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Hybrid method for equilibrium problems and variational inclusions

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Abstract

By providing a new iterative method our aim is finding a common element of the set of fixed points of two nonexpansive mappings, the set of solutions to a variational inclusion and the set of solutions of a generalized equilibrium problem in a real Hilbert space. We review the strong convergence of the new iterative method in the framework of Hilbert spaces. Finally, we show that our main result is a generalization for some known theorems in this field.

MSC: Primary 46N10; secondary 47N10

Keywords: Maximal monotone operator; Nonexpansive map; Variational inclusion; Generalized equilibrium problem; Inverse strongly monotone map

1 Introduction

Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, let $A : C \to H$ be a nonlinear map, and let F be a bifunction from $C \times C$ to $\mathbb{R}$. In 2008, Takahashi et al. [1] considered the generalized equilibrium problem: Finding $x \in C$ such that

$$F(x,y) + \langle Ax, y - x\rangle \geq 0 \quad (\forall y \in C). \tag{1}$$

The set of solutions of (1) is denoted by $\mathrm{GEP}(F,A)$. If $A = 0$, then problem (1) becomes the equilibrium problem: Finding $x \in C$ such that

$$F(x,y) \geq 0 \quad (\forall y \in C). \tag{2}$$

Problem (2) was studied by Blum et al. [2] in 1994. The set of solutions of (2) is denoted by $\mathrm{EP}(F)$. If $F(x,y) = 0$ for all $x, y \in C$, then problem (1) becomes the variational inequality problem: Finding $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad (\forall y \in C). \tag{3}$$

Problem (3) was studied by Hartman et al. [3] in 1966 and has been extensively in the literature (see, e.g., [4–10]). The set of solutions of (3) is denoted by $\mathrm{VI}(C,A)$. If $C = H$, then $\mathrm{VI}(H,A) = A^{-1}(0) = \{x \in H : Ax = 0\}$. Recall that a mapping $A : C \to H$ is said to be monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in C$ [6, 7, 11]. A mapping A is said to be α-strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2$ for all $u, v \in C$ [6, 7, 11]. A mapping A is said to be α-inverse strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$ [6, 7, 11]. In such a case, A is said to be α-inverse-strongly monotone.

Let $T : C \to C$ be a mapping. We denote by $F(T)$ the fixed-point set of T, that is, $F(T) = \{x \in C : T(x) = x\}$. A mapping T is said to be L-Lipschitz if there exists $L \geq 0$ such that $\|Tu - Tv\| \leq L\|u - v\|$ for all $u, v \in C$. The mapping T is called nonexpansive if $L = 1$. It is also called contraction if $L < 1$. Note that any α-inverse strongly monotone mapping A is Lipschitz and $\|Au - Av\| \leq \frac{1}{\alpha}\|u - v\|$ [6, 7, 11]. There are a lot works associated with the fixed point algorithms for nonexpansive mappings (see, e.g., [12–23]).

Let $A : H \to H$ be a single-valued nonlinear map, and let $B : H \to 2^H$ be a set-valued mapping. The variational inclusion is finding $p \in H$ such that

$$\theta \in A(p) + B(p), \tag{4}$$

where θ is the zero vector in H. For $A = 0$, (4) becomes the inclusion problem introduced by Rockafellar [24]. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. The graph of B is $G(B) = \{(u, v) \in H \times H : v \in Bu\}$. A set-valued mapping B is said to be monotone if $\langle x - y, f - h\rangle \geq 0$ for all $x, y \in D(B)$, $f \in Bx$, and $h \in By$ [25]. A monotone operator B is maximal if the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping [25]. Also, a monotone mapping B is maximal if and only if $\langle x - y, f - h\rangle \geq 0$ ($(x, f) \in H \times H$ $(y, h) \in G(B)$) implies $f \in Bx$ [25]. For a maximal monotone operator B on H and $r > 0$, we define the single-valued operator $J_r^B x = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r. It is well known that $J_r^B x$ is firmly nonexpansive, that is, $\langle x - y, J_r^B x - J_r^B y\rangle \geq \|J_r^B x - J_r^B y\|^2$ for all $x, y \in H$ (see [13]), and that a solution of (4) is a fixed point of $J_r^B(I - rA)$ for all $r > 0$ [25].

A basic problem for maximal monotone operator B is finding

$$x \in H \quad \text{such that} \quad 0 \in Bx. \tag{5}$$

A known method for solving problem (5) is the proximal point algorithm: $x_1 = x \in H$, and

$$x_{n+1} = J_{r_n}^B x_n \quad (n \geq 1),$$

where $J_{r_n}^B = (I + r_n B)^{-1}$ and $\{r_n\} \subset (0, \infty)$. Then Rockafellar [24, 26] proved that the sequence $\{x_n\}$ converges weakly to an element of $B^{-1}(0)$ (see also [27]). In the literature, there are a large number references associated with the proximal point algorithm [27–29]. In 2011, Shehu [8] suggested the following iterative sequence. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad (\forall y \in C), \\ x_{n+1} = a_n x_n + (1 - a_n)T[\beta_n f(x_n) + (1 - \beta_n)J_{r_n}^B(y_n - r_n Ay_n)] \quad (\forall n \geq 1). \end{cases}$$

Under appropriate conditions, the author proved that the sequence $\{x_n\}$ converges strongly to a point $P_{F(T)\cap(A+B)^{-1}(0)\cap \mathrm{GEP}(F,A)}u$ [8]. Our goal in this paper is to present an iterative method that converges strongly to a common element of the fixed point set of two nonexpansive mappings and the zero set of the sums of maximal monotone operators in Hilbert spaces. Our results extend and improve some related old results.

2 Preliminaries

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. It is well known that, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| = \inf_{y\in C} \|x - y\| =: d(x, C)$. It is well known that P_C is nonexpansive monotone mapping from H onto C, $\langle x - P_C x, z - P_C x\rangle \le 0$, $\|x - z\|^2 \ge \|x - P_C x\|^2 + \|z - P_C x\|^2$ for all $x \in H$ and $z \in C$, and $\langle P_C x - P_C z, x - z\rangle \ge \|P_C x - P_C z\|^2$ for all $z, x \in H$ (see [13]). Let A be a monotone mapping from C into H. In the context of the variational inequality problem, it is easy to see that from the relation $\langle x - P_C x, z - P_C x\rangle \le 0$ we have

$$p \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad p = P_C(p - \lambda Ap) \quad \text{for some } \lambda > 0.$$

For solving the equilibrium problem for a a bifunction $F : C \times C \to \mathbb{R}$, we assume that F satisfy the following conditions:

(A_1) $F(x, x) = 0$ for all $x \in C$,

(A_2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$,

(A_3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1 - t)x, y) \le F(x, y)$,

(A_4) for each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Put $F(x, y) = \langle Ax, y - x\rangle$ for $x, y \in C$. Then we see that the equilibrium problem (2) is reduced to the variational inequality (3). We need the following results.

Lemma 2.1 ([2, 30]) *Let C be a nonempty closed convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). For $r > 0$ and $x \in H$, consider the map $T_r : H \to C$ defined by $T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \text{ for all } y \in C\}$. Then $T_r(x) \neq \emptyset$ for all $x \in H$, T_r is single-valued, $\mathrm{EP}(F)$ is closed and convex, $F(T_r) = \mathrm{EP}(F)$, and T_r is firmly nonexpansive, that is, $\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle$ for all $x, y \in H$.*

Lemma 2.2 ([31]) *Let C be a nonempty closed convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). Define the multivalued mapping A_F from H into itself by $A_F x = \{z \in C : F(z, y) \le \langle y - x, z\rangle \text{ for all } y \in C\}$ whenever $x \in C$ and $A_F x = \emptyset$ otherwise. Then A_F is a maximal monotone operator with the domain $T_r(x) = (I + rA_F)^{-1}x$ for all $x \in H$ and $r > 0$.*

Lemma 2.3 ([32]) *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $T : C \to C$ be a nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $x = T(x)$.*

Lemma 2.4 ([33]) *Let $\{x_n\}$ be a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \le (1 - \lambda_n)x_n + \gamma_n,$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$, and γ_n is a sequence with $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \gamma_n \le 0$ or $\sum_{n=0}^{\infty} |\gamma_n \lambda_n| < \infty$. Then $\lim_{n\to\infty} x_n = 0$.

