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Large-domain stability of random attractors for stochastic g-Navier–Stokes equations with additive noise

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Abstract

This paper concerns the long term behavior of the stochastic two-dimensional g-Navier–Stokes equations with additive noise defined on a sequence of expanding domains, where the ultimate domain is unbounded and of Poincaré type. We prove that the weak continuity is uniform with respect to all expanding cocycles, which yields the equi-asymptotic compactness by using an energy equation method. Finally, we show the existence of a random attractor for the equation on each domain and the upper semi-continuity of random attractors as the bounded domain is expanded to the unbounded ultimate domain.

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1 Introduction

The fluid dynamics of deterministic or stochastic Navier–Stokes (NS) equations has been extensively studied. For example, many properties such as existence, upper semi-continuity, regularity, and fractal dimension of an attractor were studied in the literature [4, 5, 8, 13, 19]. However, we find that most of the above-mentioned studies are given in a two-dimensional situation rather than three-dimensional one, which encourages us to do more in-depth research about the dynamic behavior of Navier–Stokes equations.

The g-NS equations in spatial dimension 2 were introduced by Roh [18] as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(x), \\ \nabla \cdot (gu) = 0, \end{cases} \tag{1}$$

where g is a suitable smooth real-valued function. The uniqueness and existence of solutions for the g-NS equations in $\mathbb{R}^n$ ($n = 2, 3$) were proved by Bae and Roh [3]. When $g = 1$, Eq. (1) becomes the usual 2D Navier–Stokes equations. In the last decade, the limiting behavior of solutions in terms of the existence of attractors for 2D g-NS equations has been

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studied in both autonomous and non-autonomous cases without the stochastic situations, see [10, 11, 18].

As described in [18], the 2D g-NS equations arise in a natural way when we study the standard 3D Navier–Stokes problem in a 3D thin domain $\mathcal{O}_g = \mathcal{O} \times (0, \varepsilon g)$, $(\mathcal{O} \subset \mathbb{R}^2)$ which was introduced by [9, 17], and we do not claim that the g-NS equations form a model of any fluid flow. They may, or may not. That they are derived from a standard 3D problem is the basis for our study. However, as we know, there are no results related to the long-time behavior of solutions for the 2D stochastic g-NS equations.

In this paper, we consider both the existence and large-domain stability of a random attractor for the stochastic 2D g-NS equations on an unbounded Poincaré-type domain $\mathcal{O}_\infty \subset \mathbb{R}^2$, which is regarded as a limit of the sequence of expanding domains $\mathcal{O}_k = \{x \in \mathcal{O}_\infty : |x| < k\}$.

We write the sequence of stochastic 2D g-NS equations on $\mathcal{O}_k$ ($k \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$) as a unified form:

$$
\begin{cases}
du_k - (\nu \Delta u_k - (u_k \cdot \nabla)u_k - \nabla p)\, dt = f(x,t)\, dt + \varepsilon h(x)\, dW(t), & x \in \mathcal{O}_k, \\
\nabla \cdot (g u_k) = 0, & x \in \mathcal{O}_k, \\
u_k = 0, & x \in \partial \mathcal{O}_k, \\
u_k(\tau, x) = u_{\tau,k}(x), & x \in \mathcal{O}_k, \tau \in \mathbb{R},
\end{cases}
\tag{2}
$$

where $\nu, \varepsilon > 0$, p is the pressure, u_k is the velocity vector, W is a scalar Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. $h(x)$ is a given time-independent two-dimensional vector function belonging to some Sobolev spaces which will be specified later.

The first subject is to show the existence of a random attractor $\mathcal{A}_k$ in $H_g(\mathcal{O}_k)$ (a special subspace of $\mathbb{L}^2(\mathcal{O}_k)$) for each $k \in \overline{\mathbb{N}}$. Due to both non-autonomy and randomness of model (2), the attractor is actually a bi-parametric set $\mathcal{A}_k = \{\mathcal{A}_k(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $H_g(\mathcal{O}_k)$ (see [21]). Even for this existence of a pullback attractor, the assumption of small noise ($\varepsilon \leq \varepsilon_0$) seems to be necessary.

To study problem (2), the real-valued function $g = g(x) \in W^{1,\infty}(\mathcal{O}_\infty)$ satisfies the following basic assumption:

$$
0 < m_0 \leq g(x) \leq M_0, \quad \forall x = (x_1, x_2) \in \mathcal{O}_\infty.
\tag{3}
$$

Using the famous energy equation method [19], we establish the existence of random attractors. More precisely, for each $k \in \overline{\mathbb{N}}$, the stochastic g-NS equations (2) have a random attractor $\mathcal{A}_k$ in $H_g(\mathcal{O}_k)$.

The second subject is to investigate *large-domain stability* of the attractor, which means that $\mathcal{A}_k$ is stable (upper semi-continuous) at $\mathcal{A}_\infty$ under a suitable Hausdorff semi-distance.

Such an expanding-domain problem is contrary to the thin-domain problem, the latter was extensively investigated in the literature (see [14, 15]) and time-varying domains problem [20]. However, the same difficulty arises from the fact that both $\mathcal{A}_k$ and $\mathcal{A}_\infty$ lie in different phase spaces, compared with the same phase space in time-dependent stability of a pullback attractor [6, 7, 12].

In order to define a distance between two subsets lying in different spaces, we consider the null-expansion $\widetilde{u}_k$ of the solution $u_k \in H_g(\mathcal{O}_k)$ defined by

$$
\widetilde{u}_k = u_k, \quad x \in \mathcal{O}_k; \qquad \widetilde{u}_k = 0, \quad x \in \mathcal{O}_\infty \setminus \mathcal{O}_k.
$$

One can show easily that $\widetilde{u}_k \in H_g(\mathcal{O}_\infty)$ if $u_k \in H_g(\mathcal{O}_k)$. However, in general, $\widetilde{u}_k \neq u_\infty$. So, the attractor $\mathcal{A}_k$ can be expanded to a bi-parametric set $\widetilde{\mathcal{A}}_k$ in $H_g(\mathcal{O}_\infty)$, defined by

$$\widetilde{\mathcal{A}}_k(\tau, \omega) = \left\{ u \in H_g(\mathcal{O}_\infty) : \exists v \in \mathcal{A}_k(\tau, \omega), \text{s.t. } u = \widetilde{v} \right\}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.$$

In this way, the Hausdorff semi-distance between $\widetilde{\mathcal{A}}_k$ and $\mathcal{A}_\infty$ lies in the same space $H_g(\mathcal{O}_\infty)$ where the Hausdorff semi-distance can be understood in the following sense:

$$\text{dist}_{H_g(\mathcal{O}_\infty)}\big(\widetilde{\mathcal{A}}_k(\tau, \omega), \mathcal{A}_\infty(\tau, \omega)\big) = \sup_{u \in \widetilde{\mathcal{A}}_k(\tau, \omega)} \inf_{v \in \mathcal{A}_\infty(\tau, \omega)} \|u - v\|_{H_g(\mathcal{O}_\infty)}. \tag{4}$$

Then our aim is to prove

$$\text{dist}_{H_g(\mathcal{O}_\infty)}\big(\widetilde{\mathcal{A}}_k(\tau, \omega), \mathcal{A}_\infty(\tau, \omega)\big) \to 0 \quad \text{as } k \to \infty. \tag{5}$$

However, the usual energy equation method is not sufficient to prove the large-domain stability from $\mathcal{A}_k$ to $\mathcal{A}_\infty$ as $k \to \infty$. We will expand each cocycle Φ_k (on $H_g(\mathcal{O}_k)$) to the null-expansion cocycle $\widetilde{\Phi}_k$ on a subspace of $H_g(\mathcal{O}_\infty)$ and prove that the sequence of expanding cocycles $\{\widetilde{\Phi}_k\}_k$ is equi-asymptotically compact (uniformly in k) in $H_g(\mathcal{O}_\infty)$.

For this end, we develop the usual energy equation method from a single system to a sequence of systems, and prove weak equi-continuity of expanding cocycles $\{\widetilde{\Phi}_k\}_k$, which together with an energy equality can help us to establish the equi-asymptotic compactness of expanding cocycles $\{\widetilde{\Phi}_k\}_k$.

Finally, by proving the convergence from $\widetilde{\Phi}_k$ to Φ_∞ in $H_g(\mathcal{O}_\infty)$, we establish the large-domain stability (5) as desired.

This paper is organized as follows. In the next section, the functional spaces and a continuous cocycle for the stochastic g-Navier–Stokes equations are defined. In Sect. 3, we define the expanding cocycles and prove the convergence of the expanding cocycles for stochastic g-Navier–Stokes equations. We derive the uniform estimates and weak equi-continuity of the solution sequence $\{v_k\}$ in Sect. 4, which yields the equi-asymptotic compactness of the sequence $\{\widetilde{\Phi}_k\}_k$ of expanding cocycles in Sect. 5. In the last section, we show the existence and large-domain stability of the attractor when the domain changes from bounded to unbounded.

2 Random attractor on varying domain

2.1 Functional spaces and operators

As pointed out in Sect. 1, the unbounded domain $\mathcal{O}_\infty$ is of Poincaré type, and thus there exists $\lambda_\infty > 0$ such that

$$\int_{\mathcal{O}_\infty} |\nabla \zeta(x)|^2 \, dx \geq \lambda_\infty \int_{\mathcal{O}_\infty} |\zeta(x)|^2 \, dx, \quad \forall \zeta \in H_0^1(\mathcal{O}_\infty). \tag{6}$$

Let $\mathcal{O}_k = \{x \in \mathcal{O}_\infty : |x| \leq k\}$, and for each $k \in \overline{\mathbb{N}}$, we use $(L_g^2(\mathcal{O}_k), \|\cdot\|_g)$ to denote the space $L^2(\mathcal{O}_k)$ with the following norm:

$$\|\zeta\|_g^2 := \int_{\mathcal{O}_k} g(x) |\zeta(x)|^2 \, dx, \quad \forall \zeta \in L^2(\mathcal{O}_k), k \in \overline{\mathbb{N}}.$$

By (3), one can show that $m_0\|\zeta\|^2 \le \|\zeta\|_g^2 \le M_0\|\zeta\|^2$. So, both norms $\|\cdot\|_g$ and $\|\cdot\|$ are indeed equivalent.

Also, we use $(H_{0,g}^1(\mathcal{O}_k), \|\cdot\|_{H_{0,g}^1})$ to denote the space $H_0^1(\mathcal{O}_k)$ with the following norm:

$$\|\zeta\|_{H_{0,g}^1(\mathcal{O}_k)}^2 := \int_{\mathcal{O}_k} g(x)\big|\nabla\zeta(x)\big|^2 dx, \quad \forall \zeta \in H_0^1(\mathcal{O}_k), k \in \overline{\mathbb{N}}.$$

Then, by [16], there exists $\lambda_0 > 0$ (independent of k) such that

$$\|\zeta\|_{H_{0,g}^1(\mathcal{O}_k)}^2 \ge \lambda_0\|\zeta\|_g^2, \quad \forall \zeta \in H_0^1(\mathcal{O}_k), k \in \overline{\mathbb{N}}, \tag{7}$$

which implies that the new norm is (uniformly) equivalent to the original $H_0^1(\mathcal{O}_k)$-norm.

To reformulate system (2), we introduce some function space:

$$\mathcal{V}(\mathcal{O}_k) = \big\{u \in \mathbb{C}_0^\infty(\mathcal{O}_k) : \nabla \cdot (gu) = 0\big\},$$

$$H_g(\mathcal{O}_k) = \mathrm{cl}_{\mathbb{L}_g^2(\mathcal{O}_k)}\,\mathcal{V}, \qquad V_g(\mathcal{O}_k) = \mathrm{cl}_{\mathbb{H}_{0,g}^1(\mathcal{O}_k)}\,\mathcal{V},$$

where cl_X denotes the closure taken in X and $\mathbb{C}_0^\infty(\mathcal{O}_k) = C_0^\infty(\mathcal{O}_k)^2$, $\mathbb{L}_g^2(\mathcal{O}_k) := L_g^2(\mathcal{O}_k)^2$, $\mathbb{H}_{0,g}^1(\mathcal{O}_k) = H_{0,g}^1(\mathcal{O}_k)^2$, respectively.

