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The convergence rate of truncated hypersingular integrals generated by the modified Poisson semigroup

Melih Eryiğit[1], Sinem Sezer Evcan[2*] and Selim Çobanoğlu[3]

*Correspondence:
sinemsezer@akdeniz.edu.tr
[2]Faculty of Education, Akdeniz University, Antalya, Turkey
Full list of author information is available at the end of the article

Abstract

Hypersingular integrals have appeared as effective tools for inversion of multidimensional potential-type operators such as Riesz, Bessel, Flett, parabolic potentials, etc. They represent (at least formally) fractional powers of suitable differential operators. In this paper the family of the so-called "truncated hypersingular integral operators" $\mathbf{D}_\varepsilon^\alpha f$ is introduced, that is generated by the modified Poisson semigroup and associated with the Flett potentials $\mathcal{F}^\alpha\varphi = (E+\sqrt{-\Delta})^{-\alpha}\varphi$ $(0<\alpha<\infty, \varphi\in L_p(\mathbb{R}^n))$. Then the relationship between the order of "L_p-smoothness" of a function f and the "rate of L_p-convergence" of the families $\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha f$ to the function f as $\varepsilon \to 0^+$ is also obtained.

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1 Introduction

For a sufficiently "good function" f on $\mathbb{R}^n$, the Riesz and Bessel potentials of order α are defined by

$$\big(I^\alpha f\big)(x) = \frac{1}{\gamma_n(\alpha)}\int_{\mathbb{R}^n} |y|^{\alpha-n} f(x-y)\,dy, \quad 0<\alpha<n, \tag{1}$$

where

$$\gamma_n(\alpha) = \pi^{\frac{n}{2}} 2^\alpha \Gamma(\alpha/2)/\Gamma\big((n-\alpha)/2\big), \quad \operatorname{Re}\alpha>0, \alpha\neq n, n+2, n+4, \dots,$$

and

$$\big(J^\alpha f\big)(x) = \frac{1}{\beta_n(\alpha)}\int_{\mathbb{R}^n} G_\alpha(y) f(x-y)\,dy, \quad \operatorname{Re}\alpha>0, \tag{2}$$

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with the kernel

$$G_\alpha(y) = \int_0^\infty e^{-\xi-\frac{|y|^2}{4\xi}}\xi^{\frac{\alpha-n}{2}-1}\,d\xi, \quad \beta_n(\alpha) = 2^n\pi^{\frac{n}{2}}\Gamma(\alpha/2),$$

respectively.

These operators can be regarded (in a certain sense) as negative "fractional powers" of $-\Delta$ and $(E-\Delta)$, i.e.,

$$I^\alpha = (-\Delta)^{-\alpha/2}, \qquad J^\alpha = (E-\Delta)^{-\alpha/2}, \qquad \Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}, \quad \text{and}$$

E is the identity operator.

If $f \in L_p(\mathbb{R}^n)$ then the integral (1) converges a.e. for $1 \le p < \frac{n}{\operatorname{Re}\alpha}$, and the integral (2) converges for $1 \le p < \infty$, and the conditions are sharp. The references [10, 12, 19, 20, 22, 28] can be recommended for further reading on these potentials.

There are also "one-dimensional" integral representations of the Riesz and Bessel potentials via Poisson integral (see [18], [19, pp. 224 and 262]).

$$\big(I^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(P_t f)(x)\,dt, \tag{3}$$

$$\big(J^\alpha f\big)(x) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}\alpha)}\int_0^\infty \left(\frac{t}{2}\right)^{\frac{1}{2}(\alpha-1)} J_{\frac{1}{2}(\alpha-1)}(t)(P_t f)(x)\,dt \tag{4}$$

(J_ν is the Bessel function of the first kind of order ν).

As seen from (3) and (4), the Riesz potentials are better suited to Poisson integral than the Bessel potentials. There is, however, another kind of fractional integral operators which are compatible with Poisson integral and whose kernels behavior roughly takes place between the behaviors of the kernels of the Bessel and Riesz potentials. These potentials, called the Flett potentials, were first introduced by T.M. Flett in [11] (see also [23, pp. 541–542]).

