in Theorem 2.2. For the proof of the second inequality in (16), since $\phi$ is a GA-convex function on $[a,b]$, by Lemma 2.1, we have

$$\begin{aligned}\phi\left(\left(\frac{ab}{x}\right)^t\left(\frac{ab}{y}\right)^{1-t}\right) &= \phi\left(\frac{ab}{x^t y^{1-t}}\right) \\ &\le \phi(a)+\phi(b)-\phi\left(x^t y^{1-t}\right),\end{aligned}$$

and similarly

$$\phi\left(\left(\frac{ab}{x}\right)^{1-t}\left(\frac{ab}{y}\right)^{t}\right) \le \phi(a)+\phi(b)-\phi\left(x^{1-t} y^{t}\right)$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. By adding these inequalities, we have

$$\begin{aligned}&\phi\left(\left(\frac{ab}{x}\right)^t\left(\frac{ab}{y}\right)^{1-t}\right) + \phi\left(\left(\frac{ab}{x}\right)^{1-t}\left(\frac{ab}{y}\right)^{t}\right) \\ &\quad\le 2\phi(a)+2\phi(b)-\left[\phi\left(x^t y^{1-t}\right)+\phi\left(x^{1-t} y^{t}\right)\right].\end{aligned} \tag{17}$$

Multiplying both sides of (17) by $\frac{1}{2}g(\frac{ab}{x^t y^{1-t}})$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}&\frac{1}{2}\left\{\int_{\frac{ab}{y}}^{\frac{ab}{x}}\phi(u)g(u)\frac{du}{u}+\int_{\frac{ab}{y}}^{\frac{ab}{x}}\phi(u)g\left(\frac{(ab)^2}{xyu}\right)\frac{du}{u}\right\}\\&\quad\le\left[\phi(a)+\phi(b)\right]\int_x^y g\left(\frac{ab}{u}\right)\frac{du}{u}-\frac{1}{2}\left[\int_x^y\phi(u)g\left(\frac{ab}{u}\right)\frac{du}{u}+\int_x^y\phi(u)g\left(\frac{abu}{xy}\right)\frac{du}{u}\right]\end{aligned}$$

for all $x,y\in[a,b]$.Thus the second inequality is proved. For the proof of the last inequality in (16), by the GA-convexity of $\phi$, we have

$$\phi(\sqrt{xy})=\phi\left(\sqrt{\left(x^t y^{1-t}\right)\left(x^{1-t}y^t\right)}\right)\le\frac{\phi(x^t y^{1-t})+\phi(x^{1-t}y^t)}{2}$$

for all $x,y\in[a,b]$. So, we get

$$\phi(a)+\phi(b)-\frac{1}{2}\left[\phi\left(x^t y^{1-t}\right)+\phi\left(x^{1-t}y^t\right)\right]\le\phi(a)+\phi(b)-\phi(\sqrt{xy})\tag{18}$$

for all $x,y\in[a,b]$. Multiplying both sides of (18) by $g(\frac{ab}{x^t y^{1-t}})$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$\begin{aligned}&\left[\phi(a)+\phi(b)\right]\int_x^y g\left(\frac{ab}{u}\right)\frac{du}{u}-\frac{1}{2}\left[\int_x^y\phi(u)g\left(\frac{ab}{u}\right)\frac{du}{u}+\int_x^y\phi(u)g\left(\frac{abu}{xy}\right)\frac{du}{u}\right]\\&\quad\le\left[\phi(a)+\phi(b)-\phi(\sqrt{xy})\right]\int_x^y g\left(\frac{ab}{u}\right)\frac{du}{u}\end{aligned}$$

for all $x,y\in[a,b]$. □

If we take $x=a$ and $y=b$ in Theorem 2.3, then we can derive the following weighted Hermite–Hadamard inequalities for GA-convex functions.