Then $H_g(\mathcal{O}_k)$, $V_g(\mathcal{O}_k)$ are Hilbert spaces with the inner products $(\cdot,\cdot)_g$ and $((\cdot,\cdot))_g$ of $\mathbb{L}_g^2(\mathcal{O}_k)$ given by, for $u = (u_1, u_2)$, $v = (v_1, v_2) \in H_g(\mathcal{O}_k)$,

$$(u,v)_g = \int_{\mathcal{O}_k} u \cdot v g\, dx = \sum_{i=1}^2 (u_i, v_i)_g \quad \text{and} \quad \|u\|_g = (u,u)_g^{1/2},$$

and for all $u = (u_1, u_2), v = (v_1, v_2) \in V_g(\mathcal{O}_k)$,

$$((u,v))_g := (Du, Dv)_g = \sum_{i,j=1}^2 \int_{\mathcal{O}_k} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} g\, dx \quad \text{and} \quad \|u\|_{V_g} = (Du, Du)_g^{1/2},$$

where $Du = ((\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}), (\frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}))$.

Now, we can define the g-Laplace operator as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - \frac{1}{g}(\nabla g \cdot \nabla)u.$$

Then, the first equation of (2) can be rewritten as

$$du_k - \left(\nu\Delta_g u_k - \frac{\nu}{g}(\nabla g \cdot \nabla)u_k - (u_k \cdot \nabla)u_k - \nabla p\right)dt = f(x,t)\,dt + \varepsilon h(x)\,dW(t). \tag{8}$$

Consider the g-orthogonal projection $P_{g,k} : \mathbb{L}_g^2(\mathcal{O}_k) \to H_g(\mathcal{O}_k)$ and define the g-Stokes operator[18] by

$$A_{g,k}u = -P_{g,k}\left(\frac{1}{g}(\nabla \cdot g\nabla)u\right), \qquad \langle A_{g,k}u, v\rangle_g = \langle A_{g,k}u, gv\rangle = ((u,v))_g$$

for all $u, v \in V_g(\mathcal{O}_k)$. Let $V_g^*(\mathcal{O}_k)$ be the dual space of $V_g(\mathcal{O}_k)$. Then $A_{g,k} : V_g(\mathcal{O}_k) \to V_g^*(\mathcal{O}_k)$ is a homomorphism with $\|A_{g,k}\| \leq 1/m_0$ (see [16]) and the bound is independent of $k \in \overline{\mathbb{N}}$.

Furthermore, we consider the uniform bound of operator $R_{g,k} : V_g(\mathcal{O}_k) \to V_g^*(\mathcal{O}_k)$ given by

$$R_{g,k} u = P_{g,k}\left(\frac{1}{g}(\nabla g \cdot \nabla)u\right), \quad \forall u \in V_g(\mathcal{O}_k).$$

In this case, by [16], we have the following result.

Lemma 2.1 *For each $k \in \overline{\mathbb{N}}$ and $u \in V_g(\mathcal{O}_k)$, we have the following uniform bounds*:

$$\|u\|_{H_g(\mathcal{O}_k)}^2 \leq \frac{1}{\lambda_0} \|u\|_{V_g(\mathcal{O}_k)}^2 \quad and \quad \|R_{g,k} u\|_{V_g^*(\mathcal{O}_k)} \leq \frac{\|\nabla g\|_\infty}{m_0^2 \lambda_0^{1/2}} \|u\|_{V_g(\mathcal{O}_k)}, \tag{9}$$

where λ_0 is given by (7) and $\|\cdot\|_\infty$ is the norm in $L^\infty(\mathcal{O}_\infty)$.

In the sequel, we will define the bilinear operator $B_{g,k} : V_g(\mathcal{O}_k) \times V_g(\mathcal{O}_k) \to V_g^*(\mathcal{O}_k)$ and the trilinear form $b_{g,k} : V_g(\mathcal{O}_k) \times V_g(\mathcal{O}_k) \times V_g(\mathcal{O}_k) \to \mathbb{R}$ by

$$\left\langle B_{g,k}(u,v), w\right\rangle_g = b_{g,k}(u,v,w) = \sum_{i,j=1}^2 \int_{\mathcal{O}_k} u_i \frac{\partial v_j}{\partial x_i} w_j g \, dx, \quad \forall u, v, w \in V_g(\mathcal{O}_k),$$

and we write $B_{g,k}(u) = B_{g,k}(u,u)$ without confusion. By Roh [18], we have

$$b_{g,k}(u,v,v) = 0, \qquad b_{g,k}(u,v,w) = -b_{g,k}(u,w,v), \tag{10}$$

$$\left\|B_{g,k}(u)\right\|_{V_g^*} \leq c\|u\|_g \|u\|_{V_g}, \tag{11}$$

$$\left|b_{g,k}(u,v,w)\right| \leq c\|u\|_g^{\frac{1}{2}} \|u\|_{V_g}^{\frac{1}{2}} \|v\|_{V_g} \|w\|_g^{\frac{1}{2}} \|w\|_{V_g}^{\frac{1}{2}}. \tag{12}$$

Therefore, we can rewrite (8) in the sense of abstract equation

$$du_k + \left(\nu A_{g,k} u_k + B_{g,k}(u_k, u_k) + \nu R_{g,k} u_k\right) dt = P_{g,k} f(x,t)|_{\mathcal{O}_k} \, dt + \varepsilon h(x) \, dW(t). \tag{13}$$

Definition 2.2 For a function $u : \mathcal{O}_k \to \mathbb{R}^2$, its null-expansion $\tilde{u} : \mathcal{O}_\infty \to \mathbb{R}^2$ is defined by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \mathcal{O}_k, \\ 0 & \text{for } x \in \mathcal{O}_\infty \setminus \mathcal{O}_k. \end{cases} \tag{14}$$

Conversely, for a function $v : \mathcal{O}_\infty \to \mathbb{R}^2$, the restriction $v|_{\mathcal{O}_k} : \mathcal{O}_k \to \mathbb{R}^2$ is given by

$$v|_{\mathcal{O}_k}(x) = v(x), \quad x \in \mathcal{O}_k.$$

We need to estimate the norms of both expansion and restriction in H_g, V_g, and V_g^*.

Lemma 2.3 ([16])
(1) *If $u \in H_g(\mathcal{O}_k)$, then $\tilde{u} \in H_g(\mathcal{O}_\infty)$ and $\|\tilde{u}\|_{H_g(\mathcal{O}_\infty)} = \|u\|_{H_g(\mathcal{O}_k)}$.*

(2) *If $u \in V_g(\mathcal{O}_k)$, then $\tilde{u} \in V_g(\mathcal{O}_\infty)$ and $\|\tilde{u}\|_{V_g(\mathcal{O}_\infty)} = \|u\|_{V_g(\mathcal{O}_k)}$.*

(3) *If $v \in H_g(\mathcal{O}_\infty)$, then $v|_{\mathcal{O}_k} \in H_g(\mathcal{O}_k)$ and $\|v|_{\mathcal{O}_k}\|_{H_g(\mathcal{O}_k)} \leq \|v\|_{H_g(\mathcal{O}_\infty)}$.*

(4) *If $w \in V_g^*(\mathcal{O}_\infty)$, then the restriction $w|_{\mathcal{O}_k} \in V_g^*(\mathcal{O}_\infty)$ and $\|w|_{\mathcal{O}_k}\|_{V_g^*(\mathcal{O}_k)} \leq \|w\|_{V_g^*(\mathcal{O}_\infty)}$.*

2.2 Cocycles for stochastic *g*-NS equations

The standard probability space $(\Omega, \mathcal{F}, P)$ will be used in this paper where

$$\Omega = \left\{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \right\} \quad \text{and} \quad \lim_{t \to \pm\infty} \frac{\omega(t)}{t} = 0, \tag{15}$$

$\mathcal{F}$ is the Borel algebra induced by the compact-open topology of Ω, and P is the Wiener measure on $(\Omega, \mathcal{F})$. Given $t \in \mathbb{R}$, define $\theta_t : \Omega \to \Omega$ by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad (\omega, t) \in \Omega \times \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a parametric dynamical system. Let $z(\theta_t \omega) = -\int_{-\infty}^0 e^s(\theta_t \omega)(s) \, ds$, which solves the stochastic Ornstein–Uhlenbeck equation $dz + z \, dt = dW(t)$. It follows from [1] that there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that $z(\theta_t \omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$, and we have the following limits:

$$\lim_{t \to \pm\infty} \frac{z(\theta_t \omega)}{t} = \lim_{t \to \pm\infty} \frac{1}{t} \int_{-t}^0 z(\theta_s \omega) \, ds = \mathbb{E}\big(z(\omega)\big) = 0, \tag{16}$$

$$\lim_{t \to \infty} \frac{1}{t} \int_{-t}^0 \big|z(\theta_s \omega)\big|^m \, ds = \mathbb{E}\big(|z(\omega)|^m\big) = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi}}, \quad \forall m > 0, \tag{17}$$

where $\mathbb{E}$, Γ denote expectation and gamma function, respectively.

Suppose $h \in H_g(\mathcal{O}_\infty)$, then denote by $h_k(x) := P_{g,k} h|_{\mathcal{O}_k}(x)$ for $x \in \mathcal{O}_k$, $k \in \overline{\mathbb{N}}$ and consider the change of variables:

$$v_k(t, \tau, \omega, v_{\tau,k}) = u_k(t, \tau, \omega, u_{\tau,k}) - \varepsilon z(\theta_t \omega) h_k,$$

with $v_{\tau,k} = u_{\tau,k} - \varepsilon z(\theta_\tau \omega) h_k$, where we understand $h_\infty = P_{g,\infty} h = h$.

In this case, system (13) can be rewritten as follows:

$$\frac{dv_k}{dt} + \nu A_{g,k} v_k + B_{g,k}\big(v_k + \varepsilon h_k z(\theta_t \omega)\big) + \nu R_{g,k} v_k$$

$$= -\varepsilon \nu z(\theta_t \omega) A_{g,k} h_k - \varepsilon \nu z(\theta_t \omega) R_{g,k} h_k + P_{g,k} f(t)|_{\mathcal{O}_k} + \varepsilon z(\theta_t \omega) h_k \tag{18}$$

with the initial value

$$v_k(\tau, \tau, \omega, v_{\tau,k}) = v_{\tau,k} = u_{\tau,k} - \varepsilon z(\theta_\tau \omega) h_k. \tag{19}$$

Definition 2.4 The function $v_k(\cdot, \tau, \omega, v_{\tau,k})$ is called a weak solution of Eqs. (18)–(19) if

$$v_k \in C\big([\tau, \infty), H_g(\mathcal{O}_k)\big) \cap L^2_{\text{loc}}\big((\tau, \infty), V_g(\mathcal{O}_k)\big)$$

and, for all $t \geq \tau$, P-a.s. $\omega \in \Omega$, and $w \in V_g(\mathcal{O}_k)$,

$$
\begin{aligned}
\left(v_k(t), w\right)_g &+ \nu \int_\tau^t \langle A_{g,k} v_k, w \rangle_g \, ds + \nu \int_\tau^t \langle R_{g,k} v_k, gw \rangle \, ds \\
&+ \int_\tau^t \langle B_{g,k}(v_k + \varepsilon h_k z(\theta_s\omega)), w \rangle_g \, ds + \varepsilon\nu \int_\tau^t z(\theta_s\omega) \langle A_{g,k} h_k + R_{g,k} h_k, w \rangle_g \, ds \\
&= (v_{\tau,k}, w)_g + \int_\tau^t \langle f(s)|_{\mathcal{O}_k} + \varepsilon h_k z(\theta_s\omega), gw \rangle \, ds.
\end{aligned}
\tag{20}
$$

Lemma 2.5 *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}, V_g^*(\mathcal{O}_\infty))$. Then, for each $k \in \overline{\mathbb{N}}$ and $v_{\tau,k} \in H_g(\mathcal{O}_k)$, system* (18)–(19) *has a unique weak solution v_k in the sense of Definition* 2.4. *Moreover, the solution $v_k(t, \tau, \omega; v_{\tau,k})$ is continuous in $v_{\tau,k}$ and measurable in ω.*

Then we can define a family of measurable mappings $\Phi_k : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H_g(\mathcal{O}_k) \to H_g(\mathcal{O}_k)$ corresponding to system (18). Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_{\tau,k} \in H_g(\mathcal{O}_k)$, we have

$$
\Phi_k(t, \tau, \omega) v_{\tau,k} = v_k(t + \tau, \tau, \theta_{-\tau}\omega, v_{\tau,k}),
\tag{21}
$$

where $t \geq 0$. Then, for each $k \in \overline{\mathbb{N}}$, Φ_k is a continuous cocycle [21] and we have

$$
\Phi_k(0, \tau, \omega) = I, \qquad \Phi_k(t + s, \tau, \omega) = \Phi_k(t, \tau + s, \theta_s\omega)\Phi_k(s, \tau, \omega)
$$

for all $t, s \geq 0$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$.