The Flett potentials $\mathcal{F}^\alpha f$ of a function f are defined in Fourier terms as follows:

$$\big(\mathcal{F}^\alpha f\big)^\wedge(x) = \big(1+|x|\big)^{-\alpha}\widehat{f}(x), \quad x \in \mathbb{R}^n, \alpha > 0. \tag{5}$$

These potentials are considered as the negative fractional powers of the operator $(E+\Lambda)$, where $\Lambda = (-\Delta)^{1/2}$ and Δ is the Laplacian, and have the integral representation

$$\big(\mathcal{F}^\alpha f\big)(x) = \big(\phi_\alpha(y) * f\big)(x) = \int_{\mathbb{R}^n}\phi_\alpha(y)f(x-y)\,dy. \tag{6}$$

The kernel $\phi_\alpha(y)$ is of the form

$$\phi_\alpha(y) = \frac{1}{\lambda_n(\alpha)}|y|^{\alpha-n}\int_0^\infty \frac{s^\alpha e^{-s|y|}}{(1+s^2)^{\frac{n+1}{2}}}\,ds \quad (\alpha > 0), \tag{7}$$

where $\lambda_n(\alpha) = \pi^{(n+1)/2}\Gamma(\alpha)/\Gamma((n+1)/2)$.

The potential-type operators take important place in analysis and its applications, see, for example, E. Stein [26, pp. 121–141], E. Stein and G. Weiss [27], E. Stein [25], S.G. Samko, A.A. Kilbas, and O.I. Marichev [23, pp. 538–554]. Many researchers from different areas have studied characterizations, modifications, and several properties of these potentials, see P. Lizorkin [13], R. Wheeden [28], M. Fisher [10], V. Balakrishnan [8], S. Samko [21–23], B. Rubin [16–19], V.A. Nogin [14, 15]. The wavelet approach to these potentials is given and developed by B. Rubin [19, 20], I.A. Aliev and B. Rubin [6] and I.A. Aliev [2]; see also [4, 7, 24].

In [17] B. Rubin introduced "truncated hypersingular" integrals $D_\varepsilon^\alpha f$ and $\mathfrak{D}_\varepsilon^\alpha f$ $(\varepsilon > 0)$ generated by the Poisson semigroup and metaharmonic semigroup, respectively. It has been also proved that under some conditions on function $\varphi \in L_p(\mathbb{R}^n)$ and parameter $\alpha > 0$, the expressions $D_\varepsilon^\alpha I^\alpha \varphi$ and $\mathfrak{D}_\varepsilon^\alpha J^\alpha \varphi$ converge to φ as $\varepsilon \to 0^+$, pointwise (a.e.) and in the L_p-norm.

In this work, in a similar way to [17], we first define the families of the truncated hypersingular integral operators associated with Flett potentials and generated by finite difference and modified Poisson semigroup $e^{-t}(P_t f)$,

$$\big(\mathbf{D}_\varepsilon^\alpha f\big)(x) = \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^{l} \binom{l}{k} (-1)^k e^{-k\tau} (P_{k\tau} f)(x) \right] \frac{d\tau}{\tau^{1+\alpha}}, \quad \varepsilon > 0, \tag{8}$$

secondly, we find a relationship between the "order of L_p-smoothness" of function φ and the "rate of L_p-convergence" of the families $\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi$ to φ as $\varepsilon \to 0^+$.

We note that an analogous problem for the Bessel and Riesz potentials has been investigated in [3, 5], and [9].

2 Notions and auxiliary lemmas

We denote by $L_p \equiv L_p(\mathbb{R}^n)$ the standard space of measurable functions on $\mathbb{R}^n$ with the finite norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} \big|f(x)\big|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty; \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \big|f(x)\big|.$$

The Fourier and inverse Fourier transforms of $f \in L_1(\mathbb{R}^n)$ are defined by

$$\widehat{f}(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)\, d\xi, \quad x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n; \qquad f^\vee(\xi) = (2\pi)^{-n} \widehat{f}(-\xi).$$

The Flett potentials, defined in (6), have another (one-dimensional) integral representation via modified Poisson semigroup:

$$\big(\mathcal{F}^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} (P_t f)(x)\, dt, \quad f \in L_p \ (1 \le p \le \infty). \tag{9}$$

Here the Poisson semigroup $P_t f$ is defined as

$$(P_t f)(x) = \int_{\mathbb{R}^n} p(y;t) f(x-y)\, dy \quad (t > 0), \tag{10}$$

where

$$p(y;t) = \big(e^{-t|\cdot|}\big)^\vee (y) = \frac{a_n t}{(t^2 + |y|^2)^{\frac{n+1}{2}}}, \quad a_n = \pi^{-\frac{(n+1)}{2}} \Gamma\left(\frac{n+1}{2} \right) \tag{11}$$

is the Poisson kernel.

We would like to note that the expression in (9) has the same nature of classical Balakrishnan's formulas for fractional powers of operators (see Samko et al. [23, p. 121]).

For the sake of convenience of the reader, let us give some important properties of the Poisson's semigroup $P_t\varphi$ $(t > 0)$ and its kernel $p(y;t)$.