Lemma 2.5 ([34]) *Let H be a real Hilbert space, let $x_j \in H$, and let $a_j \in [0,1]$, $j = 1,2,3$, be such that $a_1 + a_2 + a_3 = 1$. Then we have*

$$\|a_1x_1 + a_2x_2z + a_3x_3\|^2 = a_1\|x_1\|^2 + a_2\|x_2\|^2 + a_3\|x_3\|^2 - \sum_{1\le i,j\le 3} a_ia_j\|x_i - x_j\|^2.$$

Lemma 2.6 ([31]) *Let B be a maximal monotone operator on H. Then we have*

$$\frac{\lambda - r}{r}\langle J_\lambda^B x - J_r^B x, J_\lambda^B x - x\rangle \ge \|J_\lambda^B x - J_r^B x\|^2 \quad (\forall \lambda, r > 0 \text{ and } x \in H).$$

3 Main results

Now we are ready to state and prove our main results.

Theorem 3.1 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let A be an α-inverse strongly monotone mapping from C into H, let M be a β-inverse strongly monotone map from C into H, and let B be a maximal monotone operator on H with domain contained in C. Assume that $S, T : C \to C$ are two nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap (M + B)^{-1}(0) \cap \mathrm{GEP}(F, A) \ne \emptyset$ and $f : C \to C$ is a contraction map with the constant $\rho \in (0,1)$. Suppose that $\{b_n\}$, $\{a_n\}$, and $\{\mu_n\}$ are some sequences in $(0,1)$ and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \ge 0 \quad (\forall y \in C), \\ z_n = \mu_n x_n + (1 - \mu_n) J_{\lambda_n}^B (y_n - \lambda_n M y_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1 - b_n)(a_n S z_n + (1 - a_n) T y_n)] \quad (\forall n \ge 1). \end{cases} \tag{6}$$

Suppose the following conditions hold:

(d_1) $0 < c \le \lambda_n \le d < 2\beta$, $\lim_{n\to\infty} |\lambda_n - \lambda_{n-1}| = 0$,

(d_2) $0 < a \le r_n \le b < 2\alpha$, $\lim_{n\to\infty} |r_n - r_{n-1}| = 0$,

(d_3) $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^\infty b_n = \infty$, $\sum_{n=1}^\infty |b_n - b_{n-1}| < \infty$,

(d_4) $\sum_{n=1}^\infty |\mu_n - \mu_{n-1}| < \infty$, $\sum_{n=1}^\infty |a_n - a_{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution to the variational inequality $\langle (I - f)q, x - q\rangle \ge 0$ for all $x \in \Omega$.

Proof First, we show that $I - \lambda_n M$ is nonexpansive. Let $x, y \in C$ and $0 < \lambda_n < 2\beta$. Then

$$\begin{aligned} \|(I - \lambda_n M)x - (I - \lambda_n M)y\|^2 &= \|(x - y) - \lambda_n(Mx - My)\|^2 \\ &\le \|x - y\|^2 - 2\lambda_n\langle x - y, Mx - My\rangle + \lambda_n^2\|Mx - My\|^2 \\ &\le \|x - y\|^2 - \lambda_n\beta\|Mx - My\|^2 + \lambda_n^2\|Mx - My\|^2 \\ &= \|x - y\|^2 + \lambda_n(\lambda_n - 2\beta)\|Mx - My\|^2 \\ &\le \|x - y\|^2. \end{aligned} \tag{7}$$

Thus $I - \lambda_n M$ is nonexpansive. Note that y_n can be rewritten as $y_n = T_{r_n}(x_n - r_n A x_n)$ for $n \ge 1$. Let $q \in \Omega$. From (d_2) and Lemma 2.1 we have

$$\|y_n - q\|^2 = \|T_{r_n}(x_n - r_n A x_n) - q\|^2$$

$$\begin{aligned}
&= \left\| T_{r_n}(x_n - r_n A x_n) - T_{r_n}(q - r_n A q)\right\|^2 \\
&\le \left\| (x_n - r_n A x_n) - (q - r_n A q)\right\|^2 \\
&= \|x_n - q\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Aq\|^2 \le \|x_n - q\|^2. \qquad (8)
\end{aligned}$$

By (6) and (7), since $J^B_{\lambda_n}$ is nonexpansive, we have

$$\begin{aligned}
\|z_n - q\|^2 &= \left\| \mu_n x_n + (1-\mu_n) J^B_{\lambda_n}(y_n - \lambda_n M y_n) - q\right\|^2 \\
&= \left\| \mu_n (x_n - q) + (1-\mu_n)\bigl(J^B_{\lambda_n}(y_n - \lambda_n M y_n) - q\bigr)\right\|^2 \\
&\le \mu_n \|x_n - q\|^2 + (1-\mu_n)\left\| J^B_{\lambda_n}(y_n - \lambda_n M y_n) - q\right\|^2 \\
&\le \mu_n \|x_n - q\|^2 + (1-\mu_n)\left\| J^B_{\lambda_n}(y_n - \lambda_n M y_n) - J^B_{\lambda_n}(q - \lambda_n M q)\right\|^2 \\
&\le \mu_n \|x_n - q\|^2 + (1-\mu_n)\left\| (y_n - \lambda_n M y_n) - (q - \lambda_n M q)\right\|^2 \\
&\le \mu_n \|x_n - q\|^2 + (1-\mu_n)\bigl(\|y_n - q\|^2 + \lambda_n(\lambda_n - 2\beta)\|My_n - Mq\|^2\bigr) \\
&\le \|x_n - q\|^2 + (1-\mu_n)\lambda_n(\lambda_n - 2\beta)\|My_n - Mq\|^2 \le \|x_n - q\|^2. \qquad (9)
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\| P_C\bigl[b_n f(x_n) + (1-b_n)\bigl(a_n S z_n + (1-a_n) T y_n\bigr)\bigr] - P_C(q)\right\| \\
&\le \left\| \bigl[b_n f(x_n) + (1-b_n)\bigl(a_n S z_n + (1-a_n) T y_n\bigr)\bigr] - q\right\| \\
&\le b_n \left\| f(x_n) - q\right\| + (1-b_n)\bigl[a_n \|S z_n - q\| + (1-a_n)\|T y_n - q\|\bigr] \\
&\le b_n \left\| f(x_n) - q\right\| + (1-b_n)\bigl[a_n \|x_n - q\| + (1-a_n)\|y_n - q\|\bigr] \\
&\le b_n \left\| f(x_n) - q\right\| + (1-b_n)\|x_n - q\| \\
&\le b_n\bigl(\left\| f(x_n) - f(q)\right\| + \left\| f(q) - q\right\|\bigr) + (1-b_n)\|x_n - q\| \\
&\le b_n\bigl(\rho\|x_n - q\| + \left\| f(q) - q\right\|\bigr) + (1-b_n)\|x_n - q\| \\
&\le \bigl(1 - b_n(1-\rho)\bigr)\|x_n - q\| + b_n\left\| f(q) - q\right\| \\
&\le \max\left\{ \|x_n - q\|, \frac{\|f(q) - q\|}{(1-\rho)} \right\} \\
&\ \ \vdots \\
&\le \max\left\{ \|x_1 - q\|, \frac{\|f(q) - q\|}{(1-\rho)} \right\}.
\end{aligned}$$