**Corollary 2.5** *Let $\phi: I\subseteq(0,\infty)\to\mathbb{R}$ be a function such that $\phi\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\phi$ is a GA-convex function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is nonnegative and integrable, then the following inequalities hold*:

$$\begin{aligned}&\phi(\sqrt{ab})\int_a^b\frac{g(u)}{u}\,du\\&\quad\le\frac{1}{2}\left\{\int_a^b\phi(u)g(u)\frac{du}{u}+\int_a^b\phi(u)g\left(\frac{ab}{u}\right)\frac{du}{u}\right\}\\&\quad\le\frac{\phi(a)+\phi(b)}{2}\int_a^b\frac{g(u)}{u}\,du-\frac{1}{2}\left[\int_a^b\phi(u)g\left(\frac{ab}{u}\right)\frac{du}{u}+\int_a^b\phi(u)g(u)\frac{du}{u}\right]\\&\quad\le\left[\phi(a)+\phi(b)-\phi(\sqrt{ab})\right]\int_a^b g\left(\frac{ab}{u}\right)\frac{du}{u}.\end{aligned}\tag{19}$$

If we choose $g(u)=\frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{ab}{xu})$ or $g(u)=\frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{yu}{ab})$ in Theorem 2.3, then we obtain other Hermite–Hadamard–Mercer inequalities for GA-convex functions via Hadamard fractional integrals as follows.

**Corollary 2.6** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a,b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$, then the following inequalities hold*:

$$\begin{aligned}\phi\left(\frac{ab}{\sqrt{xy}}\right) &\le \frac{\Gamma(\alpha+1)}{2\ln^{\alpha}(y/x)}\left\{J^{\alpha}_{\frac{ab}{y}+}\phi\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi\left(\frac{ab}{y}\right)\right\} \\ &\le \phi(a)+\phi(b)-\frac{\Gamma(\alpha+1)}{2\ln^{\alpha}(y/x)}\left[J^{\alpha}_{y-}\phi(x)+J^{\alpha}_{x+}\phi(y)\right] \\ &\le \left[\phi(a)+\phi(b)\right]-\phi(\sqrt{xy}) \end{aligned} \tag{20}$$

*for all $x,y \in [a,b]$ with $x < y$ and $\alpha > 0$. Specially, we take $\alpha = 1$ in the above inequalities, then we get*

$$\begin{aligned}\phi\left(\frac{ab}{\sqrt{xy}}\right) &\le \frac{1}{\ln(y/x)}\int_x^y \phi\left(\frac{ab}{u}\right)\frac{du}{u} \le \phi(a)+\phi(b)-\frac{1}{\ln(y/x)}\int_x^y \phi(u)\frac{du}{u} \\ &\le \left[\phi(a)+\phi(b)\right]-\phi(\sqrt{xy})\end{aligned}$$

*for all $x,y \in [a,b]$ with $x < y$. Let $w : [a,b] \to \mathbb{R}$ be a nonnegative and integrable function.*

If we choose $g(u) = \frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{ab}{xu})w(u)$ or $g(u) = \frac{1}{\Gamma(\alpha)}\ln^{\alpha-1}(\frac{yu}{ab})w(u)$ in Theorem 2.3, then we obtain weighted Hermite–Hadamard–Mercer inequalities for GA-convex functions via Hadamard fractional integrals as follows.

**Corollary 2.7** *Let $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\phi \in L[a,b]$, where $a,b \in I$ with $a < b$. If $\phi$ is a GA-convex function on $[a,b]$ and $w : [a,b] \to \mathbb{R}$ is nonnegative and integrable, then the following inequalities for Hadamard fractional integrals hold*:

$$\begin{aligned}\phi\left(\frac{ab}{\sqrt{xy}}\right)J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right) &\le \frac{1}{2}\left\{J^{\alpha}_{\frac{ab}{y}+}\phi w\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi w\left(\frac{ab}{y}\right)\right\} \\ &\le \left[\phi(a)+\phi(b)\right]J^{\alpha}_{y-}w(x)-\frac{1}{2}\left[J^{\alpha}_{y-}\phi w(x)+J^{\alpha}_{x+}\phi w(y)\right] \\ &\le \left[\phi(a)+\phi(b)-\phi(\sqrt{xy})\right]J^{\alpha}_{y-}w(x)\end{aligned}$$

*and*

$$\begin{aligned}\phi\left(\frac{ab}{\sqrt{xy}}\right)J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right) &\le \frac{1}{2}\left\{J^{\alpha}_{\frac{ab}{y}+}\phi w\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi w\left(\frac{ab}{y}\right)\right\} \\ &\le \left[\phi(a)+\phi(b)\right]J^{\alpha}_{x+}w(y)-\frac{1}{2}\left[J^{\alpha}_{x+}\phi w(y)+J^{\alpha}_{y-}\phi w(x)\right] \\ &\le \left[\phi(a)+\phi(b)-\frac{\phi(x)+\phi(y)}{2}\right]J^{\alpha}_{x+}w(y)\end{aligned}$$