We now take a universe $\mathfrak{D}$ on $H_g(\mathcal{O}_\infty)$, which consists of all set-valued mappings $D : \mathbb{R} \times \Omega \to 2^{H_g(\mathcal{O}_\infty)} \setminus \{\emptyset\}$ satisfying

$$
\lim_{t \to +\infty} e^{-\frac{3}{2}\lambda t} \left\| D(\tau - t, \theta_{-t}\omega) \right\|^2_{H_g(\mathcal{O}_\infty)} = 0, \quad \tau \in \mathbb{R}, \omega \in \Omega,
\tag{22}
$$

where $\|D\|$ denotes the supremum of norms of all elements and $\lambda = \frac{1}{3}\lambda_0 \nu$.

Denote $\mathfrak{D}|_{\mathcal{O}_k}$ to be the restriction universe of $\mathfrak{D}$ on $H_g(\mathcal{O}_k)$. By Lemma 2.3, we have $D_k \in \mathfrak{D}|_{\mathcal{O}_k}$ if and only if

$$
\lim_{t \to +\infty} e^{-\frac{3}{2}\lambda t} \left\| D_k(\tau - t, \theta_{-t}\omega) \right\|^2_{H_g(\mathcal{O}_k)} = 0 \quad \text{for } k \in \mathbb{N}.
\tag{23}
$$

It is easy to check that if $D_k \in \mathfrak{D}|_{\mathcal{O}_k}$, then $\widetilde{D_k} \in \mathfrak{D}$. Furthermore, $\widetilde{D|_{\mathcal{O}_k}} \neq D$ for $D \in \mathfrak{D}$.

In order to obtain the $\mathfrak{D}|_{\mathcal{O}_k}$-pullback attractor $\mathcal{A}_k$ for all $k \in \overline{\mathbb{N}}$, we make further assumptions.

Assumption G We further assume $g \in W^{1,\infty}(\mathcal{O}_k)$ and

$$
\|\nabla g\|_\infty \leq \frac{1}{4} m_0 \lambda_0^{\frac{1}{2}}, \quad \forall k \in \overline{\mathbb{N}},
\tag{24}
$$

where λ_0 is given by (7).

Assumption H We take the function $h(x) \in (W^{1,\infty}(\mathcal{O}_k))^2 \cap V_g(\mathcal{O}_k) \cap \mathbb{H}^2(\mathcal{O}_k)$, $\forall k \in \overline{\mathbb{N}}$.

Assumption F $f \in L^2_{\text{loc}}(\mathbb{R}, V^*_g(\mathcal{O}_\infty))$, we assume that, for $\lambda = \frac{1}{3}\lambda_0 \nu$,

$$\int_{-\infty}^{0} e^{\lambda s} \|f(s + \tau)\|^2_{V^*_g(\mathcal{O}_\infty)} \, ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{25}$$

By Lemma 2.3(4) and Assumption F, the restriction $f|_{\mathcal{O}_k}$ is still tempered:

$$\int_{-\infty}^{0} e^{\lambda s} \|f|_{\mathcal{O}_k}(s + \tau)\|^2_{V^*_g(\mathcal{O}_k)} \, ds < \infty, \quad \forall \tau \in \mathbb{R}, k \in \mathbb{N}. \tag{26}$$

Assumption S (Small noise) The density of noise $\varepsilon \in (0, \varepsilon_0]$ is small enough, where

$$\varepsilon_0 = \min\left\{ \frac{\sqrt{\pi}\lambda}{4c_0}, 1 \right\} \quad \text{and} \quad c_0 = 2M_0\big(\|h\| + 1\big)\|\nabla h\|_{\mathbb{L}^\infty(\mathcal{O}_\infty)}.$$

3 Random attractor and convergence of expanding cocycles
3.1 Expanding cocycles

In this section, we need to expand the cocycle Φ_k from $H_g(\mathcal{O}_k)$ to $H_g(\mathcal{O}_\infty)$. For this end, we define the corresponding null-expansion of an operator $\Phi_k(t, \tau, \omega): H_g(\mathcal{O}_k) \to H_g(\mathcal{O}_k)$ by $\widetilde{\Phi}_k : H_g(\mathcal{O}_\infty) \to H_g(\mathcal{O}_\infty)$,

$$(\widetilde{\Phi}_k u)(x) = \big(\Phi_k(u|_{\mathcal{O}_k})\big)(x), \quad \forall x \in \mathcal{O}_k, \qquad (\widetilde{\Phi}_k u)(x) = 0, \quad \forall x \in \mathcal{O}_\infty \setminus \mathcal{O}_k. \tag{27}$$

However, in general, $\widetilde{u|_{\mathcal{O}_k}} \neq u$ for $u \in H_g(\mathcal{O}_\infty)$ and so $\widetilde{\Phi}_k(0, \tau, \omega)$ is not an identical operator on $H_g(\mathcal{O}_\infty)$. Hence, $\widetilde{\Phi}_k$ is not a cocycle on the whole space $H_g(\mathcal{O}_\infty)$.

Fortunately, we can show that the null-expansion $\widetilde{\Phi}_k$ is a cocycle on the closed linear subspace $H_k(\mathcal{O}_\infty)$ defined by

$$H_k(\mathcal{O}_\infty) := \big\{u \in H_g(\mathcal{O}_\infty) : u(x) = 0 \text{ for } x \in \mathcal{O}_\infty \setminus \mathcal{O}_k\big\},$$

and thus $H_k(\mathcal{O}_\infty)$ is a Banach space with the same norm as in $H_g(\mathcal{O}_\infty)$.

A $\mathfrak{D}$-pullback random attractor means a bi-parametric set which is measurable, compact, invariant, and $\mathfrak{D}$-pullback attracting. For the concept and existence theorem, the reader can refer to[21].

Theorem 3.1 *For each $k \in \overline{\mathbb{N}}$, the null-expansion $\widetilde{\Phi}_k$ of Φ_k is a cocycle on $H_k(\mathcal{O}_\infty)$ and it has a $\widetilde{\mathfrak{D}}_k$-pullback random attractor in $H_k(\mathcal{O}_\infty)$, given by the null-expansion $\widetilde{\mathcal{A}}_k$ of the random attractor $\mathcal{A}_k$.*

Proof By (27), for each $u \in H_k(\mathcal{O}_\infty)$, we have

$$\widetilde{\Phi}_k(0, \tau, \omega)u = \text{expansion of } \Phi_k(0, \tau, \omega)u|_{\mathcal{O}_k} = \widetilde{u|_{\mathcal{O}_k}} = u.$$

So, $\widetilde{\Phi}_k(0, \tau, \omega)$ is the identical operator on $H_k(\mathcal{O}_\infty)$. Other properties of cocycle are easily verified.

Next, we prove that $\widetilde{\mathcal{A}}_k$ is a $\widetilde{\mathfrak{D}}_k$-pullback random attractor for $\widetilde{\Phi}_k$ in four steps.

Step 1. We show that $\widetilde{\mathcal{A}}_k$ is measurable on $H_k(\mathcal{O}_\infty)$. Let $u \in H_k(\mathcal{O}_\infty)$, which means $u \equiv 0$ on $\mathcal{O}_\infty \setminus \mathcal{O}_k$ and thus

$$\|u\|^2_{H_k(\mathcal{O}_\infty)} = \int_{\mathcal{O}_\infty} |u(x)|^2 g \, dx = \int_{\mathcal{O}_k} |u(x)|^2 g \, dx = \|u|_{\mathcal{O}_k}\|^2_{H_g(\mathcal{O}_k)}.$$

This equality implies that

$$d_{H_k(\mathcal{O}_\infty)}\big(u, \widetilde{\mathcal{A}}_k(\tau, \omega)\big) = d_{H_g(\mathcal{O}_k)}\big(u|_{\mathcal{O}_k}, \mathcal{A}_k(\tau, \omega)\big). \tag{28}$$

Since $\mathcal{A}_k(\tau, \omega)$ is measurable on $H_g(\mathcal{O}_k)$, the above equality implies the measurability of the mapping $\omega \to d_{H_k(\mathcal{O}_\infty)}(u, \widetilde{\mathcal{A}}_k(\tau, \omega))$ for each $u \in H_k(\mathcal{O}_\infty)$, that is, $\widetilde{\mathcal{A}}_k$ is measurable on $H_k(\mathcal{O}_\infty)$.

Step 2. We show that $\widetilde{\mathcal{A}}_k(\tau, \omega)$ is a compact set in $H_k(\mathcal{O}_\infty)$. Let $\{u_n\}$ be a sequence of $\widetilde{\mathcal{A}}_k(\tau, \omega)$. Then there exists $v_n \in \mathcal{A}_k(\tau, \omega)$ such that $u_n = \widetilde{v}_n$. Due to $\mathcal{A}_k(\tau, \omega)$ is compact in $H_g(\mathcal{O}_k)$, passing to a subsequence, we have $v_{n'} \to v \in \mathcal{A}_k(\tau, \omega)$. Denote $\widetilde{v}$ by the null-expansion of v, then $\widetilde{v} \in \widetilde{\mathcal{A}}_k$. By Lemma 2.3(1),

$$\|u_{n'} - \widetilde{v}\|_{H_g(\mathcal{O}_\infty)} = \|\widetilde{v}_{n'} - \widetilde{v}\|_{H_g(\mathcal{O}_\infty)} = \|v_{n'} - v\|_{H_g(\mathcal{O}_k)} \to 0 \quad \text{as } n' \to \infty,$$

which implies that $u_{n'} \to \widetilde{v} \in \widetilde{\mathcal{A}}_k(\tau, \omega)$. Therefore, $\widetilde{\mathcal{A}}_k(\tau, \omega)$ is compact in $H_k(\mathcal{O}_\infty)$.

Step 3. We show the invariance. Given $u \in \widetilde{\mathcal{A}}_k(t + \tau, \theta_t \omega)$, there is $v \in \mathcal{A}_k(t + \tau, \theta_t \omega)$ such that the null-expansion $\widetilde{v} = u$. By the negative invariance of $\mathcal{A}_k(\tau, \omega)$, there is $w \in \mathcal{A}_k(\tau, \omega)$ such that

$$v = \Phi_k(t, \tau, \omega) w.$$

Let $\widetilde{w}$ be the null-expansion of w, then $\widetilde{w}_{|\mathcal{O}_k} = w$. It follows from (27) that we have

$$u = \widetilde{v} = \widetilde{\Phi}_k(t, \tau, \omega) \widetilde{w} \in \widetilde{\Phi}_k(t, \tau, \omega) \widetilde{\mathcal{A}}_k(\tau, \omega),$$

which proves the negative invariance of $\widetilde{\mathcal{A}}_k$. Similarly, one can prove the positive invariance.

Step 4. We show that $\widetilde{\mathcal{A}}_k$ is $\widetilde{\mathfrak{D}}_k$-pullback attracting. Let $D \in \widetilde{\mathfrak{D}}_k$, then there is $D_k \in \mathfrak{D}_k$ such that $D = \widetilde{D}_k$. By the same method as given in (28), we know that

$$\text{dist}_{H_k(\mathcal{O}_\infty)}\big(\widetilde{\Phi}_k(t, \tau - t, \theta_{-t}\omega) D(\tau - t, \theta_{-t}\omega), \widetilde{\mathcal{A}}_k(\tau, \omega)\big)$$
$$= \text{dist}_{H_g(\mathcal{O}_k)}\big(\Phi_k(t, \tau - t, \theta_{-t}\omega) D_k(\tau - t, \theta_{-t}\omega), \mathcal{A}_k(\tau, \omega)\big).$$

Then the attraction of $\widetilde{\mathcal{A}}_k$ follows from the attraction of $\mathcal{A}_k$. □

3.2 Convergence of expanding cocycles
In this subsection, we prove the convergence of the expanding cocycles as follows.

Lemma 3.2 *Let* $v_{0,k} \in H_g(\mathcal{O}_k)$ $(k \in \overline{\mathbb{N}})$ *such that* $\|\widetilde{v_{0,k}} - v_{0,\infty}\|_{H_g(\mathcal{O}_\infty)} \to 0$ *as* $k \to \infty$. *Then*

$$\big\|\widetilde{v}_k(t, \tau, \omega; v_{0,k}) - v_\infty(t, \tau, \omega; v_{0,\infty})\big\|_{H_g(\mathcal{O}_\infty)} \to 0, \quad \text{as } k \to \infty, \tag{29}$$

uniformly in $t \in [\tau, \tau + T]$ *for any* $T > 0$.