Thus $\{x_n\}$ is bounded, and the sequences $\{y_n\}$ and $\{z_n\}$ are bounded as well. From $y_n = T_{r_n}(x_n - r_n A x_n)$ and $y_{n-1} = T_{r_{n-1}}(x_{n-1} - r_{n-1} A x_{n-1})$ we obtain

$$F(y_n, y) + \langle A x_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \ge 0 \quad (\forall y \in C) \qquad (10)$$

and

$$F(y_{n-1}, y) + \langle A x_{n-1}, y - y_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - y_{n-1}, y_{n-1} - x_{n-1}\rangle \ge 0 \quad (\forall y \in C). \qquad (11)$$

By substituting $y = y_{n-1}$ into (10) and $y = y_n$ into (11), we find

$$F(y_n, y_{n-1}) + \langle Ax_n, y_{n-1} - y_n\rangle + \frac{1}{r_n}\langle y_{n-1} - y_n, y_n - x_n\rangle \geq 0$$

and

$$F(y_{n-1}, y_n) + \langle Ax_{n-1}, y_n - y_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y_n - y_{n-1}, y_{n-1} - x_{n-1}\rangle \geq 0.$$

Now from (A_2) we get $\langle Ax_{n-1} - Ax_n, y_n - y_{n-1}\rangle + \langle y_{n-1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n-1} - x_{n-1}}{r_{n-1}}\rangle \geq 0$, and so

$$\begin{aligned}
0 &\leq \left\langle y_n - y_{n-1}, r_n(Ax_{n-1} - Ax_n) + \frac{r_n}{r_{n-1}}(y_{n-1} - x_{n-1}) - (y_n - x_n)\right\rangle \\
&= \left\langle y_{n-1} - y_n, y_n - y_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)y_{n-1} + (x_{n-1} - r_{n-1}Ax_{n-1})\right\rangle \\
&\quad - \left\langle y_{n+1} - y_n, (x_n - r_nAx_n) + x_{n-1} - \frac{r_n}{r_{n-1}}x_{n-1}\right\rangle \\
&= \left\langle y_{n-1} - y_n, y_n - y_{n-1} + \left(1 - \frac{r_n}{r_{n-1}}\right)(y_{n-1} - x_{n-1})\right. \\
&\quad \left. + (x_{n-1} - r_{n-1}Ax_{n-1}) - (x_n - r_nAx_n)\right\rangle.
\end{aligned}$$

This implies that $\|y_n - y_{n-1}\|^2 \leq \|y_n - y_{n-1}\|[|1 - \frac{r_n}{r_{n-1}}|\|y_{n-1} - x_{n-1}\| + \|x_n - x_{n-1}\|]$, and so

$$\|y_n - y_{n-1}\| \leq \frac{|r_n - r_{n-1}|}{r_{n-1}}\|y_{n-1} - x_{n-1}\| + \|x_n - x_{n-1}\|. \tag{12}$$

Set $w_n = J^B_{\lambda_n}(y_n - \lambda_n My_n)$ and $u_n = y_n - \lambda_n My_n$ for $n \geq 1$. By using Lemma 2.6 we obtain

$$\begin{aligned}
\|w_n - w_{n-1}\| &= \left\|J^B_{\lambda_n}(y_n - \lambda_n My_n) - J^B_{\lambda_{n-1}}(y_{n-1} - \lambda_{n-1}My_{n-1})\right\| \\
&= \left\|J^B_{\lambda_n}u_n - J^B_{\lambda_{n-1}}u_{n-1} + J^B_{\lambda_n}u_{n-1} - J^B_{\lambda_n}u_{n-1}\right\| \\
&\leq \left\|(y_n - \lambda_n My_n) - (y_{n-1} - \lambda_{n-1}My_{n-1})\right\| \\
&\quad + \left\|J^B_{\lambda_n}u_{n-1} - J^B_{\lambda_{n-1}}u_{n-1}\right\| \\
&\leq \left\|(y_n - \lambda_n My_n) - (y_{n-1} - \lambda_n My_{n-1}) + (\lambda_{n-1} - \lambda_n)My_{n-1}\right\| \\
&\quad + \frac{|\lambda_{n-1} - \lambda_n|}{\lambda_{n-1}}\left\|J^B_{\lambda_{n-1}}u_{n-1} - u_{n-1}\right\| \\
&\leq \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n|\|My_{n-1}\| + \frac{|\lambda_{n-1} - \lambda_n|}{\lambda_{n-1}}\left\|J^B_{\lambda_{n-1}}u_{n-1} - u_{n-1}\right\| \\
&\leq \|x_n - x_{n-1}\| + \frac{|r_n - r_{n+1}|}{r_{n+1}}\|y_{n+1} - x_{n+1}\| + |\lambda_{n-1} - \lambda_n|\|My_{n-1}\| \\
&\quad + \frac{|\lambda_{n-1} - \lambda_n|}{\lambda_{n-1}}\left\|J^B_{\lambda_{n-1}}u_{n-1} - u_{n-1}\right\|,
\end{aligned}$$

which gives

$$\|z_n - z_{n-1}\| = \left\|\left[\mu_n x_n + (1 - \mu_n)w_n\right] - \left[\mu_{n-1}x_{n-1} + (1 - \mu_{n-1})w_{n-1}\right]\right\|$$

$$
\begin{aligned}
&= \big\| \mu_n(x_n - x_{n-1}) + (1-\mu_n)(w_n - w_{n-1}) + (\mu_n - \mu_{n-1})(x_{n-1} - w_{n-1}) \big\| \\
&\leq \mu_n \|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}| \|x_{n-1} - w_{n-1}\| + (1-\mu_n)\|w_n - w_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\mu_n - \mu_{n-1}| \|x_{n-1} - w_{n-1}\| \\
&\quad + (1-\mu_n)\bigg(|\lambda_{n-1} - \lambda_n| \|My_{n-1}\| + \frac{|r_n - r_{n+1}|}{r_{n+1}} \|y_{n+1} - x_{n+1}\| \\
&\quad + \frac{|\lambda_{n-1} - \lambda_n|}{\lambda_{n-1}} \big\| J^B_{\lambda_{n-1}} u_{n-1} - u_{n-1} \big\| \bigg).
\end{aligned}
$$

Set $t_n = a_n S z_n + (1-a_n) T y_n$ for $n \geq 1$. By using (12) and last inequality we have