Lemma 2.1 (cf. B. Rubin [19, p. 217]) *Let $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, and P_tf be the Poisson integral with the kernel $p(y;t)$ defined as in (11). Then*

$$\text{(a)}\quad \int_{\mathbb{R}^n} p(y;t)\,dy = 1, \qquad \big(p(\cdot;t)\big)^{\wedge}(y) = e^{-t|y|}, \quad \textit{for all } t > 0; \tag{12}$$

$$\text{(b)}\quad \|P_tf\|_p \le \|f\|_p; \tag{13}$$

$$\text{(c)}\quad \sup_{x\in\mathbb{R}^n}\big|(P_tf)(x)\big| \le ct^{-\frac{n}{p}}\|f\|_p, \quad 1 \le p < \infty, c = c(n,p); \tag{14}$$

$$\text{(d)}\quad \sup_{t>0}\big|(P_tf)(x)\big| \le (\boldsymbol{M}f)(x), \tag{15}$$

where $(\boldsymbol{M}f)$ is the Hardy–Littlewood maximal function;

$$\text{(e)}\quad P_\alpha\big[P_\beta f(\cdot)\big](x) = (P_{\alpha+\beta}f)(x), \quad \textit{for all } \alpha, \beta > 0; \tag{16}$$

$$\text{(f)}\quad \lim_{t\to 0}(P_tf)(x) = f(x), \tag{17}$$

where the limit is understood in L_p-norm or pointwise a.e. Moreover, if $f \in C^0$ then convergence is uniform on $\mathbb{R}^n$.

Definition 2.2 Let $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$ and Poisson integral P_tf be as in (10). The modified Poisson semigroup is defined as

$$(S_tf)(x) = e^{-t}(P_tf)(x), \quad 0 \le t < \infty. \tag{18}$$

It is evident that the semigroup property

$$\big(S_\alpha(S_\beta f)\big)(x) = (S_{\alpha+\beta}f)(x)$$

holds, and, according to Lemma 2.1(f), it is assumed that

$$\big(e^{-t}P_tf\big)(x)\big|_{t=0} = f(x) = S_0f.$$

Definition 2.3 The finite difference of order $l \in \mathbb{N}$ and step $\tau \in \mathbb{R}^1$ of the function $g(t)$, $t \in \mathbb{R}^1$ is defined by

$$\Delta_\tau^l[g](t) = \sum_{k=0}^{l}\binom{l}{k}(-1)^k g(t + k\tau). \tag{19}$$

In the special case, for $t = 0$,

$$\Delta_\tau^l[g](0) = \sum_{k=0}^{l}\binom{l}{k}(-1)^k g(k\tau). \tag{20}$$

Using the modified Poisson semigroup $S_t f$ and finite difference of order $l \in \mathbb{N}$, we introduce the following truncated integral operators (cf. [19, p. 261]).

Definition 2.4 Let $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$, $\alpha > 0$ and $l > \alpha$ ($l \in \mathbb{N}$). The constructions

$$\begin{aligned}\left(\mathbf{D}_\varepsilon^\alpha f\right)(x) &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \Delta_\tau^l\left[(S_\cdot f)(x)\right](0) \frac{d\tau}{\tau^{1+\alpha}} \\ &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^l \binom{l}{k} (-1)^k e^{-k\tau} (P_{k\tau} f)(x)\right] \frac{d\tau}{\tau^{1+\alpha}}, \quad \varepsilon > 0, \end{aligned} \tag{21}$$

will be called truncated hypersingular integrals or, briefly, truncated integrals with parameter $\varepsilon > 0$. Here the normalized coefficient $\chi_l(\alpha)$ is defined by

$$\chi_l(\alpha) = \int_0^\infty \left(1 - e^{-t}\right)^l t^{-1-\alpha}\, dt. \tag{22}$$

By applying Minkowski integral inequality, it is easy to see that $\mathbf{D}_\varepsilon^\alpha f \in L_p(\mathbb{R}^n)$ for all $\varepsilon > 0$.