*for all $x,y \in [a,b]$ with $x < y$. Also, we obtain the following inequalities from both inequalities above*:

$$\begin{aligned}&\phi\left(\frac{ab}{\sqrt{xy}}\right)\left[J^{\alpha}_{\frac{ab}{x}-}w\left(\frac{ab}{y}\right)+J^{\alpha}_{\frac{ab}{y}+}w\left(\frac{ab}{x}\right)\right] \\ &\quad\le J^{\alpha}_{\frac{ab}{y}+}\phi w\left(\frac{ab}{x}\right)+J^{\alpha}_{\frac{ab}{x}-}\phi w\left(\frac{ab}{y}\right)\end{aligned}$$

$$\le \left[\phi(a)+\phi(b)\right]\left[J_{x+}^{\alpha}w(y)+J_{y-}^{\alpha}w(x)\right]-\left[J_{y-}^{\alpha}\phi w(x)+J_{x+}^{\alpha}\phi w(y)\right]$$
$$\le \left[\phi(a)+\phi(b)-\phi(\sqrt{xy})\right]\left[J_{x+}^{\alpha}w(y)+J_{y-}^{\alpha}w(x)\right] \tag{21}$$

*for all* $x, y \in [a, b]$ *with* $x < y$.

Specially, if we choose $x = a$ and $y = b$ in (21), then we get the following inequalities.

**Corollary 2.8** *Let* $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ *be a function such that* $\phi \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $\phi$ *is a GA-convex function on* $[a,b]$ *and* $w : [a,b] \to \mathbb{R}$ *is nonnegative and integrable, then the following inequalities for Hadamard fractional integrals hold*:

$$\begin{aligned}
&\phi(\sqrt{ab})\left[J_{b-}^{\alpha}w(a)+J_{a+}^{\alpha}w(b)\right]\\
&\quad\le \left[J_{b-}^{\alpha}\phi w(a)+J_{a+}^{\alpha}\phi w(b)\right]\\
&\quad\le \left[\phi(a)+\phi(b)\right]\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right]-\left[J_{b-}^{\alpha}\phi w(a)+J_{a+}^{\alpha}\phi w(b)\right]\\
&\quad\le \left[\phi(a)+\phi(b)-\phi(\sqrt{ab})\right]\left[J_{a+}^{\alpha}w(b)+J_{b-}^{\alpha}w(a)\right].
\end{aligned}$$

If we choose $g(u) = 1$ in Theorem 2.3, then we obtain the following Hermite–Hadamard–Mercer inequalities for GA-convex functions.

**Corollary 2.9** *Let* $\phi : I \subseteq (0,\infty) \to \mathbb{R}$ *be a function such that* $\phi \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $\phi$ *is a GA-convex function on* $[a,b]$, *then the following inequalities hold*:

$$\begin{aligned}
\phi\left(\frac{ab}{\sqrt{xy}}\right) &\le \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du\\
&\le \left[\phi(a)+\phi(b)\right]-\frac{1}{\ln y/x}\int_x^y \phi(u)\frac{du}{u}\\
&\le \left[\phi(a)+\phi(b)-\phi(\sqrt{xy})\right]
\end{aligned}$$

*for all* $x, y \in [a, b]$ *with* $x < y$.

## 3 Some Hermite–Hadamard type inequalities via Jensen–Mercer inequality for GA-convex functions

We will use the following notations throughout this section:

$$\begin{aligned}
C_1(u) &= \int_0^1 |2t-1|u^t\,dt = \frac{(u-1)\ln u-2u+4\sqrt{u}-2}{\ln^2 u},\\
C_2(u) &= \int_0^1 t|2t-1|u^t\,dt\\
&= \frac{u\ln^2 u+(-3u+2\sqrt{u}+1)\ln u+4u-8\sqrt{u}+4}{\ln^3 u},\\
C_3(u) &= \int_0^1 (1-t)|2t-1|u^t\,dt = uC_2\left(u^{-1}\right)\\
&= \frac{-\ln^2 u+(u+2\sqrt{u}-3)\ln u-4u+8\sqrt{u}-4}{\ln^3 u},
\end{aligned}$$

$$\begin{aligned}
C_4(u) &= \int_0^1 u^t\,dt = \frac{u-1}{\ln u}, \\
C_5(u) &= \int_0^1 tu^t\,dt = \frac{u\ln u - u + 1}{\ln^2 u}, \\
C_6(u) &= \int_0^1 (1-t)u^t\,dt = \frac{u - \ln u - 1}{\ln^2 u}.
\end{aligned}$$

In order to prove our main results, we need the following identity which is related to the second inequality in (15).