$$
\begin{aligned}
\|t_n - t_{n-1}\| &= \big\| a_n(Sz_n - Sz_{n-1}) + (1-a_n)(Ty_n - Ty_{n-1}) \\
&\quad + (a_n - a_{n-1})(Sz_{n-1} - Ty_{n-1}) \big\| \\
&\leq a_n \|Sz_n - Sz_{n-1}\| + (1-a_n)\|Ty_n - Ty_{n-1}\| \\
&\quad + |a_n - a_{n-1}| \|Sz_{n-1} - Ty_{n-1}\| \\
&\leq a_n \|z_n - z_{n-1}\| + (1-a_n)\|y_n - y_{n-1}\| + |a_n - a_{n-1}| \|Sz_{n-1} - Ty_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \frac{|r_n - r_{n+1}|}{r_{n+1}} \|y_{n+1} - x_{n+1}\| + |a_n - a_{n-1}| \|Sz_{n-1} - Ty_{n-1}\| \\
&\quad + (1-\mu_n)a_n \bigg(|\lambda_{n-1} - \lambda_n| \|My_{n-1}\| + \frac{|\lambda_{n-1} - \lambda_n|}{\lambda_{n-1}} \big\| J^B_{\lambda_{n-1}} u_{n-1} - u_{n-1} \big\| \bigg) \\
&\quad + a_n |\mu_n - \mu_{n-1}| \|x_{n-1} - w_{n-1}\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \big\| P_C\big[b_n f(x_n) + (1-b_n)t_n\big] - P_C\big[b_{n-1} f(x_{n-1}) + (1-b_{n-1})t_{n-1}\big] \big\| \\
&\leq \big\| \big[b_n f(x_n) + (1-b_n)t_n\big] - \big[b_{n-1} f(x_{n-1}) + (1-b_{n-1})t_{n-1}\big] \big\| \\
&\leq \big\| b_n\big(f(x_n) - f(x_{n-1})\big) + (1-b_n)(t_n - t_{n-1}) \big\| \\
&\quad + \big\| (b_n - b_{n-1})\big(f(x_{n-1}) - t_{n-1}\big) \big\| \\
&\leq b_n \big\| f(x_n) - f(x_{n-1}) \big\| + |b_n - b_{n-1}| \big\| f(x_{n-1}) - t_{n-1} \big\| \\
&\quad + (1-b_n)\|t_n - t_{n-1}\| \\
&\leq b_n \rho \|x_n - x_{n-1}\| + |b_n - b_{n-1}| \big\| f(x_{n-1}) - t_{n-1} \big\| + (1-b_n)\|t_n - t_{n-1}\| \\
&\leq \big(1 - b_n(1-\rho)\big)\|x_n - x_{n-1}\| + |b_n - b_{n-1}| \big\| f(x_{n-1}) - t_{n-1} \big\| \\
&\quad + \frac{|r_n - r_{n+1}|}{a} \|y_{n+1} - x_{n+1}\| + |\mu_n - \mu_{n-1}| \|x_{n-1} - w_{n-1}\| \\
&\quad + |\lambda_{n-1} - \lambda_n| \|My_{n-1}\| + \frac{|\lambda_{n-1} - \lambda_n|}{c} \big\| J^B_{\lambda_{n-1}} u_{n-1} - u_{n-1} \big\| \\
&\quad + |a_n - a_{n-1}| \|Sz_{n-1} - Ty_{n-1}\| \\
&\leq \big(1 - b_n(1-\rho)\big)\|x_n - x_{n-1}\| + L\bigg[|b_n - b_{n-1}| + |\mu_n - \mu_{n-1}| \\
&\quad + \frac{|r_n - r_{n+1}|}{a} + \left(1 + \frac{1}{c}\right)|\lambda_{n-1} - \lambda_n| + |a_n - a_{n-1}| \bigg],
\end{aligned}
$$

where L is a constant such that

$$L \geq \max\Big\{\sup_{n\geq 1}\|Sz_{n-1} - Ty_{n-1}\|, \sup_{n\geq 1}\|My_n\|, \sup_{n\geq 1}\big\|J^B_{\lambda_{n-1}}u_{n-1} - u_{n-1}\big\|,$$
$$\sup_{n\geq 1}\big\|f(x_{n-1}) - t_{n-1}\big\|, \sup_{n\geq 1}\|y_{n+1} - x_{n+1}\|, \sup_{n\geq 1}\|x_{n-1} - w_{n-1}\|\Big\}.$$

Now by using (d_1), (d_2), (d_3), (d_4), and Lemma 2.4 we get

$$\limsup_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{13}$$

From (6) we have

$$\begin{aligned}
\|x_n - t_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \\
&= \|x_n - x_{n+1}\| + \big\|p_C\big[b_nf(x_n) + (1-b_n)t_n\big] - P_C(t_n)\big\| \\
&\leq \|x_n - x_{n+1}\| + \big\|\big[b_nf(x_n) + (1-b_n)t_n\big] - t_n\big\| \\
&\leq \|x_n - x_{n+1}\| + b_n\big\|f(x_n) - t_n\big\|.
\end{aligned}$$

Consequently, from the condition (d_3) we obtain

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{14}$$

Let $q = P_\Omega f(q)$. In addition, from (6), (8), Lemma 2.5, and (9) we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\|P_C\big[b_nf(x_n) + (1-b_n)\big(a_nSz_n + (1-a_n)Ty_n\big)\big] - P_C(q)\big\|^2 \\
&\leq \big\|\big[b_nf(x_n) + (1-b_n)\big(a_nSz_n + (1-a_n)Ty_n\big)\big] - q\big\|^2 \\
&\leq b_n\big\|f(x_n) - q\big\|^2 + (1-b_n)\big[a_n\|Sz_n - q\|^2 + (1-a_n)\|Ty_n - q\|^2 \\
&\quad - a_n(1-a_n)\|Sz_n - Ty_n\|\big] \\
&\leq b_n\big\|f(x_n) - q\big\|^2 + (1-b_n)\big[a_n\|z_n - q\|^2 + (1-a_n)\|y_n - q\|^2 \\
&\quad - a_n(1-a_n)\|Sz_n - Ty_n\|\big] \\
&\leq b_n\big\|f(x_n) - q\big\|^2 + (1-b_n)\big(a_n\|x_n - q\|^2 - a_n(1-a_n)\|Sz_n - Ty_n\|\big) \\
&\quad + (1-b_n)a_n(1-\mu_n)\lambda_n(\lambda_n - 2\beta)\|My_n - Mq\|^2 \\
&\quad + (1-b_n)(1-a_n)\big(\|x_n - q\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Aq\|^2\big) \\
&\leq b_n\big\|f(x_n) - q\big\| + (1-b_n)\big(\|x_n - q\| - a_n(1-a_n)\|Sz_n - Ty_n\|\big) \\
&\quad + (1-b_n)a_n(1-\mu_n)\lambda_n(\lambda_n - 2\beta)\|My_n - Mq\|^2 \\
&\quad + (1-b_n)(1-a_n)r_n(r_n - 2\alpha)\|Ax_n - Aq\|^2,
\end{aligned}$$

which yields

$$(1-b_n)(1-a_n)r_n(2\alpha - r_n)\cdot\|Ax_n - Aq\|^2$$

$$\begin{aligned}
&\leq b_n\|f(x_n)-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\quad+(1-b_n)a_n(1-\mu_n)\lambda_n(\lambda_n-2\beta)\|My_n-Mq\|^2 \\
&\quad-(1-b_n)a_n(1-a_n)\|Sz_n-Ty_n\|, \\
&(1-b_n)a_n(1-\mu_n)\lambda_n(2\beta-\lambda_n)\cdot\|My_n-Mq\|^2 \\
&\leq b_n\|f(x_n)-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\quad+(1-b_n)(1-a_n)r_n(r_n-2\alpha)\|Ax_n-Aq\|^2 \\
&\quad-(1-b_n)a_n(1-a_n)\|Sz_n-Ty_n\|,
\end{aligned}$$

and

$$\begin{aligned}
(1-b_n)a_n(1-a_n)\|Sz_n-Ty_n\| \leq{}& b_n\|f(x_n)-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&+(1-b_n)(1-a_n)r_n(r_n-2\alpha)\|Ax_n-Aq\|^2 \\
&+(1-b_n)a_n(1-\mu_n)\lambda_n(\lambda_n-2\beta)\|My_n-Mq\|^2.
\end{aligned}$$

By using (d_1), (d_2), (d_3), and (13) we get

$$\lim_{n\to\infty}\|My_n-Mq\|=\lim_{n\to\infty}\|Ax_n-Aq\|=\lim_{n\to\infty}\|Sz_n-Ty_n\|=0. \tag{15}$$

Since $\|t_n-Ty_n\|\leq a_n\|Sz_n-Ty_n\|$ and $\|t_n-Sz_n\|\leq(1-a_n)\|Ty_n-Sz_n\|$, we obtain

$$\lim_{n\to\infty}\|t_n-Ty_n\|=\lim_{n\to\infty}\|t_n-Sz_n\|=0. \tag{16}$$

Note that

$$\begin{aligned}
\|w_n-q\|^2 &= \|J^B_{\lambda_n}(y_n-\lambda_nMy_n)-J^B_{\lambda_n}(q-\lambda_nMq)\|^2 \\
&\leq \langle (y_n-\lambda_nMy_n)-(q-\lambda_nMq), w_n-q\rangle \\
&= \frac{1}{2}\|(y_n-\lambda_nMy_n)-(q-\lambda_nMq)\|^2+\frac{1}{2}\|w_n-q\|^2 \\
&\quad-\frac{1}{2}\|(y_n-\lambda_nMy_n)-(q-\lambda_nMq)-(w_n-q)\|^2 \\
&\leq \frac{1}{2}\big[\|y_n-q\|^2+\|w_n-q\|^2-\|(y_n-w_n)-\lambda_n(My_n-Mq)\|^2\big] \\
&= \frac{1}{2}\big[\|y_n-q\|^2+\|w_n-q\|^2-\|y_n-w_n\|^2+2\lambda_n\langle y_n-w_n, My_n-Mq\rangle \\
&\quad-\lambda_n^2\|My_n-Mq\|^2\big],
\end{aligned}$$

and so

$$\|w_n-q\|^2\leq\|y_n-q\|^2-\|y_n-w_n\|^2+2\lambda_n\|y_n-w_n\|\|My_n-Mq\|. \tag{17}$$