Lemma 2.5 (cf. Rubin [19, p. 224]) *Let $\varphi \in L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$, $0 < \alpha < \infty$, and truncated integral operators $\mathbf{D}_\varepsilon^\alpha$ be defined as in (21). If $\mathcal{F}^\alpha \varphi$ are the Flett potentials of $\varphi \in L_p(\mathbb{R}^n)$, and $P_t\varphi$, $(t > 0)$ is the Poisson integral of φ, then the following equation holds in pointwise (a.e.) sense*:

$$\left(\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi\right)(x) = \int_0^\infty K_\alpha^{(l)}(\eta) e^{-\varepsilon\eta} (P_{\varepsilon\eta}\varphi)(x)\, d\eta, \quad \varepsilon > 0. \tag{23}$$

Here the function $K_\alpha^{(l)}(\eta)$ is defined as

$$K_\alpha^{(l)}(\eta) = \left[\Gamma(1+\alpha)\chi_l(\alpha)\right]^{-1} \eta^{-1} \sum_{k=0}^l \binom{l}{k} (-1)^k (\eta - k)_+^\alpha, \quad l > \alpha,$$

with $a_+^\alpha = \begin{cases} a^\alpha, & \text{if } a>0, \\ 0, & \text{if } a \le 0. \end{cases}$

Proof For a function $h(t)$ $(0 < t < \infty)$, let

$$I_-^\alpha h(t) = \left(\Gamma(\alpha)\right)^{-1} \int_t^\infty \frac{h(r)}{(r-t)^{1-\alpha}}\, dr = \left(\Gamma(\alpha)\right)^{-1} \int_0^\infty \frac{h(r+t)}{r^{1-\alpha}}\, dr, \quad \alpha > 0. \tag{24}$$

Then by making use of Rubin's method [19, p. 224], it can be shown that

$$S_t\left[\mathcal{F}^\alpha f\right](x) = I_-^\alpha\left[(S_\cdot f)(x)\right](t) \tag{25}$$

holds for all $t > 0$ and a.e. $x \in \mathbb{R}^n$.

Now, by using (25), we have

$$\begin{aligned}\left(\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi\right)(x) &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^l \binom{l}{k} (-1)^k e^{-k\tau} \left(S_{k\tau} \mathcal{F}^\alpha \varphi\right)(x)\right] \frac{d\tau}{\tau^{1+\alpha}} \\ &\overset{(25)}{=} \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^l \binom{l}{k} (-1)^k I_-^\alpha\left[(S_\cdot \varphi)(x)\right](k\tau)\right] \frac{d\tau}{\tau^{1+\alpha}}. \end{aligned} \tag{26}$$

Further,

$$\begin{aligned}
&\sum_{k=0}^{l}\binom{l}{k}(-1)^k I_-^\alpha\big[(S.\varphi)(x)\big](k\tau)\\
&\quad\overset{(24)}{=}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\frac{1}{\Gamma(\alpha)}\int_{k\tau}^{\infty}(r-k\tau)^{\alpha-1}(S_r\varphi)(x)\,dr\\
&\quad=\int_0^\infty h_\tau(r)(S_r\varphi)(x)\,dr,
\end{aligned}\tag{27}$$

where

$$h_\tau(r)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{l}\binom{l}{k}(-1)^k(r-k\tau)_+^{\alpha-1}\tag{28}$$

with

$$(r-k\tau)_+^{\alpha-1}=\begin{cases}(r-k\tau)^{\alpha-1}, & \text{if } r>k\tau,\\ 0, & \text{if } r\le k\tau.\end{cases}$$

Now, by taking into account (27) in (26), we get

$$\begin{aligned}
&\big(\mathbf{D}_\varepsilon^\alpha\mathcal{F}^\alpha\varphi\big)(x)\\
&\quad=\frac{1}{\chi_l(\alpha)}\int_\varepsilon^\infty\frac{1}{\tau^{1+\alpha}}\left(\int_0^\infty h_\tau(r)(S_r\varphi)(x)\,dr\right)d\tau\\
&\quad=\frac{1}{\chi_l(\alpha)}\int_0^\infty(S_r\varphi)(x)\left(\int_\varepsilon^\infty\frac{1}{\tau^{1+\alpha}}h_\tau(r)\,d\tau\right)dr\\
&\qquad(\text{change of variables } r=\varepsilon\eta, 0<\eta<\infty)\\
&\quad=\frac{\varepsilon}{\chi_l(\alpha)}\int_0^\infty(S_{\varepsilon\eta}\varphi)(x)\left(\int_\varepsilon^\infty\frac{1}{\tau^{1+\alpha}}h_\tau(\varepsilon\eta)\,d\tau\right)d\eta\\
&\quad\overset{(28)}{=}\frac{\varepsilon}{\Gamma(\alpha)\chi_l(\alpha)}\int_0^\infty(S_{\varepsilon\eta}\varphi)(x)\left(\sum_{k=0}^{l}\binom{l}{k}(-1)^k\int_\varepsilon^\infty\frac{1}{\tau^{1+\alpha}}(\varepsilon\eta-k\tau)_+^{\alpha-1}\,d\tau\right)d\eta.
\end{aligned}\tag{29}$$

