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Wavelet approximation of a function using Chebyshev wavelets

H.K. Nigam[1*], R.N. Mohapatra[2] and K. Murari[1]

*Correspondence:
hknigam@cusb.ac.in
[1] Department of Mathematics, Central University of South Bihar, Panchanpur, Gaya, Bihar 824236, India
Full list of author information is available at the end of the article

Abstract

In this paper, we estimate the best wavelet approximations of a function f having bounded second derivatives and bounded higher-order derivatives using Chebyshev wavelets of third and fourth kinds.

MSC: 41A30; 42C40

Keywords: Wavelet approximation; Chebyshev wavelets; Admissibility conditions; Functions of bounded derivatives

1 Introduction

In recent years, wavelets have found their way into many different fields of science and engineering; particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentation, time-frequency analysis, and fast algorithms for easy implementation. Wavelets allow an accurate representation of variety of functions and operators.

The wavelet approximation technique is a recent tool to detect and analyze abrupt change in seismic signal processing. The wavelet approximation of a function by Haar wavelet has been determined by Devore [2], Debnath [1], Meyer [7], Morlet [11], and Lal and Kumar [4].

Chebyshev polynomials have become increasingly crucial in approximation theory. It is well known that there are four kinds of Chebyshev polynomials, and they all are particular cases of the more widely known class of Jacobi polynomials. The first and second kind Chebyshev polynomials are particular cases of symmetric Jacobi polynomials (i.e., ultraspherical polynomials), whereas third and fourth kinds of Chebyshev polynomials are particular cases of the nonsymmetric Jacobi polynomials (see Mastroianni and Milovanović [6, pp. 131–140]).

Note that a good amount of work on Chebyshev polynomials of the first kind $T_n(x)$ and the second kind $U_n(x)$ and their applications has already been done. But a very few research work has appeared on the Chebyshev polynomials of third and fourth kinds. We see that the Chebyshev polynomials of third kind $V_n(x)$ and fourth kind $W_n(x)$ and their applications are highly important in many areas, including wavelet approximation of certain functions.

It is important to note that $V_n(x)$ and $W_n(x)$ can be useful in situations in which singularities occur at one end point (+1 or -1) but not at the other.

The Chebyshev wavelet approximation method provides the best approximation of a certain function belonging to an approximate class. This motivates us to consider the Chebyshev wavelets of third and fourth kinds to estimate the error of approximation of a function.

Therefore, in this paper, we obtain the best wavelet approximation of a function f by shifted Chebyshev wavelets. In fact, we prove four theorems. In the first two theorems, we obtain the approximation of a function f having bounded second-order derivative and bounded mth derivative using shifted third kind Chebyshev wavelets. In the other two theorems, we obtain the best wavelet approximation of a function f having second-order derivative and bounded mth derivative using shifted fourth kind Chebyshev wavelets. It is important to note that the estimate of a function having more bounded derivatives is better and sharper than the estimate having less bounded derivatives, so comparison of estimated approximation has a significant importance in wavelet analysis.

The outline of the paper is as follows. In Sect. 2, we describe the Chebyshev polynomials and shifted Chebyshev polynomials of third and fourth kinds. In this section, we also define the functional approximation, projection, and wavelet approximation. Four our main theorems are given in Sect. 3. In Sect. 4, we present their proofs. Two corollaries are deduced in Sec. 5. In the last Sect. 6, we conclude our results.

2 Definitions

2.1 Chebyshev polynomials of third and fourth kinds

The Chebyshev polynomial of third kind is a polynomial of degree n given by

$$V_n(x) = \frac{\cos(m + \frac{1}{2})\theta}{\cos(\frac{\theta}{2})}, \tag{1}$$

and the Chebyshev polynomial of fourth kind is a polynomial of degree n given by

$$W_n(x) = \frac{\sin(m + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})}, \tag{2}$$

where $x = \cos\theta$.

Examples of Chebyshev polynomials of third and fourth kinds

Using (1), we get

$$\begin{aligned} &V_0(x) = 1, \qquad V_1(x) = 2x - 1, \qquad V_2(x) = 4x^2 - 2x - 1, \\ &V_3(x) = 8x^3 - 4x^2 - 4x + 1, \quad \text{etc.} \end{aligned}$$

and using (2), we get

$$\begin{aligned} &W_0(x) = 1, \qquad W_1(x) = 2x + 1, \qquad W_2(x) = 4x^2 + 2x - 1, \\ &W_3(x) = 8x^3 + 4x^2 - 4x - 1, \quad \text{etc.} \end{aligned}$$

Remark 1 The polynomials $V_n(x)$ and $W_n(x)$ are, in fact, rescalings of two particular Jacobi polynomials $P_n^{\alpha,\beta}(x)$ with $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$ and vice versa. Explicitly,

$$\binom{2n}{n} V_n(x) = 2^{2n} P_n^{(-\frac{1}{2},\frac{1}{2})}(x);$$
$$\binom{2n}{n} W_n(x) = 2^{2n} P_n^{(\frac{1}{2},-\frac{1}{2})}(x).$$

These polynomials also may be efficiently generated by using the recurrence relation $W_n(x) = (-1)^n V_n(-x)$ (see [3, 8, 10] for application in numerical integration).