By using Lemma 2.1 and (6) we have

$$\|y_n-q\|^2 = \|T_{r_n}(x_n-r_nAx_n)-T_{r_n}(q-r_nAq)\|^2$$

$$
\begin{aligned}
&\le \big\langle (x_n - r_nAx_n) - (q - r_nAq), y_n - q\big\rangle \\
&= \frac{1}{2}\big\|(x_n - r_nAx_n) - (q - r_nAq)\big\|^2 + \frac{1}{2}\|y_n - q\|^2 \\
&\quad - \frac{1}{2}\big\|(x_n - r_nAx_n) - (q - r_nAq) - (y_n - q)\big\|^2 \\
&\le \frac{1}{2}\big[\|x_n - q\|^2 + \|y_n - q\|^2 - \big\|(x_n - y_n) - 2r_n(Ax_n - Aq)\big\|^2\big] \\
&= \frac{1}{2}\big[\|x_n - q\|^2 + \|y_n - q\|^2 - \|x_n - y_n\|^2 + 2r_n\langle x_n - y_n, Ax_n - Aq\rangle \\
&\quad - r_n^2\|Ax_n - Aq\|^2\big].
\end{aligned}
$$

It follows that

$$
\|y_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - y_n\|^2 + 2r_n\langle x_n - y_n, Ax_n - Aq\rangle. \tag{18}
$$

From (8), (17), and (18) we have

$$
\begin{aligned}
\|t_n - q\|^2 &= \big\|a_n(Sz_n - q) + (1 - a_n)(Ty_n - q)\big\|^2 \\
&\le a_n\|Sz_n - q\|^2 + (1 - a_n)\|Ty_n - q\|^2 \\
&\le a_n\|z_n - q\|^2 + (1 - a_n)\|y_n - q\|^2 \\
&\le a_n\big(\mu_n\|x_n - q\|^2 + (1 - \mu_n)\|w_n - q\|^2\big) + (1 - a_n)\|y_n - q\|^2 \\
&\le a_n\big(\mu_n\|x_n - q\|^2 + (1 - \mu_n)\big(\|y_n - q\|^2 - \|y_n - w_n\|^2\big)\big) \\
&\quad + (1 - \mu_n)a_n2\lambda_n\|y_n - w_n\|\|My_n - Mq\| \\
&\quad + (1 - a_n)\big(\|x_n - q\|^2 - \|x_n - y_n\|^2 + 2r_n\langle x_n - y_n, Ax_n - Aq\rangle\big) \\
&\le \|x_n - q\|^2 - a_n(1 - \mu_n)\|y_n - w_n\|^2 \\
&\quad + (1 - \mu_n)a_n2\lambda_n\|y_n - w_n\|\|My_n - Mq\| \\
&\quad + (1 - a_n)\big(2r_n\|x_n - y_n\|\|Ax_n - Aq\| - \|x_n - y_n\|^2\big).
\end{aligned}
$$

By using (6) and last inequality we see that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\|P_C\big[b_nf(x_n) + (1 - b_n)t_n\big] - P_C(q)\big\|^2 \\
&\le b_n\big\|f(x_n) - q\big\|^2 + (1 - b_n)\|t_n - q\|^2 \\
&\le b_n\big\|f(x_n) - q\big\|^2 + (1 - b_n)\big(\|x_n - q\|^2 - a_n(1 - \mu_n)\|y_n - w_n\|^2 \\
&\quad + (1 - \mu_n)a_n2\lambda_n\|y_n - w_n\|\|My_n - Mq\| \\
&\quad + (1 - a_n)\big(2r_n\|x_n - y_n\|\|Ax_n - Aq\| - \|x_n - y_n\|^2\big)\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(1 - b_n)(1 - a_n)\|x_n - y_n\|^2 &\le b_n\big\|f(x_n) - q\big\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\quad + (1 - \mu_n)a_n2\lambda_n\|y_n - w_n\|\|My_n - Mq\| - \|y_n - w_n\|^2 \\
&\quad + (1 - a_n)\big(2r_n\|x_n - y_n\|\|Ax_n - Aq\| - \|x_n - y_n\|^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(1-b_n)(1-\mu_n)a_n\|y_n-w_n\|^2 \leq{} & b_n\|f(x_n)-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
& +(1-\mu_n)a_n2\lambda_n\|y_n-w_n\|\|My_n-Mq\| \\
& +(1-a_n)2r_n\|x_n-y_n\|\|Ax_n-Aq\|.
\end{aligned}
$$

From (d_3), (15), and $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$ we find $\lim_{n\to\infty}\|x_n-y_n\|=0$ and also $\lim_{n\to\infty}\|y_n-w_n\|=0$. Since $\|x_n-w_n\|\leq\|x_n-y_n\|+\|y_n-w_n\|$, we get $\|x_n-w_n\|\to 0$. Consequently, by using (6) we get $\lim_{n\to\infty}\|z_n-x_n\|=\lim_{n\to\infty}(1-\mu_n)\|w_n-x_n\|=0$. Moreover, from (14) and (16) we get $\|z_n-Sz_n\|\leq\|z_n-x_n\|+\|x_n-t_n\|+\|t_n-Sz_n\|\to 0$ and $\|y_n-Ty_n\|\leq\|y_n-x_n\|+\|x_n-t_n\|+\|t_n-Ty_n\|\to 0$. Hence

$$
\begin{aligned}
\|x_n-Sx_n\| &\leq \|x_n-z_n\|+\|z_n-Sz_n\|+\|Sz_n-Sx_n\| \\
&\leq \|x_n-z_n\|+\|z_n-Sz_n\|+\|z_n-x_n\|\to 0
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-y_n\|+\|y_n-Ty_n\|+\|Ty_n-Tx_n\| \\
&\leq \|x_n-y_n\|+\|y_n-Ty_n\|+\|y_n-x_n\|\to 0,
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty}\|x_n-Tx_n\|=\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{19}
$$

Now we show that $\limsup_{n\to\infty}\langle f(q)-q,x_n-q\rangle\leq 0$, where $q=P_\Omega f(q)$. The existence of q is justified since P_Ω is nonexpansive and f is a contraction. Hence $P_\Omega\circ f$ is a contraction and so has a fixed point. We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty}\langle f(q)-q,x_n-q\rangle=\lim_{i\to\infty}\langle f(q)-q,x_{n_i}-q\rangle. \tag{20}
$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ that converges weakly to v. Without loss of generality, we can assume that $x_{n_i}\rightharpoonup v$. Since $\|x_n-y_n\|\to 0$ and $\|x_n-z_n\|\to 0$, we find $y_{n_i}\rightharpoonup v$ and $y_{n_i}\rightharpoonup v$. Since $\{y_{n_i}\}$ and $\{z_{n_i}\}$ lie in C and C is closed and convex, we obtain $v\in C$. It is easy to check that $v\in F(T)$ and $v\in F(S)$. By using (19) and Lemma 2.3 we get $v\in F(T)\cap F(S)$. Now we show $v\in \mathrm{GEP}(F)$. Since $y_n=T_{r_n}(x_n-r_nAx_n)$, we obtain

$$
F(y_n,y)+\langle Ax_n,y-y_n\rangle+\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\geq 0 \quad (\forall y\in C).
$$