In (29), using the equality (see [5, p. 355])

$$\int_\varepsilon^\infty\tau^{-(1+\alpha)}(\varepsilon\eta-k\tau)_+^{\alpha-1}\,d\tau=\frac{1}{\varepsilon\eta\alpha}(\eta-k)_+^\alpha,\quad k=0,1,\dots,l,\tag{30}$$

we obtain

$$\big(\mathbf{D}_\varepsilon^\alpha\mathcal{F}^\alpha\varphi\big)(x)=\int_0^\infty K_\alpha^{(l)}(\eta)e^{-\varepsilon\eta}(P_{\varepsilon\eta}\varphi)(x)\,d\eta,$$

as desired. □

The following lemma shows that the function $K_\alpha^{(l)}(\eta)$ is an "averaging kernel".

Lemma 2.6 (see [23, p. 125], [19, p. 158]) *The following is true*:

$$\text{(i)} \quad K_\alpha^{(l)}(\eta) \in L_1(0,\infty) \quad \textit{and} \quad \int_0^\infty K_\alpha^{(l)}(\eta)\,d\eta = 1;$$

$$\text{(ii)} \quad K_\alpha^{(l)}(\eta) = \begin{cases} O(\eta^{\alpha-1}), & \textit{if } \eta \to 0^+, \\ O(\eta^{\alpha-l-1}), & \textit{if } \eta \to \infty. \end{cases}$$

Definition 2.7 (cf. [1]) Let $\rho \in (0,1)$ be a fixed parameter and a function $\mu(r)$ $(0 \le r \le \rho)$ be continuous on $[0,\rho]$, positive on $(0,\rho]$, and $\mu(0) = 0$. We say that a function $\varphi \in L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$ has "μ-smoothness property in L_p-sense" if

$$\mathcal{M}_\mu \equiv \sup_{0<r\le\rho} \frac{1}{r^n \mu(r)} \int_{|x|\le r} \left\| \varphi(t-x) - \varphi(t) \right\|_p dx < \infty. \tag{31}$$

Note that if $\mu_\varphi(r)$ is the L_p-modulus of continuity of φ, i.e.,

$$\mu_\varphi(r) = \sup_{|x|\le r} \left\| \varphi(t-x) - \varphi(t) \right\|_p \quad \left(|x| = \sqrt{x_1^2 + \cdots + x_n^2} \right),$$

then condition (31) is satisfied for $\mu(r) = \mu_\varphi(r)$. Also, it is clear that if the L_p-modulus of continuity of φ satisfies $\mu_\varphi(r) \le \mu(r)$ $(0 \le r \le \rho)$ then the expression $\mathcal{M}_\mu$ in (31) is finite.

Remark 2.8 From now on it will be assumed that $\mu(t) \ge at$ $(0 \le t \le \rho)$, for some $a > 0$ and $\mu(t) = \mu(\rho)$ for $\rho \le t < \infty$.

Lemma 2.9 (cf. [5]; see also [9]) *Let a function $\varphi \in L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$ have μ-smoothness property in L_p-sense, and the function $\psi(r)$ $(0 \le r \le \rho)$ be decreasing, nonnegative, and continuously differentiable on* $[0,\rho]$. *Then*

$$\int_{|x|\le\rho} \left\| \varphi(t-x) - \varphi(t) \right\|_p \psi\left(|x|\right) dx \le \mathcal{M}_\mu \left[\rho^n \mu(\rho)\psi(\rho) + \int_0^\rho r^n \mu(r) \left(-\psi'(r)\right) dr \right]. \tag{32}$$

Proof Set $g(x) = \|\varphi(t-x) - \varphi(t)\|_p$ and $x = r\theta$; $r = |x|$, $\theta \in \Sigma^{n-1}$. Then

$$\begin{aligned} I &\equiv \int_{|x|\le\rho} \left\| \varphi(t-x) - \varphi(t) \right\|_p \psi\left(|x|\right) dx = \int_{|x|\le\rho} g(x)\psi\left(|x|\right) dx \\ &= \int_0^\rho r^{n-1}\psi(r) \left(\int_{|\theta|=1} g(r\theta)\,d\sigma(\theta) \right) dr. \end{aligned}$$

Let us define the functions

$$\lambda(r) = \int_{|\theta|=1} g(r\theta)\,d\sigma(\theta) \quad \text{and} \quad \Omega(r) = \int_0^r \lambda(t) t^{n-1}\,dt.$$

Then we have

$$I \equiv \int_0^\rho \psi(r)\lambda(r)r^{n-1}\,dr = \int_0^\rho \psi(r)\,d\Omega(r) = \psi(r)\Omega(r)|_0^\rho - \int_0^\rho \Omega(r)\psi'(r)\,dr$$
$$= \psi(\rho)\Omega(\rho) + \int_0^\rho \Omega(r)\bigl(-\psi'(r)\bigr)\,dr.$$