Proof By considering the expansion, we rewrite Eq. (18) on the domain $\mathcal{O}_\infty$ for all $t \in [\tau, \tau + T]$, $k \in \mathbb{N}$:

$$\frac{d\widetilde{v_k}}{dt} + \nu A_{g,\infty}\widetilde{v_k} + \varepsilon \nu z(\theta_t\omega)A_{g,\infty}\widetilde{h_k} + B_{g,\infty}(\widetilde{v_k} + \varepsilon\widetilde{h_k}z(\theta_t\omega))$$
$$+ \nu R_{g,\infty}\widetilde{v_k} + \varepsilon\nu z(\theta_t\omega)R_{g,\infty}\widetilde{h_k} = \widetilde{f(t)|_{\mathcal{O}_k}} + \varepsilon z(\theta_t\omega)\widetilde{h_k}, \tag{30}$$

where $\widetilde{v_\infty} = v_\infty$ and $\widetilde{f(t)|_{\mathcal{O}_k}}$ is regarded as the null-expansion of the restriction of $f(t)$.

Let $V_k = \widetilde{v_k} - v_\infty \in H_g(O_\infty)$. Subtracting Eq. (18) for $k = \infty$ from (30) and multiplying the result by gV_k, we have

$$\frac{d}{dt}\|V_k\|_g^2 + 2\nu\|DV_k\|_g^2 = -2b_{g,\infty}(\widetilde{v_k} + \varepsilon\widetilde{h_k}z(\theta_t\omega), \widetilde{v_k} + \varepsilon\widetilde{h_k}z(\theta_t\omega), V_k)$$
$$+ 2b_{g,\infty}(v_\infty + \varepsilon hz(\theta_t\omega), v_\infty + \varepsilon hz(\theta_t\omega), V_k) - 2\nu\langle R_{g,k}V_k, gV_k\rangle$$
$$- \varepsilon\nu z(\theta_t\omega)\langle(A_{g,\infty} + R_{g,\infty})(\widetilde{h_k} - h), gV_k\rangle$$
$$+ 2\langle\widetilde{f(t)|_{\mathcal{O}_k}} - f(t), gV_k\rangle + \varepsilon z(\theta_t\omega)(\widetilde{h_k} - h, gV_k). \tag{31}$$

By (10) and the trilinear property of $b_{g,\infty}$,

$$I_1 := -2(b_{g,\infty}(\widetilde{v_k} + \varepsilon\widetilde{h_k}z(\theta_t\omega), \widetilde{v_k} + \varepsilon\widetilde{h_k}z(\theta_t\omega), V_k)$$
$$+ 2b_{g,\infty}(v_\infty + \varepsilon hz(\theta_t\omega), v_\infty + \varepsilon hz(\theta_t\omega), V_k)$$
$$= -2b_{g,\infty}(V_k + \varepsilon z(\theta_t\omega)(\widetilde{h_k} - h), \widetilde{v_k} + \varepsilon\widetilde{h_k}z(\theta_t\omega), V_k)$$
$$- 2b_{g,\infty}(v_\infty + \varepsilon z(\theta_t\omega)h, V_k + \varepsilon z(\theta_t\omega)(\widetilde{h_k} - h), V_k)$$
$$= -2b_{g,\infty}(V_k + \varepsilon z(\theta_t\omega)(\widetilde{h_k} - h), v_\infty + \varepsilon\widetilde{h_k}z(\theta_t\omega), V_k)$$
$$- 2b_{g,\infty}(v_\infty + \varepsilon z(\theta_t\omega)h, \varepsilon z(\theta_t\omega)(\widetilde{h_k} - h), V_k). \tag{32}$$

Notice that $\varepsilon z(\theta_t\omega)$ is bounded in $t \in [\tau, \tau + T]$, $\varepsilon \leq \varepsilon_0$ and the sequence $\{\widetilde{h_k} : k \in \overline{\mathbb{N}}\}$ is bounded in $V_g(\mathcal{O}_\infty)$. We infer from (12) and (32) that

$$I_1 \leq 2\|V_k\|_g\|DV_k\|_g(\|Dv_\infty\|_g + c) + c\|D(\widetilde{h_k} - h)\|_g(\|Dv_\infty\|_g + c)\|DV_k\|_g$$
$$\leq \frac{\nu}{2}\|DV_k\|_g^2 + c(\|Dv_\infty\|_g^2 + 1)\|V_k\|_g^2 + c(\|Dv_\infty\|_g^2 + 1)\|D(\widetilde{h_k} - h)\|_g^2. \tag{33}$$

By Assumption G,

$$I_2 := -2\nu\langle R_{g,k}V_k, gV_k\rangle \leq 2\nu\left|\left\langle\left(\frac{\nabla g}{g}\cdot\nabla\right)V_k, gV_k\right\rangle\right|$$
$$\leq \frac{2\nu\|\nabla g\|_\infty}{m_0\lambda_0^{1/2}}\|DV_k\|_g^2 \leq \frac{\nu}{2}\|DV_k\|_g^2. \tag{34}$$

Since $A_{g,\infty}$ and $R_{g,\infty}$ are bounded linear operators from $V_g(\mathcal{O}_\infty)$ to V_g^*, we have

$$I_3 := -\varepsilon\nu z(\theta_t\omega)\langle(A_{g,\infty} + R_{g,\infty})(\widetilde{h_k} - h), gV_k\rangle$$
$$\leq \frac{\nu}{2}\|DV_k\|_g^2 + c\|D(\widetilde{h_k} - h)\|_g^2. \tag{35}$$

For the forcing term, since $V_k = \widetilde{v}_k - v_\infty = -v_\infty$ on $\mathcal{O}_\infty \setminus \mathcal{O}_k$, we have

$$
\begin{aligned}
I_4 &:= 2\left|\left\langle \widetilde{f(t)|_{\mathcal{O}_k}} - f(t), gV_k \right\rangle\right| = 2\left|\int_{\mathcal{O}_\infty \setminus \mathcal{O}_k} f(t)v_\infty g\, dx\right| \\
&= 2\left|\int_{\mathcal{O}_\infty} f(t)(v_\infty - \widetilde{v_\infty|_{\mathcal{O}_k}})g\, dx\right| \leq 2\|f(t)\|_{V_g^*(\mathcal{O}_\infty)}\|v_\infty - \widetilde{v_\infty|_{\mathcal{O}_k}}\|_{V_g(\mathcal{O}_\infty)}.
\end{aligned}
\tag{36}
$$

By the Holder inequality and Poincaré inequality,

$$
I_5 := \varepsilon z(\theta_t\omega)(\widetilde{h}_k - h, gV_k) \leq \|V_k\|_g^2 + c\left\|D(\widetilde{h}_k - h)\right\|_g^2.
\tag{37}
$$

It follows from (33) to (37) that

$$
\begin{aligned}
\frac{d}{dt}\|V_k\|_g^2 &\leq c\|Dv_\infty\|_g^2\|V_k\|_g^2 + c\|f(t)\|_{V_g^*(\mathcal{O}_\infty)}\|v_\infty - \widetilde{v_\infty|_{\mathcal{O}_k}}\|_{V_g(\mathcal{O}_\infty)} \\
&\quad + c\big(\|Dv_\infty\|_g^2 + 1\big)\left\|D(\widetilde{h}_k - h)\right\|_g^2.
\end{aligned}
\tag{38}
$$

By Gronwall's lemma we get, for all $t \in [\tau, \tau + T]$,

$$
\begin{aligned}
\left\|V_k(t)\right\|^2 &\leq Ce^{C\int_\tau^{\tau+T}\|v_\infty(r)\|_{V_g}^2\, dr}\bigg(\left\|V_k(\tau)\right\|^2 + \int_\tau^{\tau+T}\|f(r)\|_{V_g^*}\|v_\infty - \widetilde{v_\infty|_{\mathcal{O}_k}}\|_{V_g(\mathcal{O}_\infty)}\, dr \\
&\quad + \left\|D(\widetilde{h}_k - h)\right\|_g^2 \int_\tau^{\tau+T}\big(\|v_\infty(r)\|_{V_g}^2\, dr + 1\big)\, dr\bigg).
\end{aligned}
\tag{39}
$$

By Lemma 2.5, $v_\infty \in L^2(\tau, \tau + T; V_g(\mathcal{O}_\infty))$ and thus

$$
\int_\tau^{\tau+T}\int_{\mathcal{O}_\infty}\left|Dv_\infty(r,x)\right|^2\, dx\, dr < +\infty.
\tag{40}
$$

Since $f \in L^2(\tau, \tau + T; V_g^*(\mathcal{O}_\infty))$, it follows from the Holder inequality that, as $k \to \infty$,

$$
\begin{aligned}
&\int_\tau^{\tau+T}\|f(r)\|_{V_g^*(\mathcal{O}_\infty)}\left\|D\big(v_\infty(r) - \widetilde{v_{\infty|\mathcal{O}_k}}(r)\big)\right\|_g\, dr \\
&\leq \left(\int_\tau^{\tau+T}\|f(r)\|_{V_g^*(\mathcal{O}_\infty)}^2\right)^{\frac{1}{2}}\big(\left\|D\big(v_\infty(r) - \widetilde{v_{\infty|\mathcal{O}_k}}(r)\big)\right\|_g^2\, dr\big)^{\frac{1}{2}} \\
&\leq c\left(\int_\tau^{\tau+T}\int_{\mathcal{O}_\infty \setminus \mathcal{O}_k}\left|Dv_\infty(r,x)\right|^2\, dx\, dr\right)^{\frac{1}{2}} \to 0
\end{aligned}
\tag{41}
$$

in view of (40) and the Lebesgue controlled convergence theorem. By Assumption H and the absolute continuity of the integrals, by the convergence of the initial value, we have

$$
\left\|D(\widetilde{h}_k - h)\right\|_g^2 \to 0 \quad \text{as } k \to \infty.
\tag{42}
$$

By the assumption that $\|V_k(\tau)\|_g^2 \to 0$ as $k \to \infty$, we infer from (39)–(42) that $\|V_k(t)\|_g^2 \to 0$ as $k \to \infty$, uniformly in $t \in [\tau, \tau + T]$. □

4 Weak equi-continuity of the sequence of expanding cocycles

4.1 Uniform absorption

Lemma 4.1 *There is a random variable $C_0(\omega)$ such that*

$$c_0 \varepsilon \int_{-t}^0 \left| z(\theta_r \omega) \right| dr \le \frac{1}{2} \lambda t + C_0(\omega), \quad \forall t \ge 0, \varepsilon \le \varepsilon_0. \tag{43}$$

Proof By (17) and the ergodic theorem,

$$\frac{1}{t} \int_{-t}^0 \left| z(\theta_r \omega) \right| dr \to \mathbb{E}\left(\left| z(\omega) \right| \right) = \frac{1}{\sqrt{\pi}} \quad \text{as } t \to +\infty.$$

We choose large $t_0(\omega) \ge 0$ and use Assumption S to obtain, for all $\varepsilon \le \varepsilon_0$,

$$c_0 \varepsilon \int_{-t}^0 \left| z(\theta_r \omega) \right| dr \le c_0 \varepsilon_0 \frac{2}{\sqrt{\pi}} t \le \frac{1}{2} \lambda t, \quad \forall t \ge t_0, \tag{44}$$

while, for all $t \le t_0$,

$$c_0 \varepsilon \int_{-t}^0 \left| z(\theta_r \omega) \right| dr \le c_0 \varepsilon_0 t_0 \max_{r \in [-t_0, 0]} \left| z(\theta_r \omega) \right| := C_0(\omega).$$

Let $\mathbb{Q}$ be the set of rational numbers, then $[-t_0, 0] \cap \mathbb{Q} = \{r_1, r_2, \dots\}$ is a countable set. Hence,

$$\max_{r \in [-t_0, 0]} \left| z(\theta_r \omega) \right| = \sup_{n \in \mathbb{N}} \left| z(\theta_{r_n} \omega) \right|,$$

which is measurable in ω and so is $C_0(\omega)$. Therefore, (43) holds true. $\qquad\square$