From (A_2) we get $\langle Ax_n,y-y_n\rangle+\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\geq F(y,y_n)$ for all $y\in C$. Hence

$$
\langle Ax_{n_i},y-y_{n_i}\rangle+\left\langle y-y_{n_i},\frac{y_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle\geq F(y,y_{n_i}) \tag{21}
$$

for all $y \in C$. For $0 < t \leq 1$ and $y \in C$, put $y_t = ty + (1-t)v$. Since $y \in C$ and $v \in C$, we obtain $y_t \in C$. From (21) we conclude that

$$\begin{aligned}
\langle y_t - y_{n_i}, Ay_t\rangle &\geq \langle y_t - y_{n_i}, Ay_t\rangle - \langle y_t - y_{n_i}, Ax_{n_i}\rangle \\
&\quad - \left\langle y_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, y_{n_i}) \\
&= \langle y_t - y_{n_i}, Ay_t - Ay_{n_i}\rangle + \langle y_t - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle \\
&\quad - \left\langle y_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F(y_t, y_{n_i}).
\end{aligned}$$

Since $\|y_{n_i} - x_{n_i}\| \to 0$, we have $\|Ay_{n_i} - Ax_{n_i}\| \to 0$. Further, from the inverse-strongly monotonicity of A we have $\langle y_t - y_{n_i}, Ay_t - Ay_{n_i}\rangle \geq 0$. By using (A_4), $\frac{y_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, and $y_{n_i} \rightharpoonup v$ we get $\langle y_t - v, Ay_t\rangle \geq F(y_t, v)$. From (A_1)–(A_4) we have

$$\begin{aligned}
0 &= F(y_t, y_t) = tF(y_t, y) + (1-t)F(y_t, v) \\
&\leq tF(y_t, y) + (1-t)\langle y_t - v, Ay_t\rangle \\
&= tF(y_t, y) + (1-t)t\langle y - v, Ay_t\rangle,
\end{aligned}$$

and so $0 \leq F(y_t, y) + (1-t)\langle y - v, Ay_t\rangle$. Thus $F(v, y) + (1-t)\langle y - v, Av\rangle \geq 0$ for all $y \in C$. This implies that $v \in \mathrm{GEP}(F, A)$. Finally, we show $v \in (M+B)^{-1}(0)$. Choose a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \tilde{\lambda} \in [c, d]$. Without loss of generality, assume that $\lambda_{n_i} \to \tilde{\lambda}$. By using Lemma 2.6 we obtain

$$\begin{aligned}
\|y_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\| &\leq \|y_{n_i} - z_{n_i}\| + \|z_{n_i} - (\mu_{n_i}x_{n_i} + (1-\mu_{n_i})J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i})\| \\
&\quad + \|(\mu_{n_i}x_{n_i} + (1-\mu_{n_i})J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}) - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\| \\
&\leq \|y_{n_i} - z_{n_i}\| + (1-\mu_{n_i})\|J^B_{\lambda_{n_i}}(I - \lambda_{n_i}M)y_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i})\| \\
&\quad + \mu_{n_i}\|x_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\| \\
&\leq \|y_{n_i} - z_{n_i}\| + (1-\mu_{n_i})\big[\|J^B_{\lambda_{n_i}}(I - \lambda_{n_i}M)y_{n_i} - J^B_{\lambda_{n_i}}(I - \tilde{\lambda}M)y_{n_i}\| \\
&\quad + \|J^B_{\lambda_{n_i}}(I - \tilde{\lambda}M)y_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\|\big] \\
&\quad + \mu_{n_i}\|x_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\| \\
&\leq \|y_{n_i} - z_{n_i}\| + (1-\mu_{n_i})|\lambda_{n_i} - \tilde{\lambda}|\|My_{n_i}\| \\
&\quad + (1-\mu_{n_i})\left|\frac{\lambda_{n_i} - \tilde{\lambda}}{\tilde{\lambda}}\right|\|J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i} - (I - \tilde{\lambda}M)y_{n_i}\| \\
&\quad + \mu_{n_i}\big[\|x_{n_i} - y_{n_i}\| + \|y_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\|\big].
\end{aligned}$$

Thus

$$\begin{aligned}
(1-\mu_{n_i})\|y_{n_i} - J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i}\| &\leq \|y_{n_i} - z_{n_i}\| + (1-\mu_{n_i})|\lambda_{n_i} - \tilde{\lambda}|\|My_{n_i}\| \\
&\quad + (1-\mu_{n_i})\left|\frac{\lambda_{n_i} - \tilde{\lambda}}{\tilde{\lambda}}\right|\|J^B_{\tilde{\lambda}}(I - \tilde{\lambda}M)y_{n_i} - (I - \tilde{\lambda}M)y_{n_i}\| \\
&\quad + \mu_{n_i}\|x_{n_i} - y_{n_i}\|.
\end{aligned}$$

This implies that $\lim_{k\to\infty}\|y_{n_i} - J^B_{\tilde\lambda}(I-\tilde\lambda M)y_{n_i}\| = 0$. Since $J^B_{\tilde\lambda}(I-\tilde\lambda M)$ is nonexpansive, it is demiclosed, and so $v \in F(J^B_{\tilde\lambda}(I-\tilde\lambda M))$, that is, $v \in (M+B)^{-1}(0)$. This implies $v \in \Omega$. By using (20) we get $\limsup_{n\to\infty}\langle f(q)-q, x_n-q\rangle = \lim_{i\to\infty}\langle f(q)-q, x_{n_i}-q\rangle = \langle f(q)-q, v-q\rangle \le 0$. Now we show that $x_n \to q$. From(8) and (9) we have

$$\begin{aligned}
\|t_n - q\| &= \big\|\big(a_nSz_n + (1-a_n)Ty_n\big) - q\big\| \\
&\le a_n\|Sz_n - q\| + (1-a_n)\|Ty_n - q\| \\
&\le a_n\|z_n - q\| + (1-a_n)\|y_n - q\| \le \|x_n - q\|.
\end{aligned}$$

Set $v_n = b_nf(x_n) + (1-b_n)t_n$ for all $n \ge 1$. By using (6) and the property of metric projection we obtain

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\langle P_C(v_n) - v_n, P_C(v_n) - q\big\rangle + \langle v_n - q, x_{n+1} - q\rangle \\
&\le \big\langle \big(b_nf(x_n) + (1-b_n)t_n\big) - q, x_{n+1} - q\big\rangle \\
&= b_n\big\langle f(x_n) - q, x_{n+1} - q\big\rangle + (1-b_n)\langle t_n - q, x_{n+1} - q\rangle \\
&\le b_n\big\langle f(x_n) - f(q), x_{n+1} - q\big\rangle + b_n\big\langle f(q) - q, x_{n+1} - q\big\rangle \\
&\quad + (1-b_n)\|t_n - q\|\|x_{n+1} - q\| \\
&\le b_n\rho\|x_n - q\|\|x_{n+1} - q\| + b_n\big\langle f(q) - q, x_{n+1} - q\big\rangle \\
&\quad + (1-b_n)\|x_n - q\|\|x_{n+1} - q\| \\
&\le \big(1 - b_n(1-\rho)\big)\|x_n - q\|\|x_{n+1} - q\| + b_n\big\langle f(q) - q, x_{n+1} - q\big\rangle \\
&\le \frac{(1 - b_n(1-\rho))}{2}\big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\big) + b_n\big\langle f(q) - q, x_{n+1} - q\big\rangle,
\end{aligned}$$

which implies that $\|x_{n+1} - q\|^2 \le (1 - b_n(1-\rho))\|x_n - q\|^2 + 2b_n\langle f(q) - q, x_{n+1} - q\rangle$. Now by using (d_3) and Lemma 2.4 we get $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

Let $u \in C$ and $f(x) = u \in C$ for all x. By using Theorem 3.1 we obtain the following result.