Using condition (31), we have

$$\Omega(r) = \int_0^r \lambda(t)t^{n-1}\,dt = \int_{|x|\le r} g(x)\,dx = \int_{|x|\le r} \bigl\|\varphi(t-x)-\varphi(t)\bigr\|_p\,dx$$
$$\le r^n\mu(r)\mathcal{M}_\mu,$$

hence,

$$I \le \mathcal{M}_\mu\left[\rho^n\mu(\rho)\psi(\rho) + \int_0^\rho r^n\mu(r)\bigl(-\psi'(r)\bigr)\,dr\right]. \qquad \square$$

Lemma 2.10 *Let $p(x;\varepsilon)$ be the Poisson kernel, defined as in (11), i.e.,*

$$p(x;\varepsilon) = \frac{a_n\varepsilon}{(\varepsilon^2+|x|^2)^{\frac{n+1}{2}}}, \qquad a_n = \pi^{-\frac{(n+1)}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

Then there exists a constant $c>0$ such that

$$\int_{|x|\le\rho} \bigl\|\varphi(t-x)-\varphi(t)\bigr\|_p p(x;\varepsilon)\,dx \le c\mathcal{M}_\mu\left[\varepsilon + \int_0^\infty \mu(\varepsilon t)\frac{dt}{1+t^2}\right]. \tag{33}$$

Proof By setting $\psi(|x|) = p(x;\varepsilon) \equiv a_n\varepsilon(\varepsilon^2+|x|^2)^{-\frac{n+1}{2}}$ in equality (32), we have

$$\int_{|x|\le\rho} \bigl\|\varphi(t-x)-\varphi(t)\bigr\|_p p(x;\varepsilon)\,dx$$
$$\le \mathcal{M}_\mu\left[\rho^n\mu(\rho)\frac{a_n\varepsilon}{(\varepsilon^2+\rho^2)^{\frac{n+1}{2}}} + \int_0^\rho r^n\mu(r)\left(-\frac{a_n\varepsilon}{(\varepsilon^2+r^2)^{\frac{n+1}{2}}}\right)'\,dr\right]. \tag{34}$$

A simple calculation yields

$$\rho^n\mu(\rho)\frac{a_n\varepsilon}{(\varepsilon^2+\rho^2)^{\frac{n+1}{2}}} \le c_1\varepsilon \quad \left(c_1 = a_n\frac{\mu(\rho)}{\rho}\right),$$

and

$$\left(-\frac{a_n\varepsilon}{(\varepsilon^2+r^2)^{\frac{n+1}{2}}}\right)' = c_2\frac{\varepsilon r}{(\varepsilon^2+r^2)^{\frac{n+3}{2}}} \quad \bigl(c_2 = a_n(n+1)\bigr).$$

Using of these calculations in (34) and denoting $c = \max\{c_1, c_2\}$, we have

$$\int_{|x|\le\rho} \bigl\|\varphi(t-x)-\varphi(t)\bigr\|_p p(x;\varepsilon)\,dx \le c\mathcal{M}_\mu\left[\varepsilon + \int_0^\rho \frac{\varepsilon r^{n+1}}{(\varepsilon^2+r^2)^{\frac{n+3}{2}}}\mu(r)\,dr\right]$$

$$= c\mathcal{M}_\mu\left[\varepsilon + \int_0^{\frac{\rho}{\varepsilon}} \frac{t^{n+1}}{(1+t^2)^{\frac{n+3}{2}}}\mu(\varepsilon t)\,dt\right]$$
$$\le c\mathcal{M}_\mu\left[\varepsilon + \int_0^{\infty} \frac{\mu(\varepsilon t)}{1+t^2}\,dt\right]. \qquad \square$$

Corollary 2.11 *Let the function $\mu(r)$ $(0 \le r \le \rho < 1)$ be continuous on $[0,\rho]$, positive on $(0,\rho]$, and $\mu(0) = 0$. Let, further, $\mu(t) \ge at$, $0 \le t \le \rho$ for some $a > 0$ and $\mu(t) = \mu(\rho)$ for $\rho \le t < \infty$. If there exists a locally bounded function $\omega(t) > 0$ such that*

$$\mu(\varepsilon t) \le \mu(\varepsilon)\omega(t), \quad \varepsilon \in (0,\rho), t \in (0,\infty), \quad \textit{and} \quad \int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty, \tag{35}$$

then there exists $A > 0$, which does not depend on $\varepsilon \in (0,\rho)$ and satisfies

$$\int_{|x|\le\rho} \|\varphi(t-x) - \varphi(t)\|_p p(x;\varepsilon)\,dx \le A\mu(\varepsilon), \quad \textit{for all } \varepsilon \in (0,\rho). \tag{36}$$