Lemma 4.2 *For each $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathfrak{D}$, there exists $T = T(\tau, \omega, \sigma, D) > 0$ such that, for all $t \ge T$, $\sigma \in [\tau - t, \tau]$, and $v_{\tau-t,k} \in D|_{\mathcal{O}_k}(\tau - t, \theta_{-t}\omega)$,*

$$\sup_{k \in \overline{\mathbb{N}}} \left\| v_k(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,k}) \right\|_{H_g(\mathcal{O}_k)}^2 \le e^{2\lambda(\tau - \sigma)} \rho(\sigma, \tau, \omega), \tag{45}$$

where, from Assumptions F, S,

$$\rho(\sigma, \tau, \omega) = c \int_{-\infty}^{\sigma - \tau} e^{2\lambda s + c_0 \varepsilon \int_s^0 |z(\theta_r \omega)| \, dr} \left\| f(s + \tau) \right\|_{V_g^*(\mathcal{O}_\infty)}^2 ds$$

$$+ c \int_{-\infty}^{\sigma - \tau} e^{2\lambda s + c_0 \varepsilon \int_s^0 |z(\theta_r \omega)| \, dr} \left(\left| z(\theta_s \omega) \right|^2 + \left| z(\theta_s \omega) \right|^4 \right) ds < +\infty.$$

Proof We multiply Eq. (18) by $g v_k$ and integrate over $\mathcal{O}_k$ to obtain

$$\frac{d}{dt} \| v_k \|_g^2 + 2\nu \| v_k \|_{V_g}^2$$

$$= 2\langle f(t)|_{\mathcal{O}_k}, g v_k \rangle + 2\varepsilon z(\theta_t \omega)(h_k, g v_k) - 2\varepsilon \nu z(\theta_t \omega)\langle A_{g,k} h_k, g v_k \rangle$$

$$- 2\varepsilon \nu z(\theta_t \omega)\langle R_{g,k} h_k, g v_k \rangle - 2\nu \langle R_{g,k} v_k, g v_k \rangle - 2\langle B_{g,k}(v_k + \varepsilon h_k z(\theta_t \omega)), v_k \rangle_g. \tag{46}$$

For the nonlinear term, by Lemma 2.3(4), we have

$$
\begin{aligned}
2\big|\langle f(t)|_{\mathcal{O}_k}, gv_k\rangle\big| &\leq 2M_0\big\|f(t)|_{\mathcal{O}_k}\big\|_{V_g^*(\mathcal{O}_k)}\|v_k\|_{V_g} \\
&\leq \frac{\nu}{2}\|v_k\|_{V_g}^2 + \frac{2M_0^2}{\nu}\big\|f(t)|_{\mathcal{O}_k}\big\|_{V_g^*(\mathcal{O}_k)}^2 \\
&\leq \frac{\nu}{2}\|v_k\|_{V_g}^2 + \frac{2M_0^2}{\nu}\big\|f(t)\big\|_{V_g^*(\mathcal{O}_\infty)}^2.
\end{aligned}
\tag{47}
$$

By Assumption G and Lemma 2.1, we have

$$
-2\nu\langle R_{g,k}v_k, gv_k\rangle \leq \frac{2\nu\|\nabla g\|_\infty}{m_0}\|v_k\|_{V_g}\|v_k\|_g \leq \frac{2\nu\|\nabla g\|_\infty}{m_0\lambda_0^{1/2}}\|v_k\|_{V_g}^2.
\tag{48}
$$

By Assumption H and Lemma 2.3,

$$
\begin{aligned}
2\varepsilon z(\theta_t\omega)(h_k, gv_k) &\leq \frac{\lambda}{4}\|v_k\|_g^2 + c\big|z(\theta_t\omega)\big|^2\|h_k\|_{H_g(\mathcal{O}_k)}^2 \\
&\leq \frac{\lambda}{4}\|v_k\|_g^2 + c\big|z(\theta_t\omega)\big|^2,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
-2\varepsilon\nu z(\theta_t\omega)\langle A_{g,k}h_k, gv_k\rangle &\leq 2\nu z(\theta_t\omega)\frac{\|\nabla g\|_\infty}{m_0}\|\nabla h\|_{\mathbb{L}^\infty}\|v_k\|_g \\
&\leq \frac{\lambda}{4}\|v_k\|_g^2 + c\big|z(\theta_t\omega)\big|^2,
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
-2\varepsilon\nu z(\theta_t\omega)\langle R_{g,k}h_k, gv_k\rangle &\leq 2\nu z(\theta_t\omega)\frac{\|\nabla g\|_\infty}{m_0}\|\nabla h\|_{\mathbb{L}^\infty}\|v_k\|_g \\
&\leq \frac{\lambda}{4}\|v_k\|_g^2 + c\big|z(\theta_t\omega)\big|^2.
\end{aligned}
\tag{51}
$$

Using (10), (12), and Assumption H, we have

$$
\begin{aligned}
&2\big|\langle B_{g,k}\big(v_k + \varepsilon h_k z(\theta_t\omega), v_k + \varepsilon h_k z(\theta_t\omega)\big), v_k\rangle_g\big| \\
&= 2\big|b_{g,k}\big(v_k + \varepsilon h_k z(\theta_t\omega), \varepsilon h_k z(\theta_t\omega), v_k\big)\big| \\
&\leq 2M_0\varepsilon\|\nabla h\|_{\mathbb{L}^\infty}\big|z(\theta_t\omega)\big|\|v_k\|_g^2 + 2M_0\varepsilon^2\|h\|\|\nabla h\|_{\mathbb{L}^\infty}\big|z(\theta_t\omega)\big|^2\|v_k\|_g \\
&\leq c_0\varepsilon\big(\big|z(\theta_t\omega)\big|\|v_k\|_g^2 + \big|z(\theta_t\omega)\big|^2\|v_k\|_g\big) \\
&\leq c_0\varepsilon\big|z(\theta_t\omega)\big|\|v_k\|_g^2 + \frac{\lambda}{4}\|v_k\|_g^2 + c\big|z(\theta_t\omega)\big|^4,
\end{aligned}
\tag{52}
$$

where $c_0 = 2M_0(\|h\| + 1)\|\nabla h\|_{\mathbb{L}^\infty} > 0$. It follows from (46) to (52) and the Poincaré inequality in (9)

$$
2\nu\|v_k\|_{V_g}^2 \geq \lambda_0\nu\|v_k\|_g^2 + \nu\|v_k\|_{V_g}^2 = 3\lambda\|v_k\|_g^2 + \nu\|v_k\|_{V_g}^2.
$$

We obtain that

$$
\begin{aligned}
&\frac{d}{ds}\|v_k\|_g^2 + \big(2\lambda - c_0\varepsilon\big|z(\theta_s\omega)\big|\big)\|v_k\|_g^2 + C_g\|v_k\|_{V_g}^2 \\
&\leq c\big\|f(s)\big\|_{V_g^*(\mathcal{O}_\infty)}^2 + c\big(\big|z(\theta_s\omega)\big|^4 + \big|z(\theta_s\omega)\big|^2\big),
\end{aligned}
\tag{53}
$$

where $C_g := \frac{\nu}{2}\big(1 - \frac{4\nu\|\nabla g\|_\infty}{m_0\lambda_0^{1/2}}\big) > 0$ in view of Assumption G.

Multiplying (53) by $e^{\int_s^\tau -2\lambda+c_0\varepsilon|z(\theta_r\omega)|\,dr}$, then integrating the result w.r.t. $s \in (\tau - t, \sigma)$ and replacing ω by $\theta_{-\tau}\omega$, we find that

$$
\begin{aligned}
&\left\| v_k(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t,k}) \right\|_g^2 + C_g \int_{\tau-t}^\sigma e^{\int_s^\sigma -2\lambda+c_0\varepsilon|z(\theta_{r-\tau}\omega)|\,dr} \left\| v_k(s) \right\|_{V_g}^2 ds \\
&\quad \leq e^{2\lambda(\tau-\sigma)} e^{-2\lambda t + c_0\varepsilon \int_{-t}^{\sigma-\tau} |z(\theta_r\omega)|\,dr} \left\| v_{\tau-t,k} \right\|_{H_g(\mathcal{O}_k)}^2 \\
&\qquad + c e^{2\lambda(\tau-\sigma)} \int_{-\infty}^{\sigma-\tau} e^{2\lambda s + c_0\varepsilon \int_s^0 |z(\theta_r\omega)|\,dr} \left\| f(s+\tau) \right\|_{V_g^*(\mathcal{O}_\infty)}^2 ds \\
&\qquad + c e^{2\lambda(\tau-\sigma)} \int_{-\infty}^{\sigma-\tau} e^{2\lambda s + c_0\varepsilon \int_s^0 |z(\theta_r\omega)|\,dr} \left(|z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^4 \right) ds \\
&\quad \leq e^{2\lambda(\tau-\sigma)} \left(e^{-2\lambda t + c_0\varepsilon \int_{-t}^{\sigma-\tau} |z(\theta_r\omega)|\,dr} \left\| v_{\tau-t,k} \right\|_{H_g(\mathcal{O}_k)}^2 + \rho(\sigma, \tau, \omega) \right).
\end{aligned}
\tag{54}
$$

Using (43), (23), and $\mathbb{P}$-a.s. $\rho(\sigma, \tau, \omega) > 0$, there is $T > 0$ such that, for all $t \geq T$ and $\sigma \in [\tau - t, \tau]$,

$$
\begin{aligned}
&e^{-2\lambda t + c_0\varepsilon \int_{-t}^{\sigma-\tau} |z(\theta_r\omega)|\,dr} \left\| v_{\tau-t,k} \right\|_{H_g(\mathcal{O}_k)}^2 \\
&\quad \leq e^{-2\lambda t + c_0\varepsilon \int_{-t}^0 |z(\theta_r\omega)|\,dr} \left\| D(\tau - t, \theta_{-t}\omega) \right\|_{H_g(\mathcal{O}_\infty)}^2 \\
&\quad \leq e^{C_0(\omega)} e^{-\frac{3}{2}\lambda t} \left\| D(\tau - t, \theta_{-t}\omega) \right\|_{H_g(\mathcal{O}_\infty)}^2 \leq \rho(\sigma, \tau, \omega).
\end{aligned}
\tag{55}
$$

We substitute (55) into (54) to obtain (45) as desired. In addition, by (43),

$$
\begin{aligned}
\rho(\sigma, \tau, \omega) &\leq c \int_{-\infty}^0 e^{2\lambda s + c_0\varepsilon \int_s^0 |z(\theta_r\omega)|\,dr} \left(\left\| f(s+\tau) \right\|_{V_g^*(\mathcal{O}_\infty)}^2 + |z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^4 \right) ds \\
&\leq c e^{C_0(\omega)} \int_{-\infty}^0 e^{\frac{3}{2}\lambda s} \left(\left\| f(s+\tau) \right\|_{V_g^*(\mathcal{O}_\infty)}^2 + |z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^4 \right) ds,
\end{aligned}
$$

then both (16) and Assumption F imply $\rho(\sigma, \tau, \omega) < +\infty$. □

4.2 Weak equi-continuity

In this subsection, we show the weak equi-continuity of the solution sequence $\{v_k\}$. This is different from the weak continuity for a single system as given by Rosa [19].