Corollary 3.2 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let A be an α-inverse strongly monotone mapping from C into H, let M be a β-inverse strongly monotone map from C into H, and let B be a maximal monotone operator on H with domain contained in C. Assume that $S, T : C \to C$ are two nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap (M+B)^{-1}(0) \cap \mathrm{GEP}(F,A) \ne \emptyset$. Suppose that $\{b_n\}$, $\{a_n\}$, and $\{\mu_n\}$ are some sequences in $(0,1)$ and that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are the sequences generated by*

$$\begin{cases}
x_1 \in C, \\
F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \ge 0 \quad (\forall y \in C), \\
z_n = \mu_n x_n + (1-\mu_n)J^B_{\lambda_n}(y_n - \lambda_n My_n), \\
x_{n+1} = P_C[b_nu + (1-b_n)(a_nSz_n + (1-a_n)Ty_n)] \quad (\forall n \ge 1).
\end{cases}$$

If (d_1)–(d_4) hold, then the sequence $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution to the variational inequality $\langle q - u, x - q\rangle \ge 0$ for all $x \in \Omega$.

Corollary 3.3 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let A be an α-inverse strongly monotone mapping from C into H, let M be a β-inverse strongly monotone map from C into H, and let B be a maximal monotone operator on H with domain contained in C. Assume that $S, T : C \to C$ are two nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap (M+B)^{-1}(0) \cap \mathrm{GEP}(F, A) \neq \emptyset$. Suppose that $\{b_n\}$, $\{a_n\}$, and $\{\mu_n\}$ are some sequences in $(0,1)$ and that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad (\forall y \in C), \\ z_n = \mu_n x_n + (1-\mu_n) J^B_{\lambda_n}(y_n - \lambda_n M y_n), \\ x_{n+1} = P_C[(1-b_n)(a_n S z_n + (1-a_n)T y_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)–(d_4) hold, then the sequence $\{x_n\}$ converges strongly to a point $q = P_\Omega(0)$, which is the minimum norm element in Ω.

Proof In Theorem 3.1, put $f(x) = 0$ for all x. Note that $x_n \to q = P_\Omega(0)$ and $P_\Omega(0)$ is the minimum norm element in Ω. Since $\langle (I-f)q, x-q\rangle \geq 0$, we get $\langle q, q-x\rangle \leq 0$ for all $x \in \Omega$, that is, $\|q\|^2 \leq \langle q, x\rangle \leq \|x\|\|q\|$ for all $x \in \Omega$. Thus, the point q is the unique solution to the quadratic minimization problem $q = \arg\min_{x\in\Omega} \|x\|^2$. □

Let I_C be the indicator function of C defined by $I_C(x) = 0$ for $x \in C$ and $I_C(x) = \infty$ otherwise. Recall that the subdifferential ∂I_C is a maximal monotone operator. Note that I_C is a proper lower semicontinuous convex function on H. The resolvent $J_r^{\partial I_C}$ of ∂I_C for r is P_C, and $\mathrm{VI}(C, M) = (M + \partial I_C)^{-1}(0)$, where M is an inverse strongly monotone mapping from C into H [35].

Theorem 3.4 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let A be an α-inverse strongly monotone mapping from C into H, let M be a β-inverse strongly monotone map from C into H, and let B be a maximal monotone operator on H with domain contained in C. Assume that $S, T : C \to C$ are two nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap \mathrm{VI}(C, M) \cap \mathrm{GEP}(F, A) \neq \emptyset$ and $f : C \to C$ is a contraction map with the constant $\rho \in (0,1)$. Suppose that $\{b_n\}$, $\{a_n\}$, and $\{\mu_n\}$ are some sequences in $(0,1)$ and that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad (\forall y \in C), \\ z_n = \mu_n x_n + (1-\mu_n) P_C(y_n - \lambda_n M y_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1-b_n)(a_n S z_n + (1-a_n)T y_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)–(d_4) hold, then $\{x_n\}$ converges strongly to a point $p \in \Omega$, which is the unique solution to the variational inequality $\langle (I-f)p, x-p\rangle \geq 0$ for all $x \in \Omega$.

Proof If $B = \partial I_C$ in Theorem 3.1, then $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$. This completes the proof. □

Note that Theorem 3.4 reduces the results of [1, 36].

Theorem 3.5 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let A be an α-inverse strongly monotone mapping from C into H, let M be a β-inverse strongly monotone map from C into H, and let B be a maximal monotone operator on H with domain contained in C. Assume that $S, T : C \to C$ are two nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap \mathrm{VI}(C, M) \neq \emptyset$. Suppose that $\{b_n\}$, $\{a_n\}$, and $\{\mu_n\}$ are some sequences in $(0, 1)$ and that $\{x_n\}$ and $\{z_n\}$ are some sequences generated by*

$$\begin{cases} x_1 \in C, \\ z_n = \mu_n x_n + (1 - \mu_n) P_C(x_n - \lambda_n M x_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1 - b_n)(a_n S z_n + (1 - a_n) T x_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)–(d_4) hold, then $\{x_n\}$ converges strongly to a point $p \in \Omega$, which is the unique solution to the variational inequality $\langle (I - f)p, x - p \rangle \geq 0$ for all $x \in \Omega$.

Proof Put $F = A = 0$, $B = \partial I_C$, and $r_n = 1$ for all n in Theorem 3.1. Since $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$, we obtain the desired result. □

We can see that Theorem 3.5 extends Theorem 11 in [37]. The next result reduces the related result of [38].

Theorem 3.6 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let M be a β-inverse strongly monotone map from C into H, and let B be a maximal monotone operator on H with domain contained in C. Assume that $S : C \to C$ is a nonexpansive mapping such that $\Omega = F(S) \cap \mathrm{VI}(C, M) \neq \emptyset$ and $f : C \to C$ is a contraction map with the constant $\rho \in (0, 1)$. Suppose that $\{b_n\}$ and $\{\mu_n\}$ are some sequences in $(0, 1)$ and that $\{x_n\}$ and $\{z_n\}$ are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ z_n = \mu_n x_n + (1 - \mu_n) P_C(x_n - \lambda_n M x_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1 - b_n) S z_n] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)–(d_4) hold, then $\{x_n\}$ converges strongly to a point $p \in \Omega$, which is the unique solution to the variational inequality $\langle (I - f)p, x - p \rangle \geq 0$ for all $x \in \Omega$.

Proof Put $F = A = 0$, $B = \partial I_C$, $T = I$, and $r_n = 1$ for all n in Theorem 3.1. Since $J_{\lambda_n} = P_C$ for all $\lambda_n > 0$, we obtain the desired result. □

Theorem 3.7 *Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), let A be an α-inverse strongly monotone mapping from C into H, and let $\psi : C \to C$ be a β-strict pseudo-contraction. Assume that $S, T : C \to C$ are two nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap F(\psi) \neq \emptyset$, and $f : C \to C$ is a contraction map with the constant $\rho \in (0, 1)$. Suppose that $\{b_n\}$, $\{a_n\}$ and $\{\mu_n\}$ are some* *sequences in* $(0,1)$ *and that* $\{x_n\}$ *and* $\{z_n\}$ *are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ z_n = \mu_n x_n + (1-\mu_n)((1-\lambda_n)x_n + \lambda_n \psi x_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1-b_n)(a_n S z_n + (1-a_n)T x_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)*–*(d_4) *hold and* $0 < c < \lambda_n < d < 1-\beta$ *for all* n, *then* $\{x_n\}$ *converges strongly to a point* $p \in \Omega$, *which is the unique solution to the variational inequality* $\langle (I-f)p, x-p\rangle \geq 0$ *for all* $x \in \Omega$.