Proof By taking into account (35) in (33) and using the condition $\mu(\varepsilon) \ge a\varepsilon$ $(0 \le \varepsilon \le \rho)$, we have

$$\int_{|x|\le\rho} \|\varphi(t-x) - \varphi(t)\|_p p(x;\varepsilon)\,dx \le c\mathcal{M}_\mu\left[\varepsilon + \mu(\varepsilon)\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt\right]$$
$$\le A\mu(\varepsilon). \qquad \square$$

Example For $0 < \gamma < 1$, the function

$$\mu(r) = \begin{cases} r^\gamma, & \text{if } 0 \le r \le \rho < 1, \\ \rho^\gamma, & \text{if } r \ge \rho \end{cases}$$

satisfies all the conditions of Corollary 2.11 with $\omega(t) = t^\gamma$.

Example Let $0 < \gamma < 1$ and $0 < \beta < \infty$. Then the function

$$\mu(r) = \begin{cases} 0, & \text{if } r = 0, \\ r^\gamma|\ln r|^\beta, & \text{if } 0 < r < \rho, \\ \rho^\gamma|\ln\rho|^\beta, & \text{if } r \ge \rho \end{cases}$$

satisfies all the conditions of Corollary 2.11 with $\omega(t) = t^\gamma(1 + \frac{|\ln t|}{|\ln\rho|})^\beta$ (see [3]).

3 Formulation and proof of the main theorem

Theorem 3.1 *Let the function $\mu(r)$, $0 < r < \infty$ satisfy all the conditions of Corollary 2.11. Further, suppose function $\varphi \in L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$ has the μ-smoothness property in the L_p-sense, i.e., condition (31) is satisfied. Assume that the operator $\mathbf{D}_\varepsilon^\alpha$ is defined as in (21) and the parameter $l \in \mathbb{N}$ satisfies the condition $l > \alpha + 1$. Then we have*

$$\|\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha\varphi - \varphi\|_p = O(\mu(\varepsilon)) \quad \textit{as } \varepsilon \to 0^+. \tag{37}$$

Proof By making use of formula (23), Lemma 2.6(i), and Minkowski inequality, we have

$$\begin{aligned}\left\|\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\right\|_p &\overset{(23)}{\leq} \int_0^\infty \left|K_\alpha^{(l)}(\eta)\right| e^{-\varepsilon\eta} \|P_{\varepsilon\eta}\varphi - \varphi\|_p \, d\eta \\ &\leq \int_0^\infty \left|K_\alpha^{(l)}(\eta)\right| \|P_{\varepsilon\eta}\varphi - \varphi\|_p \, d\eta. \end{aligned} \tag{38}$$

Further, by Lemma 2.1(a),

$$\begin{aligned}\|P_{\varepsilon\eta}\varphi - \varphi\|_p &= \left\| \int_{\mathbb{R}^n} p(y;\varepsilon\eta)\big[\varphi(t-y) - \varphi(t)\big]\, dy \right\|_p \\ &\leq \int_{\mathbb{R}^n} p(y;\varepsilon\eta) \left\|\varphi(t-y) - \varphi(t)\right\|_p dy \\ &= \int_{|y|\leq\rho} p(y;\varepsilon\eta) \left\|\varphi(t-y) - \varphi(t)\right\|_p dy \\ &\quad + \int_{|y|>\rho} p(y;\varepsilon\eta) \left\|\varphi(t-y) - \varphi(t)\right\|_p dy = I_1(\varepsilon) + I_2(\varepsilon).\end{aligned}$$

Owing to (36), we have $I_1(\varepsilon) \leq A\mu(\varepsilon\eta)$, where A does not depend on ε and η.

Now, let us estimate the second integral $I_2(\varepsilon)$. We have

$$\begin{aligned} I_2(\varepsilon) &\leq 2\|\varphi\|_p \int_{|y|>\rho} p(y;\varepsilon\eta)\, dy \overset{(11)}{=} 2\|\varphi\|_p a_n \int_{|y|>\rho} \frac{\varepsilon\eta}{((\varepsilon\eta)^2 + |y|^2)^{\frac{n+1}{2}}}\, dy \\ &\qquad \big(\text{converting to spherical coordinates, i.e.,} \\ &\qquad y = r\theta; \rho < r < \infty, \theta \in \Sigma^{n-1}, dy = r^{n-1}\, dr\, d\sigma(\theta)\big) \\ &= c_1\varepsilon\eta \int_\rho^\infty \frac{r^{n-1}}{((\varepsilon\eta)^2 + r^2)^{\frac{n+1}{2}}}\, dr \leq c_1\varepsilon\eta \int_\rho^\infty \frac{r^{n-1}}{r^{n+1}}\, dr = c_2\varepsilon\eta, \end{aligned}$$

where $c_2 \equiv c_2(\rho;n)$ does not depend on ε and η.