Lemma 4.3 *Suppose $f \in L_{\text{loc}}^2(\mathbb{R}, V_g^*(\mathcal{O}_\infty))$. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_{\tau,k} \in H_g(\mathcal{O}_k)$ such that the expansion $\widetilde{v_{\tau,k}} \rightharpoonup v_{\tau,\infty}$ weakly in $H_g(\mathcal{O}_\infty)$ as $k \to \infty$. Then the sequence $\widetilde{v}_k$ of expanding solutions satisfies*

$$
\widetilde{v}_k(t, \tau, \omega, v_{\tau,k}) \rightharpoonup v_\infty(t, \tau, \omega, v_{\tau,\infty}) \quad \text{weakly in } H_g(\mathcal{O}_\infty),
\tag{56}
$$

$$
\widetilde{v}_k(\cdot, \tau, \omega, v_{\tau,k}) \rightharpoonup v_\infty(\cdot, \tau, \omega, v_{\tau,\infty}) \quad \text{weakly in } L^2(\tau, \tau + T; V_g(\mathcal{O}_\infty)), \forall T > 0.
\tag{57}
$$

Proof From Lemmas 4.2 and 2.3, we can prove that, for all $T > 0$,

$$
\{\widetilde{v}_k\}_k \text{ is bounded in } L^\infty(\tau, \tau + T; H_g(\mathcal{O}_\infty)) \cap L^2(\tau, \tau + T; V_g(\mathcal{O}_\infty)).
\tag{58}
$$

We rewrite (18) as follows:

$$\frac{dv_k}{dt} = -\nu A_{g,k}v_k - \varepsilon\nu z(\theta_t\omega)A_{g,k}h_k - B_{g,k}\big(v_k + \varepsilon h_k z(\theta_t\omega)\big)$$

$$- \nu R_{g,k}v_k - \varepsilon\nu z(\theta_t\omega)R_{g,k}h_k + \varepsilon z(\theta_t\omega)h_k + P_{g,k}f(t)|_{\mathcal{O}_k},$$

where $h_k = P_{g,k}h|_{\mathcal{O}_k}$. Since the norms of the operators $A_{g,k}$, $B_{g,k}$ are bounded in k and $R_{g,k}$ satisfies (9), respectively, it follows from Lemma 2.3 and Assumption H that

$$\left\|\frac{dv_k}{dt}\right\|_{V_g^*(\mathcal{O}_k)} \leq c\big(1 + \big|z(\theta_t\omega)\big|^2 + \big|z(\theta_t\omega)\big|^4\big)\big(\|\widetilde{v_k}\|_{V_g(\mathcal{O}_\infty)}^2 + \|f(t)\|_{V_g^*(\mathcal{O}_\infty)}^2\big).$$

This together with $\|\widetilde{v_k}'\|_{V_g^*(\mathcal{O}_\infty)} = \|v_k'\|_{V_g^*(\mathcal{O}_k)}$ and (58) implies that

$$\big\{\widetilde{v_k}'\big\}_k \text{ is bounded in } L^2\big(\tau, \tau + T, V_g^*(\mathcal{O}_\infty)\big).$$

For each $i \in \mathbb{N}$, according to Aubin's compactness theorem [2] and the compactness of $V_g(\mathcal{O}_i) \hookrightarrow \mathbb{L}^2(\mathcal{O}_i)$, we obtain that

$$\{\widetilde{v_k}|_{\mathcal{O}_i}\}_k \text{ is relatively compact in } L^2\big(\tau, \tau + T; \mathbb{L}^2(\mathcal{O}_i)\big), \quad \forall T > 0. \tag{59}$$

From (58) and (59), by a diagonal process, we can extract an index subsequence k^* of k such that

$$\widetilde{v_{k^*}} \to v \quad \text{weak-star in } L^\infty\big(\tau, \tau + T; H_g(\mathcal{O}_\infty)\big)$$

$$\text{weakly in } L^2\big(\tau, \tau + T; V_g(\mathcal{O}_\infty)\big)$$

$$\text{strongly in } L^2\big(\tau, \tau + T; \mathbb{L}^2(\mathcal{O}_i)\big), \quad \forall T > 0, \forall i \in \mathbb{N}, \tag{60}$$

for some

$$v \in L^\infty\big(\tau, \tau + T; H_g(\mathcal{O}_\infty)\big) \cap L^2\big(\tau, \tau + T; V_g(\mathcal{O}_\infty)\big), \quad \forall T > 0.$$

We now show that v is a weak solution of Eq. (18) at $k = \infty$. Indeed, by the weak formulation (20), the expansion $\widetilde{v_{k^*}}$ satisfies that, for each $w \in V_g(\mathcal{O}_\infty)$ and $t > \tau$,

$$\big(\widetilde{v_{k^*}}(t), w\big)_g = \big(\widetilde{v_{k^*}}(\tau), w\big)_g + \int_\tau^t \big\langle\widetilde{f(s)|_{\mathcal{O}_{k^*}}}, gw\big\rangle ds + \varepsilon\int_\tau^t z(\theta_s\omega)(h, g\widetilde{v_{k^*}})\, ds$$

$$- \nu\int_\tau^t \langle A_{g,\infty}\widetilde{v_{k^*}}, w\rangle_g\, ds - \varepsilon\nu\int_\tau^t z(\theta_s\omega)\langle A_{g,\infty}h, w\rangle_g\, ds$$

$$- \int_\tau^t \big\langle B_{g,\infty}\big(\widetilde{v_{k^*}} + \varepsilon hz(\theta_s\omega)\big), gw\big\rangle ds - \nu\int_\tau^t \langle R_{g,\infty}\widetilde{v_{k^*}}, gw\rangle\, ds$$

$$- \varepsilon\nu\int_\tau^t z(\theta_s\omega)\langle R_{g,\infty}h, gw\rangle\, ds, \tag{61}$$

and we only need to consider the convergence of the forcing term f as $k^* \to \infty$. Since $f \in L^2(\tau, t; V_g^*(\mathcal{O}_\infty))$ and $w \in V_g(\mathcal{O}_\infty)$, we have

$$
\begin{aligned}
\int_\tau^t \langle f(s) - \widetilde{f(s)|_{\mathcal{O}_{k^*}}}, gw \rangle \, ds &= \int_\tau^t \int_{\mathcal{O}_\infty \backslash \mathcal{O}_{k^*}} f(s,x) \cdot wg \, dx \, ds \\
&\leq M_0 \int_\tau^t \|f(s)\|_{V_g^*(\mathcal{O}_\infty)} \, ds \left(\int_{\mathcal{O}_\infty \backslash \mathcal{O}_{k^*}} |Dw(x)|^2 g \, dx \right)^{\frac{1}{2}} \\
&\leq c M_0 \left(\int_\tau^t \|f(s)\|_{V_g^*(\mathcal{O}_\infty)}^2 \, ds \right)^{\frac{1}{2}} \left(\int_{\mathcal{O}_\infty \backslash \mathcal{O}_{k^*}} |Dw(x)|^2 g \, dx \right)^{\frac{1}{2}} \\
&\leq c \left(\int_{\mathcal{O}_\infty \backslash \mathcal{O}_{k^*}} |Dw(x)|^2 g \, dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } k^* \to \infty.
\end{aligned}
$$

By taking the limit of (61) as $k^* \to \infty$ and noticing that $A_{g,\infty}, B_{g,\infty}, R_{g,\infty}$ are continuous operators, we see that v is a weak solution of Eq. (18) with $k = \infty$. By the uniqueness of the solutions, we have $v(t) = v_\infty(t, \tau, \omega, v_{\tau,\infty})$. Using a standard contradiction argument, we can show that the whole sequence $\{\widetilde{v}_k\}_k$ converges to v_∞ in the sense of (60). This proves (57).

In addition, from the strong convergence in (60), we also have

$$
\left(\widetilde{v}_k(t), \xi \right)_g \to \left(v_\infty(t), \xi \right)_g \quad \text{for a.e. } t \in [\tau, +\infty) \text{ and any } \xi \in \mathcal{V}(\mathcal{O}_\infty). \tag{62}
$$

On the other hand, for all $\xi \in \mathcal{V}(\mathcal{O}_\infty)$, $\tau \leq t \leq t + a \leq \tau + T$ with $T > 0$,

$$
\begin{aligned}
\left| \left(\widetilde{v}_k(t+a) - \widetilde{v}_k(t), \xi \right)_g \right| &= \int_t^{t+a} \langle \widetilde{v}_k'(s), \xi \rangle_g \, ds = \int_t^{t+a} \int_{\mathcal{O}_\infty} \widetilde{v}_k'(s) \xi g \, dx \, ds \\
&\leq \left(\int_t^{t+a} 1 \, ds \right)^{\frac{1}{2}} \left(\int_t^{t+a} \left(\int_{\mathcal{O}_\infty} \widetilde{v}_k'(s) \xi g \, dx \right)^2 ds \right)^{\frac{1}{2}} \\
&\leq a^{\frac{1}{2}} M_0^{\frac{1}{2}} \left(\int_{\mathcal{O}_\infty} \xi^2 g \, dx \right)^{\frac{1}{2}} \left(\int_t^{t+a} \int_{\mathcal{O}_\infty} \widetilde{v}_k'^2(s) \, dx \, ds \right)^{\frac{1}{2}} \\
&= a^{\frac{1}{2}} M_0^{\frac{1}{2}} \|\xi\|_{V_g(\mathcal{O}_\infty)} \|\widetilde{v}_k'\|_{L^2(\tau, \tau+T; V_g^*(\mathcal{O}_\infty))} \\
&\leq C_T a^{\frac{1}{2}} \|\xi\|_{V_g(\mathcal{O}_\infty)}, \tag{63}
\end{aligned}
$$

where C_T is positive and independent of k. Thus, it follows from (58) and (63), we see that $\{(\widetilde{v}_k(t), \chi)\}$ is equi-bounded and equi-continuous on $[\tau, \tau + T]$ for all $T > 0$. This together with (62) yields

$$
\left(\widetilde{v}_k(t), \xi \right) \to \left(v_\infty(t), \xi \right), \quad \forall t \in [\tau, +\infty), \forall \xi \in \mathcal{V}(\mathcal{O}_\infty). \tag{64}
$$

According to the density of $\mathcal{V}(\mathcal{O}_\infty)$ in $H_g(\mathcal{O}_\infty)$, we show (56). □

By the similar (more simple) method as given in Lemma 4.3, we have weak continuity of each v_k.

Corollary 1 *For each fixed $k \in \overline{\mathbb{N}}$, if $v_{\tau,n} \rightharpoonup v_\tau$ weakly in $H_g(\mathcal{O}_k)$, then, as $n \to \infty$,*

$$v_k(t, \tau, \omega, v_{\tau,n}) \rightharpoonup v_k(t, \tau, \omega, v_\tau) \quad \text{weakly in } H_g(\mathcal{O}_k),$$

$$v_k(\cdot, \tau, \omega, v_{\tau,n}) \rightharpoonup v_k(\cdot, \tau, \omega, v_\tau) \quad \text{weakly in } L^2(\tau, \tau + T; V_g(\mathcal{O}_k)) \ \forall T > 0.$$

5 Equi-asymptotic compactness

In this section, we establish an energy equation (as (75) below) for the expanding solution $\widetilde{v}_k$ and then we use this energy equation to verify the equi-asymptotic compactness of the sequence $\{\widetilde{\Phi}_k\}_k$ of expanding cocycles.

Theorem 5.1 *Let Assumptions F, H, G hold true. Then, for any initial data $v_{0,k} \in D|_{\mathcal{O}_k}(\tau - t_k, \theta_{-t_k}\omega)$ with $t_k \to +\infty$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D \in \mathfrak{D}$, the sequence of expanded solutions*

$$\left\{ \widetilde{\Phi}_k(t_k, \tau - t_k, \theta_{-t_k}\omega) \widetilde{v_{0,k}} \right\} = \left\{ \widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\}$$

for problem (18) has a convergent subsequence in $H_g(\mathcal{O}_\infty)$.