**Lemma 3.1** *Let $\phi : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, and $a, b \in I$, with $a < b$. If $\phi' \in L[a,b]$, then*

$$\begin{aligned}
&\frac{\phi(ab/x) + \phi(ab/y)}{2} - \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du \\
&\quad = \frac{ab\ln y/x}{2y}\int_0^1 (2t-1)\left(\frac{y}{x}\right)^t \phi'\left(\frac{ab}{x^t y^{1-t}}\right) dt
\end{aligned}$$

*for all $x, y \in [a,b]$, with $x < y$.*

*Proof* Integrating by parts and changing variables of integration yields

$$\begin{aligned}
&\frac{ab\ln y/x}{2y}\int_0^1 (2t-1)\left(\frac{y}{x}\right)^t \phi'\left(\frac{ab}{x^t y^{1-t}}\right) dt \\
&\quad = \frac{1}{2}\int_0^1 (2t-1)\,d\phi\left(\frac{ab}{x^t y^{1-t}}\right) \\
&\quad = \frac{1}{2}\left((2t-1)\phi\left(\frac{ab}{x^t y^{1-t}}\right)\Big|_0^1\right) - \int_0^1 \phi\left(\frac{ab}{x^t y^{1-t}}\right) dt \\
&\quad = \frac{\phi(ab/x) + \phi(ab/y)}{2} - \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du.
\end{aligned}$$

This completes the proof. □

**Theorem 3.1** *Let $\phi : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a, b \in I^\circ$ with $a < b$ and $\phi' \in L[a,b]$. If $|\phi'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, then*

$$\begin{aligned}
&\left|\frac{\phi(ab/x) + \phi(ab/y)}{2} - \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du\right| \\
&\quad \leq \frac{ab\ln y/x}{2y(q+1)^{1-\frac{1}{q}}}\left[C_1\left(\frac{y^q}{x^q}\right)\left(\left|\phi'(b)\right|^q + \left|\phi'(a)\right|^q\right)\right. \\
&\qquad \left. - C_2\left(\frac{y^q}{x^q}\right)\left|\phi'(x)\right|^q - C_3\left(\frac{y^q}{x^q}\right)\left|\phi'(y)\right|^q\right]^{\frac{1}{q}} \qquad (22)
\end{aligned}$$

*for all $x, y \in [a,b]$, with $x < y$.*

*Proof* Since $|\phi'|^q$ is GA-convex on $[a,b]$, from Lemma 3.1 and the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{\phi(ab/x)+\phi(ab/y)}{2}-\frac{1}{\ln y/x}\int_x^y\frac{\phi(ab/u)}{u}\,du\right| \\
&\quad\le\frac{ab\ln y/x}{2y}\int_0^1|2t-1|\left(\frac{y}{x}\right)^t\left|\phi'\left(\frac{ab}{x^ty^{1-t}}\right)\right|dt \\
&\quad\le\frac{ab\ln y/x}{2y}\left(\int_0^1|2t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1|2t-1|\left(\frac{y}{x}\right)^{qt}\left|\phi'\left(\frac{ab}{x^ty^{1-t}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le\frac{ab\ln y/x}{2y(q+1)^{1-\frac{1}{q}}}\left(\int_0^1|2t-1|\left(\frac{y}{x}\right)^{qt}\right. \\
&\qquad\left.\times\left\{\left|\phi'(b)\right|^q+\left|\phi'(a)\right|^q-t\left|\phi'(x)\right|^q-(1-t)\left|\phi'(y)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&\quad=\frac{ab\ln y/x}{2y(q+1)^{1-\frac{1}{q}}}\left[C_1\left(\frac{y^q}{x^q}\right)\left(\left|\phi'(b)\right|^q+\left|\phi'(a)\right|^q\right)\right. \\
&\qquad\left.-C_2\left(\frac{y^q}{x^q}\right)\left|\phi'(x)\right|^q-C_3\left(\frac{y^q}{x^q}\right)\left|\phi'(y)\right|^q\right]^{\frac{1}{q}}. 
\end{aligned}$$

□

If we take $q=1$ in Theorem 3.1, we can derive the following corollary.