Hence, we obtain that

$$\|P_{\varepsilon\eta}\varphi - \varphi\|_p \leq A\mu(\varepsilon\eta) + c_2\varepsilon\eta.$$

Further,

$$\begin{aligned}\left\|\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\right\|_p &\overset{(38)}{\leq} \int_0^\infty \left|K_\alpha^{(l)}(\eta)\right| \big(A\mu(\varepsilon\eta) + c_2\varepsilon\eta\big)\, d\eta \\ &\qquad \big(\text{using the condition } \mu(\varepsilon) \geq a\varepsilon, \varepsilon \in (0,\rho)\big) \\ &\leq c_3\mu(\varepsilon) \int_0^\infty \left|K_\alpha^{(l)}(\eta)\right| \big(\omega(\eta) + \eta\big)\, d\eta. \end{aligned} \tag{39}$$

The condition $\int_0^\infty \frac{\omega(\eta)}{1+\eta^2}\, d\eta < \infty$ and Lemma 2.6(ii) yield

$$\begin{aligned}\int_0^\infty \left|K_\alpha^{(l)}(\eta)\right| \omega(\eta)\, d\eta &= \int_0^1 \left|K_\alpha^{(l)}(\eta)\right| \omega(\eta)\, d\eta + \int_1^\infty \left|K_\alpha^{(l)}(\eta)\right| \omega(\eta)\, d\eta \\ &\leq c_4 + \int_1^\infty \frac{\omega(\eta)}{1+\eta^2} \big(1+\eta^2\big) \left|K_\alpha^{(l)}(\eta)\right| d\eta \end{aligned}$$

$$\begin{aligned}&\big(\text{we use the asymptotics } K_\alpha^{(l)}(\eta) = O\big(\eta^{\alpha-l-1}\big)\\ &\text{as } \eta \to \infty \text{ and the condition } l > \alpha + 1\big)\\ &\le c_4 + c_5 \int_1^\infty \frac{\omega(\eta)}{1+\eta^2}\,d\eta = c_6 < \infty.\end{aligned}$$

On the other hand, because of $K_\alpha^{(l)}(\eta) = O(\eta^{\alpha-l-1})$, $\eta \to \infty$ and $l > (\alpha + 1)$, we have

$$\begin{aligned}\int_0^\infty \big|K_\alpha^{(l)}(\eta)\big|\eta\,d\eta &= \int_0^1 \big|K_\alpha^{(l)}(\eta)\big|\eta\,d\eta + \int_1^\infty \big|K_\alpha^{(l)}(\eta)\big|\eta\,d\eta\\ &\le c_7 + \int_1^\infty \big|K_\alpha^{(l)}(\eta)\big|\eta\,d\eta \le c_8.\end{aligned}$$

Taking all of these estimates into account in (39), it follows that

$$\big\|\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\big\|_p \le c\mu(\varepsilon) \quad \text{as } \varepsilon \to 0^+,$$

where the constant c does not depend on ε. This completes the proof. □

Corollary 3.2

(i) *Let* $\mu(t) = t^\gamma$, $0 < \gamma < 1$, $t \in [0, \rho)$, *and suppose a function* $\varphi \in L_p(\mathbb{R}^n)$ *has* μ*-smoothness property in* L_p*-sense. Then*

$$\big\|\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\big\|_p = O\big(\varepsilon^\gamma\big) \quad \text{as } \varepsilon \to 0^+.$$

(ii) *Let* $\mu(t) = t^\gamma |\ln t|^\beta$, $0 < \gamma < 1$, $\beta \in (0, \infty)$, $t \in (0, \rho)$, *and suppose a function* $\varphi \in L_p(\mathbb{R}^n)$ *has* μ*-smoothness property in* L_p*-sense. Then*

$$\big\|\mathbf{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\big\|_p = O\big(\varepsilon^\gamma |\ln \varepsilon|^\beta\big) \quad \text{as } \varepsilon \to 0^+.$$

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Author details
[1]Department of Mathematics, Akdeniz University, Antalya, Turkey. [2]Faculty of Education, Akdeniz University, Antalya, Turkey. [3]Ahi Evran Vocational and Technical Anatolian High School, Şanlıurfa, Turkey.

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