Proof By taking $\sigma = \tau$ in (45), there is $k_0 \in \mathbb{N}$ such that

$$\sup_{k \geq k_0} \left\| v_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\|_{H_g(\mathcal{O}_k)}^2 \leq \rho(\tau, \tau, \omega) < +\infty, \tag{65}$$

and thus, by Lemma 2.3(1), the sequence of expanding solutions

$$\left\{ \widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\}_k \text{ is bounded in } H_g(\mathcal{O}_\infty). \tag{66}$$

Passing to subsequence, there is $v \in H_g(\mathcal{O}_\infty)$ such that

$$\widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \rightharpoonup v \quad \text{weakly in } H_g(\mathcal{O}_\infty). \tag{67}$$

By the resonance theorem,

$$\liminf_{k \to \infty} \left\| \widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\|_{H_g(\mathcal{O}_\infty)} \geq \|v\|_{H_g(\mathcal{O}_\infty)}. \tag{68}$$

Hence, in order to prove the weak convergence in (67) is strongly convergent, it suffices to prove that, for a subsequence,

$$\limsup_{k \to \infty} \left\| \widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\|_{H_g(\mathcal{O}_\infty)} \leq \|v\|_{H_g(\mathcal{O}_\infty)}. \tag{69}$$

By the cocycle property, we have, for each $m, k \in \mathbb{N}$,

$$\widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) = \widetilde{v}_k\left(\tau, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\right). \tag{70}$$

For each $m \in \mathbb{N}$, by taking $\sigma = \tau - m$ in (45), there is $K_m \in \mathbb{N}$ such that

$$\sup_{k \geq K_m} \left\| \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\|_{H_g(\mathcal{O}_\infty)}^2 \leq e^{2\lambda m} \rho(\tau - m, \tau, \omega) < +\infty,$$

which means that the sequence $\{\widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\}_k$ is bounded in $H_g(\mathcal{O}_\infty)$ and thus there are $v^m \in H_g(\mathcal{O}_\infty)$ and an index subsequence $\{k(m)\}$ of $\{k(m-1)\}$ such that

$$\widetilde{v_{k(m)}}(\tau - m, \tau - t_{k(m)}, \theta_{-\tau}\omega, v_{0,k(m)}) \rightharpoonup v^m \quad \text{weakly in } H_g(\mathcal{O}_\infty).$$

We still use $\{k\}$ to denote the index diagonal subsequence $\{k(k)\}$. Then, as $k \to \infty$,

$$\widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \rightharpoonup v^m \quad \text{weakly in } H_g(\mathcal{O}_\infty) \text{ for all } m \in \mathbb{N}. \tag{71}$$

By (70), (71), and (56), we have

$$\widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \rightharpoonup v_\infty(\tau, \tau - m, \theta_{-\tau}\omega, v^m) \quad \text{in } H_g(\mathcal{O}_\infty). \tag{72}$$

From (67), (72), and the uniqueness of weak limit, we have

$$v_\infty(\tau, \tau - m, \theta_{-\tau}\omega, v^m) = v. \tag{73}$$

Now, we infer from (46) an energy equation $\mathcal{O}_k$ for all $k \in \overline{\mathbb{N}}$:

$$\frac{d}{dt}\|v_k\|_g^2 + 2\lambda\|v_k\|_g^2 + \Psi(v_k) + 2\varepsilon v z(\theta_t\omega)\langle A_{g,k}h_k, gv_k\rangle$$
$$+ 2\varepsilon v z(\theta_t\omega)\langle R_{g,k}h_k, gv_k\rangle + b_{g,k}(v_k + \varepsilon h_k z(\theta_t\omega), v_k + \varepsilon h_k z(\theta_t\omega), v_k)$$
$$= 2\langle f(t)|_{\mathcal{O}_k}, gv_k\rangle + 2\varepsilon z(\theta_t\omega)(h_k, gv_k), \tag{74}$$

where $\Psi(v_k) = 2v\|v_k\|_{V_g(\mathcal{O}_k)}^2 + 2v\langle R_{g,k}v_k, gv_k\rangle - 2\lambda\|v_k\|_g^2$.

Since the null-expansion does not change the norm, we have

$$\Psi(v_k) = \Psi(\widetilde{v}_k), \quad \text{and} \quad \langle f(t)|_{\mathcal{O}_k}, gv_k\rangle = \langle f(t), g\widetilde{v}_k\rangle.$$

Hence, we can rewrite (74) on $\mathcal{O}_\infty$ as follows:

$$\frac{d}{dt}\|\widetilde{v}_k\|_g^2 + 2\lambda\|\widetilde{v}_k\|_g^2 + \Psi(\widetilde{v}_k) + 2\varepsilon v z(\theta_t\omega)\langle \widetilde{A_{g,k}h_k}, g\widetilde{v}_k\rangle + 2\varepsilon v z(\theta_t\omega)\langle \widetilde{R_{g,k}h_k}, g\widetilde{v}_k\rangle$$
$$+ b_{g,\infty}(\widetilde{v}_k + \varepsilon h z(\theta_t\omega), \widetilde{v}_k + \varepsilon h z(\theta_t\omega), \widetilde{v}_k) = 2\langle f(t), g\widetilde{v}_k\rangle + 2\varepsilon z(\theta_t\omega)(h, g\widetilde{v}_k) \tag{75}$$

for all $k \in \overline{\mathbb{N}}$, where we regard $\widetilde{v}_\infty$ as v_∞. Multiplying (75) by $e^{2\lambda t}$ and then integrating the result over $[s, \tau]$, we obtain

$$\left\|\widetilde{v}_k(\tau, s, \omega, v_{s,k})\right\|_g^2 = e^{2\lambda(s-\tau)}\|\widetilde{v_{s,k}}\|_g^2 - \int_s^\tau e^{\lambda(r-\tau)}\Psi(\widetilde{v}_k(r, s, \omega, v_{s,k}))\, dr$$
$$- 2v\varepsilon \int_s^\tau e^{2\lambda(r-\tau)}z(\theta_r\omega)\langle \widetilde{A_{g,k}h_k}, g\widetilde{v}_k(r, s, \omega, v_{s,k})\rangle\, dr$$
$$- 2v\varepsilon \int_s^\tau e^{2\lambda(r-\tau)}z(\theta_r\omega)\langle \widetilde{R_{g,k}h_k}, g\widetilde{v}_k(r, s, \omega, v_{s,k})\rangle\, dr$$
$$- 2\int_s^\tau e^{2\lambda(r-\tau)}b_{g,\infty}(\widetilde{v}_k(r, s, \omega, v_{s,k}) + \varepsilon h z(\theta_r\omega), \widetilde{v}_k(r, s, \omega, v_{s,k})$$

$$+ \varepsilon h z(\theta_r \omega), \widetilde{v}_k(r, s, \omega, v_{s,k})) \, dr$$

$$+ 2 \int_s^\tau e^{2\lambda(r-\tau)} \langle f(r), g\widetilde{v}_k(r, s, \omega, v_{s,k}) \rangle \, dr$$

$$+ 2\varepsilon \int_s^\tau e^{2\lambda(r-\tau)} z(\theta_r \omega) \big(h, g\widetilde{v}_k(r, s, \omega, v_{s,k}) \big) \, dr. \tag{76}$$

Let $s = \tau - m$ and $\widetilde{v_{s,k}} = \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})$ in (76), we infer from (70) that, for all $k \in \mathbb{N}$,

$$\big\| \widetilde{v}_k(\tau, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \big\|_g^2$$

$$= \big\| \widetilde{v}_k\big(\tau, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\big) \big\|_g^2$$

$$= e^{-2\lambda m} \big\| \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \big\|_g^2$$

$$- \int_{\tau-m}^\tau e^{2\lambda(r-\tau)} \Psi \big(\widetilde{v}_k\big(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\big) \big) \, dr$$

$$- 2\nu\varepsilon \int_{\tau-m}^\tau e^{2\lambda(r-\tau)}$$

$$\times z(\theta_{r-\tau}\omega) \langle \widetilde{A_{g,k} h_k}, g\widetilde{v}_k\big(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\big) \rangle \, dr$$

$$- 2\nu\varepsilon \int_{\tau-m}^\tau e^{2\lambda(r-\tau)}$$

$$\times z(\theta_{r-\tau}\omega) \langle \widetilde{R_{g,k} h_k}, g\widetilde{v}_k\big(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\big) \rangle \, dr$$

$$- 2 \int_{\tau-m}^\tau e^{2\lambda(r-\tau)} b_{g,k}\big(\widetilde{u}_k(r, \tau - m), \widetilde{u}_k(r, \tau - m), \widetilde{u}_k(r, \tau - m) - \varepsilon h z(\theta_{r-\tau}\omega) \big) \, dr$$

$$+ 2 \int_{\tau-m}^\tau e^{2\lambda(r-\tau)} \langle f(r), g\widetilde{v}_k\big(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\big) \rangle \, dr$$

$$+ 2\varepsilon \int_{\tau-m}^\tau e^{2\lambda(r-\tau)}$$

$$\times z(\theta_{r-\tau}\omega) \big(h, g\widetilde{v}_k\big(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k})\big) \big) \, dr. \tag{77}$$

We then consider the limits of the right-hand side of (77) as $k \to \infty$ one by one. For the first term, by (45) with $\sigma = \tau - m$ and $t = t_k$, we see that

$$e^{-2\lambda m} \big\| \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \big\|_{H_g(\mathcal{O}_\infty)}^2$$

$$\leq e^{-\frac{3}{2}\lambda t_k} \|\widetilde{v_{0,k}}\|_{H_g(\mathcal{O}_\infty)}^2 + c \int_{-\infty}^{-m} e^{2\lambda s + c_0\varepsilon \int_s^0 |z(\theta_r\omega)| \, dr} \big\| f(s + \tau) \big\|_{V_g^*(\mathcal{O})}^2 \, ds$$

$$+ c \int_{-\infty}^{-m} e^{2\lambda s + c_0\varepsilon \int_s^0 |z(\theta_r\omega)| \, dr} \big(\big| z(\theta_s\omega) \big|^2 + \big| z(\theta_s\omega) \big|^4 \big) \, ds.$$

Since $v_{0,k} \in D|_{\mathcal{O}_k}(\tau - t_k, \theta_{-t_k}\omega)$, we have

$$e^{-\frac{3}{2}\lambda t_k} \|\widetilde{v_{0,k}}\|_{H_g(\mathcal{O}_\infty)}^2 = e^{-\frac{3}{2}\lambda t_k} \|v_{0,k}\|_{H_g(\mathcal{O}_k)}^2 \leq e^{-\frac{3}{2}\lambda t_k} \big\| D|_{\mathcal{O}_k}(\tau - t_k, \theta_{-t_k}\omega) \big\|_{H_g(\mathcal{O}_k)}^2$$

$$\leq e^{-\frac{3}{2}\lambda t_k} \big\| D(\tau - t_k, \theta_{-t_k}\omega) \big\|_{H_g(\mathcal{O}_\infty)}^2 \to 0 \quad \text{as } k \to \infty.$$

Hence, we have the following estimate of the limit:

$$
\limsup_{k \to \infty} e^{-2\lambda m} \left\| \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right\|^2
$$

$$
\leq c \int_{-\infty}^{-m} e^{2\lambda s + c_0 \varepsilon \int_s^0 |z(\theta_r \omega)| \, dr} \left\| f(s + \tau) \right\|_{V_g^*(\mathcal{O})}^2 \, ds
$$

$$
+ c \int_{-\infty}^{-m} e^{2\lambda s + c_0 \varepsilon \int_s^0 |z(\theta_r \omega)| \, dr} \left(|z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^4 \right) ds. \tag{78}
$$

For the second term, we claim that $\Psi(v_k) = 2\nu \|v_k\|_{V_g(\mathcal{O}_k)}^2 + 2\nu \langle R_{g,k} v_k, g v_k \rangle - 2\lambda \|v_k\|_g^2$ defines a norm which is equivalent to the norm in $V_g(\mathcal{O}_k)$. Indeed, by Lemma 2.1 and Assumption G, we see that

$$
\Psi(v_k) \leq 2\nu \|v_k\|_{V_g(\mathcal{O}_k)}^2 + \frac{2\nu \|\nabla g\|_\infty}{m_0 \lambda_0^{1/2}} \|v_k\|_{V_g(\mathcal{O}_k)}^2 \leq 2\nu \|v_k\|_{V_g(\mathcal{O}_k)}^2 + \frac{\nu}{2} \|v_k\|_{V_g(\mathcal{O}_k)}^2
$$

$$
= \frac{5\nu}{2} \|v_k\|_{V_g(\mathcal{O}_k)}^2.
$$

On the other hand, by the uniform Poincaré inequality and $\lambda = \frac{1}{3}\lambda_0 \nu$,

$$
\Psi(v_k) \geq 2\nu \|v\|_{V_g(\mathcal{O}_k)}^2 - \frac{\nu}{2} \|v_k\|_{V_g(\mathcal{O}_k)}^2 - 2\lambda \frac{1}{\lambda_0} \|v_k\|_{V_g(\mathcal{O}_k)}^2 = \frac{5\nu}{6} \|v_k\|_{V_g(\mathcal{O}_k)}^2.
$$

Thus, by the Fatou lemma and weak equi-continuity (57), we obtain

$$
\liminf_{k \to \infty} \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \Psi\left(\widetilde{v}_k\left(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right) \right) dr
$$

$$
\geq \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \liminf_{k \to \infty} \Psi\left(\widetilde{v}_k\left(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right) \right) dr
$$

$$
\geq \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \Psi\left(v_\infty\left(r, \tau - m, \theta_{-\tau}\omega, v^m \right) \right) dr,
$$

which implies

$$
\limsup_{k \to \infty} - \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \Psi\left(\widetilde{v}_k\left(r, \tau - m, \theta_{-\tau}\omega, \widetilde{v}_k(\tau - m, \tau - t_k, \theta_{-\tau}\omega, v_{0,k}) \right) \right) dr
$$

$$
\leq - \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \Psi\left(v_\infty\left(r, \tau - m, \theta_{-\tau}\omega, v^m \right) \right) dr. \tag{79}
$$