**Corollary 3.1** *Let $\phi: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on $I^\circ$, and $a,b\in I^\circ$ with $a<b$ and $\phi'\in L[a,b]$. If $|\phi'|$ is geometrically convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\phi(ab/x)+\phi(ab/y)}{2}-\frac{1}{\ln y/x}\int_x^y\frac{\phi(ab/u)}{u}\,du\right| \\
&\quad\le\frac{ab\ln y/x}{2y}\left[C_1\left(\frac{y}{x}\right)\left(\left|\phi'(b)\right|+\left|\phi'(a)\right|\right)-C_2\left(\frac{y}{x}\right)\left|\phi'(x)\right|-C_3\left(\frac{y}{x}\right)\left|\phi'(y)\right|\right]
\end{aligned}$$

*for all $x,y\in[a,b]$ with $x<y$.*

If we take $x=a$ and $y=b$ in Theorem 3.1, we can derive the following corollary.

**Corollary 3.2** *Let $\phi: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be differentiable on $I^\circ$, and $a,b\in I^\circ$ with $a<b$ and $\phi'\in L[a,b]$. If $|\phi'|^q$ is GA-convex on $[a,b]$ for $q\ge1$, then*

$$\begin{aligned}
&\left|\frac{\phi(a)+\phi(b)}{2}-\frac{1}{\ln b/a}\int_a^b\frac{\phi(u)}{u}\,du\right| \\
&\quad\le\frac{a\ln b/a}{2(q+1)^{1-\frac{1}{q}}}\left[\left(C_1\left(\frac{b^q}{a^q}\right)-C_2\left(\frac{b^q}{a^q}\right)\right)\left|\phi'(a)\right|^q\right. \\
&\qquad\left.+\left(C_1\left(\frac{b^q}{a^q}\right)-C_3\left(\frac{b^q}{a^q}\right)\right)\left|\phi'(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

**Theorem 3.2** *Let $\phi : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I^\circ$ with $a < b$ and $\phi' \in L[a,b]$. If $|\phi'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\begin{aligned}
&\left|\frac{\phi(ab/x)+\phi(ab/y)}{2} - \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du\right| \\
&\quad\leq \frac{ab\ln y/x}{2y(p+1)^{1/p}}\left[C_4\left(\frac{y^q}{x^q}\right)\left[\left|\phi'(b)\right|^q + \left|\phi'(a)\right|^q\right]\right. \\
&\qquad \left. - C_5\left(\frac{y^q}{x^q}\right)\left|\phi'(x)\right|^q - C_6\left(\frac{y^q}{x^q}\right)\left|\phi'(y)\right|^q\right]^{\frac{1}{q}}
\end{aligned}$$

*for all $x,y \in [a,b]$ with $x < y$.*

*Proof* Since $|\phi'|^q$ is GA-convex on $[a,b]$, from Lemma 3.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\phi(ab/x)+\phi(ab/y)}{2} - \frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du\right| \\
&\quad\leq \frac{ab\ln y/x}{2y}\left(\int_0^1 |2t-1|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{y}{x}\right)^{qt}\left|\phi'\left(\frac{ab}{x^t y^{1-t}}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\leq \frac{ab\ln y/x}{2y(p+1)^{1/p}}\left(\int_0^1 \left(\frac{y}{x}\right)^{qt}\left\{\left[\left|\phi'(b)\right|^q + \left|\phi'(a)\right|^q\right] - t\left|\phi'(x)\right|^q - (1-t)\left|\phi'(y)\right|^q\right\}dt\right)^{\frac{1}{q}} \\
&\quad= \frac{ab\ln y/x}{2y(p+1)^{1/p}}\left[C_4\left(\frac{y^q}{x^q}\right)\left[\left|\phi'(b)\right|^q + \left|\phi'(a)\right|^q\right]\right. \\
&\qquad \left. - C_5\left(\frac{y^q}{x^q}\right)\left|\phi'(x)\right|^q - C_6\left(\frac{y^q}{x^q}\right)\left|\phi'(y)\right|^q\right]^{\frac{1}{q}}. \qquad \square
\end{aligned}$$

If we take $x = a$ and $y = b$ in Theorem 3.2, we can derive the following corollary.