Proof Put $F = A = 0$, $r_n = 1$, and $M = I - \psi$. Then M is $\frac{1-\beta}{2}$-inverse-strongly monotone map, $F(\psi) = VI(C,M)$, and $P_C(x_n - \lambda_n M x_n) = (1-\lambda_n)x_n + \lambda_n \psi x_n$ for all n. Now by using Theorem 3.5 we obtain the desired result. □

Note that Theorem 3.7 reduces the result of [39].

Theorem 3.8 *Let* C *be a nonempty closed convex subset of* H, *let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A_1)*–*(A_4), *let* A *be an* α*-inverse strongly monotone mapping from* C *into* H, *and let* $\psi : C \to C$ *be a* β*-strict pseudo-contraction. Assume that* $S, T : C \to C$ *are two nonexpansive mappings such that* $\Omega = F(T) \cap F(S) \cap F(\psi) \cap \mathrm{GEP}(F,A) \neq \emptyset$ *and* $f : C \to C$ *is a contraction map with the constant* $\rho \in (0,1)$. *Suppose that* $\{b_n\}$, $\{a_n\}$, *and* $\{\mu_n\}$ *are some sequences in* $(0,1)$ *and that* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle A x_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad (\forall y \in C), \\ z_n = \mu_n x_n + (1-\mu_n)((1-\lambda_n)y_n + \lambda_n \psi y_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1-b_n)(a_n S z_n + (1-a_n)T y_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)*–*(d_4) *hold and* $0 < c < \lambda_n < d < 1-\beta$ *for all* n, *then* $\{x_n\}$ *converges strongly to a point* $p \in \Omega$, *which is the unique solution to the variational inequality* $\langle (I-f)p, x-p\rangle \geq 0$ *for all* $x \in \Omega$.

Proof Put $M = I - \psi$. Then M is $\frac{1-\beta}{2}$-inverse-strongly monotone mapping, $F(\psi) = VI(C,M)$, and $P_C(y_n - \lambda_n M y_n) = (1-\lambda_n)y_n + \lambda_n \psi y_n$ for all n. By using Theorem 3.1 we obtain the desired result. □

We can check that Theorem 3.8 reduces the result of [40].

Theorem 3.9 *Let* C *be a nonempty closed convex subset of* H, *let* M *be a* β*-inverse strongly monotone map from* C *into* H, *and let* B *be a maximal monotone operator on* H *with domain contained in* C. *Assume that* $S, T : C \to C$ *are two nonexpansive mappings such that* $\Omega = F(T) \cap F(S) \cap (M+B)^{-1}(0) \neq \emptyset$. *Suppose that* $\{b_n\}$, $\{a_n\}$, *and* $\{\mu_n\}$ *are some sequences* *in* $(0,1)$ *and that* $\{x_n\}$ *and* $\{z_n\}$ *are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ z_n = \mu_n x_n + (1-\mu_n) J^B_{\lambda_n}(x_n - \lambda_n M x_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1-b_n)(a_n S z_n + (1-a_n)Tx_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)*–*(d_4) *hold, then* $\{x_n\}$ *converges strongly to a point* $p \in \Omega$*, which is the unique solution to the variational inequality* $\langle (I-f)p, x-p\rangle \geq 0$ *for all* $x \in \Omega$*.*

Proof It is sufficient put $F = A = 0$ and $r_n = 1$ for all n in Theorem 3.1. □

We can see that Theorem 3.9 reduces the result of [31]. Let $g : H \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous proper function. Put $B = \partial g$, where ∂ denotes subdifferential of g. Then B is a maximal monotone operator, and $0 \in \partial f(x)$ is equivalent to $g(x') = \min_{x\in C} g(x)$ [24, 26]. Recall that the subdifferential of g at x is defined by

$$\partial g(x) := \bigl\{ v \in H : g(y) \geq g(x) + \langle v, y-x\rangle \text{ for all } y \in H \bigr\}.$$

Theorem 3.10 *Let* C *be a nonempty closed convex subset of* H*, let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A_1)*–*(A_4)*,* A *be an* α*-inverse strongly monotone mapping from* C *into* H*, and let* $g : H \to (-\infty, +\infty]$ *be a proper convex lower semicontinuous function. Assume that* $S, T : C \to C$ *are two nonexpansive mappings such that* $\Omega = F(T) \cap F(S) \cap (\partial f)^{-1}(0) \cap \mathrm{GEP}(F,A) \neq \emptyset$*. Suppose that* $\{b_n\}$*,* $\{a_n\}$*, and* $\{\mu_n\}$ *be sequences in* $(0,1)$ *and that* $\{x_n\}$*,* $\{y_n\}$*,* $\{u_n\}$*, and* $\{z_n\}$ *are the sequences generated by*

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \langle A x_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad (\forall y \in C), \\ u_n = \arg\min_{w\in H}\{g(w) + \frac{\|w-y_n\|^2}{2\lambda_n}\}, \\ z_n = \mu_n x_n + (1-\mu_n) u_n, \\ x_{n+1} = P_C[b_n f(x_n) + (1-b_n)(a_n S z_n + (1-a_n)Ty_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)*–*(d_4) *hold and* $0 < c < \lambda_n < d < \infty$ *for all* n*, then* $\{x_n\}$ *converges strongly to a point* $p \in \Omega$*, which is the unique solution to the variational inequality* $\langle (I-f)p, x-p\rangle \geq 0$ *for all* $x \in \Omega$*.*

Proof Put $M = 0$. Then by Theorem 3.1 the desired result immediately follows. □

Put $A = 0$ in Theorem 3.1. Then we obtain next theorem, which reduces the result of [41].

Theorem 3.11 *Let* C *be a nonempty closed convex subset of* H*, let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A_1)*–*(A_4)*, let* M *be a* β*-inverse strongly monotone map from* C *into* H*, and let* B *be a maximal monotone operator on* H *with domain contained in* C*. Assume that* $S, T : C \to C$ *are two nonexpansive mappings such that* $\Omega = F(T) \cap F(S) \cap (M +$

$B)^{-1}(0) \cap \mathrm{GEP}(F) \neq \emptyset$ and $f : C \to C$ is a contraction map with the constant $\rho \in (0,1)$. Suppose that $\{b_n\}$, $\{a_n\}$, and $\{\mu_n\}$ are some sequences in $(0,1)$ and that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are the sequences generated by

$$\begin{cases} x_1 \in C, \\ F(y_n, y) + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ z_n = \mu_n x_n + (1-\mu_n)J^B_{\lambda_n}(y_n - \lambda_n M y_n), \\ x_{n+1} = P_C[b_n f(x_n) + (1-b_n)(a_n S z_n + (1-a_n)T y_n)] \quad (\forall n \geq 1). \end{cases}$$

If (d_1)–(d_4) hold and $0 < c < \lambda_n < d < \infty$ for all n, then $\{x_n\}$ converges strongly to a point $p \in \Omega$, which is the unique solution to the variational inequality $\langle (I-f)p, x-p\rangle \geq 0$ for all $x \in \Omega$.

Here we provide an example to illustrate Theorem 3.1.

Example 3.1 Let $H = \mathbb{R}$ with Euclidean norm and usual Euclidean inner product. Put $C := [-1,\infty)$, $Sx = \frac{x}{2}$, $Tx = \frac{x}{3}$, $Bx = \log(x+1)$, $Mx = 4x$, $\beta = \frac{1}{4}$, $F(x,y) = y-x$, $\alpha = \frac{1}{3}$, and $Ax = 3x-1$ for all x. It is clear that S and T are nonexpansive, M is a β-inverse strongly monotone mapping, B is a maximal monotone operator, F is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), A is an α-inverse strongly monotone mapping, and $0 \in \Omega = F(T) \cap F(S) \cap (M + B)^{-1}(0) \cap \mathrm{GEP}(F,A)$. Now by using Theorem 3.1 the sequence $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution to the variational inequality $\langle (I-f)q, x-q\rangle \geq 0$ for all $x \in \Omega$.

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