Similarly, from Lemma 4.3 and [5], we get

$$
\limsup_{k \to \infty} -2 \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} b_{g,k}\left(\widetilde{u}_k(r, \tau - m), \widetilde{u}_k(r, \tau - m), \widetilde{u}_k(r, \tau - m) - \varepsilon h z(\theta_{r-\tau}\omega) \right) dr
$$

$$
\leq -2 \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} b_{g,\infty}\left(u_\infty(r, \tau - m), u_\infty(r, \tau - m), v_\infty\left(r, \tau - m, \theta_{-\tau}\omega, v^m \right) \right) dr. \tag{80}
$$

From (50) and the weak equi-continuity (57), we have

$$
\begin{aligned}
\limsup_{k\to\infty} &-\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \\
&\times z(\theta_{r-\tau}\omega)\langle \widetilde{A_{g,k}h_k}, g\widetilde{v}_k\big(r,\tau-m,\theta_{-\tau}\omega,\widetilde{v}_k(\tau-m,\tau-t_k,\theta_{-\tau}\omega,v_{0,k})\big)\rangle dr \\
&\le -\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} z(\theta_{r-\tau}\omega)\langle A_{g,\infty}h, gv_{\infty}\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\rangle dr.
\end{aligned} \tag{81}
$$

By (51) and the weak equi-continuity (57), we get

$$
\begin{aligned}
\limsup_{k\to\infty} &-\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} \\
&\times z(\theta_{r-\tau}\omega)\langle \widetilde{R_{g,k}h_k}, g\widetilde{v}_k\big(r,\tau-m,\theta_{-\tau}\omega,\widetilde{v}_k(\tau-m,\tau-t_k,\theta_{-\tau}\omega,v_{0,k})\big)\rangle dr \\
&\le -\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} z(\theta_{r-\tau}\omega)\langle R_{g,\infty}h, gv_{\infty}\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\rangle dr.
\end{aligned} \tag{82}
$$

By the weak equi-continuity (57) again, it follows from (71), on $L^2(\tau-m,\tau;V_g(\mathcal{O}_{\infty}))$, that

$$
\widetilde{v}_k\big(\cdot,\tau-m,\theta_{-\tau}\omega,\widetilde{v}_k(\tau-m,\tau-t_k,\theta_{-\tau}\omega,v_{0,k})\big) \rightharpoonup v_{\infty}\big(\cdot,\tau-m,\theta_{-\tau}\omega,v^m\big).
$$

By $f \in L^2(\tau-m,\tau;V_g^*(\mathcal{O}_{\infty}))$ and Assumption H, we know

$$
r \mapsto e^{2\lambda(r-\tau)}\big(f(r)+\varepsilon z(\theta_{r-\tau}\omega)h\big)g \in L^2\big(\tau-m,\tau;V_g^*(\mathcal{O}_{\infty})\big).
$$

Thereby, we have

$$
\begin{aligned}
\lim_{k\to\infty} 2\int_{\tau-m}^{\tau} &e^{2\lambda(r-\tau)}\langle f(r)+\varepsilon z(\theta_{r-\tau}\omega)h, \\
&g\widetilde{v}_k\big(r,\tau-m,\theta_{-\tau}\omega,\widetilde{v}_k(\tau-m,\tau-t_k,\theta_{-\tau}\omega,v_{0,k})\big)\rangle dr \\
&= 2\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)}\langle f(r)+\varepsilon z(\theta_{r-\tau}\omega)h, gv_{\infty}\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\rangle dr.
\end{aligned} \tag{83}
$$

Taking the sup-limit of (77) as $k\to\infty$, from (78) to (83), we obtain

$$
\begin{aligned}
\limsup_{k\to\infty} &\big\|\widetilde{v}_k(\tau,\tau-t_k,\theta_{-\tau}\omega,v_{0,k})\big\|^2 \\
&\le c\int_{-\infty}^{-m} e^{2\lambda s+c_0\varepsilon\int_s^0 |z(\theta_r\omega)|\,dr}\big\|f(s+\tau)\big\|^2_{V_g^*(\mathcal{O})}\,ds \\
&\quad + c\int_{-\infty}^{-m} e^{2\lambda s+c_0\varepsilon\int_s^0 |z(\theta_r\omega)|\,dr}\big(|z(\theta_s\omega)|^2+|z(\theta_s\omega)|^4\big)\,ds \\
&\quad - \int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)}\Psi\big(v_{\infty}\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\big)\,dr \\
&\quad - 2\nu\varepsilon\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} z(\theta_{r-\tau}\omega)\langle A_{g,\infty}h, gv_{\infty}\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\rangle dr \\
&\quad - 2\varepsilon\nu\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} z(\theta_{r-\tau}\omega)\langle R_{g,\infty}h, gv_{\infty}\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\rangle dr
\end{aligned}
$$

$$-2\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)} b_{g,\infty}\big(u_\infty(r,\tau-m), u_\infty(r,\tau-m), v_\infty\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\big)\,dr$$

$$+2\int_{\tau-m}^{\tau} e^{2\lambda(r-\tau)}\big\langle f(r)+\varepsilon z(\theta_{r-\tau}\omega)h, gv_\infty\big(r,\tau-m,\theta_{-\tau}\omega,v^m\big)\big\rangle\,dr. \tag{84}$$

We denote the sum of last five terms in (84) by $I(m)$, let $k=\infty$, $s=\tau-m$ in (76), and then we obtain

$$\|v\|^2_{H_g(\mathcal{O}_\infty)} = \big\|v_\infty\big(\tau,\tau-m,\theta_{-\tau}\omega,v^m\big)\big\|^2_{H_g(\mathcal{O}_\infty)} = e^{-2\lambda m}\big\|v^m\big\|^2_g + I(m). \tag{85}$$

Substituting (85) into (84) yields, for all $m \in \mathbb{N}$,

$$\limsup_{k\to\infty}\big\|\widetilde{v}_k(\tau,\tau-t_k,\theta_{-\tau}\omega,v_{0,k})\big\|^2_{H_g(\mathcal{O}_\infty)}$$

$$\leq \|v\|^2_{H_g(\mathcal{O}_\infty)} - e^{-2\lambda m}\big\|v^m\big\|^2_g + c\int_{-\infty}^{-m} e^{2\lambda s+c_0\varepsilon\int_s^0 |z(\theta_r\omega)|\,dr}\big\|f(s+\tau)\big\|^2_{V_g^*(\mathcal{O})}\,ds$$

$$+ c\int_{-\infty}^{-m} e^{2\lambda s+c_0\varepsilon\int_s^0 |z(\theta_r\omega)|\,dr}\big(\big|z(\theta_s\omega)\big|^2 + \big|z(\theta_s\omega)\big|^4\big)\,ds. \tag{86}$$

By Lemma 4.1 and Assumption F, as $m \to \infty$,

$$\int_{-\infty}^{-m} e^{2\lambda s+c_0\varepsilon\int_s^0 |z(\theta_r\omega)|\,dr}\big\|f(s+\tau)\big\|^2_{V_g^*(\mathcal{O})}\,ds$$

$$\leq e^{C_0(\omega)}\int_{-\infty}^{-m} e^{\frac{3}{2}\lambda s}\big\|f(s+\tau)\big\|^2_{V_g^*(\mathcal{O})}\,ds \to 0.$$

Therefore, letting $m \to \infty$ in (86), we obtain (69) as desired. $\qquad\square$

Corollary 2 *Let $k \in \overline{\mathbb{N}}$ be fixed. Then, for any $v_{0,n} \in D|_{\mathcal{O}_k}(\tau-t_n,\theta_{-t_n}\omega)$ with $t_n \to +\infty$, $D \in \mathfrak{D}$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, the sequence $\{v_k(\tau,\tau-t_n,\theta_{-\tau}\omega)v_{0,n}\}_n$ of solutions of (18) has a convergent subsequence in $H_g(\mathcal{O}_k)$.*

6 Final conclusion

In the last section, we deduce the existence and large-domain stability of the attractor when the domain changes from bounded to unbounded.

Theorem 6.1 *For each $k \in \overline{\mathbb{N}}$, let Φ_k be the cocycle associated with the g-NS equation (18) on $\mathcal{O}_k$, and let $\mathfrak{D}_k := \mathfrak{D}|_{\mathcal{O}_k}$ be the restriction of the universe $\mathfrak{D}$ in (22). Then Φ_k has a $\mathfrak{D}_k$-pullback random attractor $\mathcal{A}_k$ in $H_g(\mathcal{O}_k)$.*

Proof By taking $\sigma = \tau$ in (45), we find that Φ_k has an absorbing set $\mathcal{M}_k$ given by

$$\mathcal{M}_k(\tau,\omega) = \big\{u \in H_g(\mathcal{O}_k) : \|u\|_{H_g(\mathcal{O}_k)} \leq \rho(\tau,\omega)\big\}, \quad \forall k \in \overline{\mathbb{N}},$$

where $\rho(\tau,\omega) := \rho(\tau,\tau,\omega) = c(\rho_1(\tau,\omega)+\rho_2(\tau,\omega))$ with

$$\rho_1(\tau,\omega) := \int_{-\infty}^{0} e^{2\lambda s+c_0\varepsilon\int_s^0 |z(\theta_r\omega)|\,dr}\big\|f(s+\tau)\big\|^2_{V_g^*(\mathcal{O}_\infty)}\,ds,$$

$$\rho_2(\tau,\omega) := \int_{-\infty}^{0} e^{2\lambda s + c_0 \varepsilon \int_s^0 |z(\theta_r \omega)|\, dr} \left(|z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^4 \right) ds.$$

We need to prove that ρ_1 and ρ_2 are tempered with the growth rate $\frac{3}{2}\lambda$. Indeed, by Lemma 4.1 and Assumption F,

$$e^{-\frac{3}{2}\lambda t} \rho_1(\tau - t, \theta_{-t}\omega)$$

$$= e^{-\frac{3}{2}\lambda t} \int_{-\infty}^{0} e^{2\lambda s + c_0 \varepsilon \int_s^0 |z(\theta_{r-t}\omega)|\, dr} \left\| f(s + \tau - t) \right\|_{V_g^*(\mathcal{O})}^2 ds$$

$$\leq e^{-\frac{3}{2}\lambda t} \int_{-\infty}^{-t} e^{\frac{3}{2}\lambda(s+t) + c_0 \varepsilon \int_s^{-t} |z(\theta_r \omega)|\, dr} \left\| f(s + \tau) \right\|_{V_g^*(\mathcal{O})}^2 ds$$

$$\leq e^{C_0(\omega)} \int_{-\infty}^{-t} e^{\lambda s} \left\| f(s + \tau) \right\|_{V_g^*(\mathcal{O})}^2 ds \to 0$$

as $t \to +\infty$. Similarly, the tempered property of $|z(\theta_s\omega)|^2 + |z(\theta_s\omega)|^4$ implies that ρ_2 is tempered. Therefore, $\mathcal{M}_k \in \mathfrak{D}_k$.

On the other hand, by Corollary 2, Φ_k is $\mathfrak{D}_k$-pullback asymptotically compact. Therefore, it follows from the abstract result [21] that Φ_k has a unique $\mathfrak{D}_k$-pullback random attractor denoted by $\mathcal{A}_k = \{\mathcal{A}_k(\tau, \omega)\}$.

In addition, by Theorem 3.1, the expanded cocycle $\widetilde{\Phi}_k$ has a $\widetilde{\mathfrak{D}}_k$-pullback random attractor in $H_k(\mathcal{O}_\infty)$. This expanded attractor is just the null-expansion $\widetilde{\mathcal{A}}_k$ of $\mathcal{A}_k$. □

Finally, we establish the large-domain stability (upper-semicontinuity) of random attractors as $k \to \infty$.

Theorem 6.2 *The sequence $\{\mathcal{A}_k\}_k$ of random attractors associated with problem* (18) *satisfies*

$$\text{dist}_{H_g(\mathcal{O}_\infty)}\left(\widetilde{\mathcal{A}}_k(\tau, \omega), \mathcal{A}_\infty(\tau, \omega) \right) \to 0 \tag{87}$$

as $k \to \infty$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, where $\widetilde{\mathcal{A}}_k$ is the null-expansion of $\mathcal{A}_k$.

Proof The proof is similar to that of [16, Theorem VI.2] and so is omitted here. □

Abbreviations
Not applicable.

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Not applicable.

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Authors' contributions
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