**Corollary 3.3** *Let $\phi : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a,b \in I^\circ$ with $a < b$ and $\phi' \in L[a,b]$. If $|\phi'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\begin{aligned}
&\left|\frac{\phi(a)+\phi(b)}{2} - \frac{1}{\ln b/a}\int_a^b \frac{\phi(u)}{u}\,du\right| \\
&\quad\leq \frac{a\ln b/a}{2(p+1)^{1/p}}\left[\left(C_4\left(\frac{b^q}{a^q}\right) - C_5\left(\frac{b^q}{a^q}\right)\right)\left|\phi'(a)\right|^q\right. \\
&\qquad \left. + \left(C_4\left(\frac{b^q}{a^q}\right) - C_6\left(\frac{b^q}{a^q}\right)\right)\left|\phi'(b)\right|^q\right]^{\frac{1}{q}} \\
&\quad= \frac{(\ln b/a)^{1/p}}{2q^{1/q}(p+1)^{1/p}}\left[\left(L\left(a^q,b^q\right)-a^q\right)\left|\phi'(a)\right|^q + \left(b^q - L\left(a^q,b^q\right)\right)\left|\phi'(b)\right|^q\right]^{\frac{1}{q}},
\end{aligned}$$

*where $L(u,v) = (u-v)/(\ln u - \ln v)$, $u \neq v$, is logarithmic mean.*

**Theorem 3.3** *Let $\phi : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a, b \in I^\circ$ with $a < b$ and $\phi' \in L[a,b]$. If $|\phi'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\begin{aligned}&\left|\frac{\phi(ab/x)+\phi(ab/y)}{2}-\frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du\right|\\&\quad\le \frac{ab\ln y/x}{2y}C_4^{1/p}\left(\frac{y^p}{x^p}\right)\left(\left[\frac{|\phi'(b)|^q+|\phi'(a)|^q}{q+1}\right]-\frac{1}{2}\left[\frac{|\phi'(x)|^q+|\phi'(y)|^q}{q+1}\right]dt\right)^{\frac{1}{q}}\end{aligned}$$

*for all $x, y \in [a,b]$ with $x < y$.*

*Proof* Using Lemma 3.1, Hölder's inequality, and the GA-convexity of $|\phi'|^q$, it is easily seen that

$$\begin{aligned}&\left|\frac{\phi(ab/x)+\phi(ab/y)}{2}-\frac{1}{\ln y/x}\int_x^y \frac{\phi(ab/u)}{u}\,du\right|\\&\quad\le \frac{ab\ln y/x}{2y}\left(\int_0^1\left(\frac{y}{x}\right)^{pt}dt\right)^{\frac{1}{p}}\\&\qquad\times\left(\int_0^1|2t-1|^q\left\{\left[\left|\phi'(b)\right|^q+\left|\phi'(a)\right|^q\right]-t\left|\phi'(x)\right|^q-(1-t)\left|\phi'(y)\right|^q\right\}dt\right)^{\frac{1}{q}}\\&\quad\le \frac{ab\ln y/x}{2y}C_4^{1/p}\left(\frac{y^p}{x^p}\right)\left(\left[\frac{|\phi'(b)|^q+|\phi'(a)|^q}{q+1}\right]-\frac{1}{2}\left[\frac{|\phi'(x)|^q+|\phi'(y)|^q}{q+1}\right]dt\right)^{\frac{1}{q}}.\qquad\square\end{aligned}$$

If we take $x = a$ and $y = b$ in Theorem 3.3, we can derive the following corollary.

**Corollary 3.4** *Let $\phi : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^\circ$, and $a, b \in I^\circ$ with $a < b$ and $\phi' \in L[a,b]$. If $|\phi'|^q$ is GA-convex on $[a,b]$ for $q > 1$, then*

$$\begin{aligned}&\left|\frac{\phi(a)+\phi(b)}{2}-\frac{1}{\ln b/a}\int_a^b \frac{\phi(u)}{u}\,du\right|\\&\quad\le \frac{b-a}{4}\frac{L^{1/p}(a^p,b^p)}{L(a,b)}\left[\frac{|\phi'(b)|^q+|\phi'(a)|^q}{q+1}\right]^{\frac{1}{q}},\end{aligned}$$

*where $L(u,v) = (u-v)/(\ln u - \ln v)$, $u \neq v$, is logarithmic mean.*

## 4 Conclusion

This article aims to investigate certain weighted Hermite–Hadamard–Mercer type inequalities for a GA-convex function, which are related to the Hermite–Hadamard–Fejér inequality and fractional Hermite–Hadamard type inequalities. It is worth mentioning that certain results proved in this article generalize parts of the results provided by İşcan [1], Kunt and İşcan [3], and Latif et al. [4]. Certain estimates related to the second Hermite–Hadamard–Mercer inequality for GA-convex functions given in (15) are obtained. For this purpose, an identity for differentiable mappings is established.

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