Since

$$\cos\left(n+\frac{1}{2}\right)\theta + \cos\left(n-2+\frac{1}{2}\right)\theta = 2\cos\theta\cos\left(n-1+\frac{1}{2}\right)\theta \tag{3}$$

and

$$\sin\left(n+\frac{1}{2}\right)\theta + \sin\left(n-2+\frac{1}{2}\right)\theta = 2\sin\theta\cos\left(n-1+\frac{1}{2}\right)\theta, \tag{4}$$

it immediately follows that

$$V_n(x) = 2xV_{n-1} - V_{n-2}(x), \quad n = 2, 3, \ldots, \tag{5}$$

with

$$V_0(x) = 1, \qquad V_1(x) = 2x - 1,$$

and

$$W_n(x) = 2xW_{n-1} - W_{n-2}(x), \quad n = 2, 3, \ldots, \tag{6}$$

with

$$W_0(x) = 1, \qquad W_1(x) = 2x + 1.$$

It is clear from (5) and (6) that both $V_n(x)$ and $W_n(x)$ are polynomials of degree n in x, in which all powers of x are present, and in which the leading coefficients (of x) are equal to 2^n.

The polynomials $V_n(x)$ and $W_n(x)$ are orthogonal on $(-1, 1)$, that is,

$$\int_{-1}^{1} w_1(x)V_k(x)V_j(x)\,dx = \int_{-1}^{1} w_2(x)W_k(x)W_j(x)\,dx \tag{7}$$
$$= \begin{cases} \pi & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

where

$$w_1(x) = \sqrt{\frac{1+x}{1-x}}, \qquad w_2(x) = \sqrt{\frac{1-x}{1+x}}. \tag{9}$$

2.2 Shifted Chebyshev polynomials of third and fourth kinds

The shifted polynomials V_n^* and W_n^* of third and fourth kinds, respectively, are defined as

$$V_n^*(x) = V_n(2x-1), \tag{10}$$

$$W_n^*(x) = W_n(2x-1). \tag{11}$$

The orthogonal relations of $V_n^*(t)$ and $W_n^*(t)$ on $[0,1]$ are given by

$$\int_0^1 w_1^* V_n^*(t) V_m^*(t)\,dx = \int_0^1 w_2^* W_n^*(t) W_m^*(t)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \tag{12}$$

where

$$w_1^* = \sqrt{\frac{t}{1-t}}, \qquad w_2^* = \sqrt{\frac{1-t}{t}} \quad \text{(see [5] and [9])}. \tag{13}$$

According to (10) and (11) and the relation $W_n(x) = (-1)^n V_n(-x)$, we can conclude that

$$W_n^*(x) = (-1)^n V_n(1-x),$$

so that the orthogonal polynomials with respect to w_2^* can be obtained from those orthogonal with respect to w_1^* by the previous simple substitution $x := 1 - x$ and the factor $(-1)^n$ (in order to get all positive leading coefficients). Therefore it suffices to consider only one of these weights, say, w_1^*.

The polynomials $V_n^*(x)$ satisfy the following three-term recurrence relation:

$$V_n^*(x) = 2(2x-1)V_{n-1}^*(x) - V_{n-2}^*(x), \quad n = 2, 3, \ldots,$$

with $V_0^*(x) = 1$ and $V_1^*(x) = 4x - 3$. The next polynomials are

$$V_2^*(x) = 16x^2 - 20x + 5,$$

$$V_3^*(x) = 64x^3 - 112x^2 + 56x - 7,$$

$$V_4^*(x) = 256x^4 - 576x^3 + 432x^2 - 120x + 9,$$

$$V_5^*(x) = 1024x^5 - 2816x^4 + 2816x^3 - 1232x^2 + 220x - 11,$$

and so on.

2.3 Shifted Chebyshev wavelets of third and fourth kind

When the dilation parameter a and the translation parameter b vary continuously, then we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0. \tag{14}$$

Each of the third and fourth kind of Chebyshev wavelets $\psi_{n,m}(t) := \psi(k,n,m,t)$ has four arguments with $k, n \in \mathbb{N}$, m is the order of the polynomial $V_m^*(t)$ or $W_m^*(t)$, and t is the normalized time. The Chebyshev wavelets of third and fourth kinds are defined explicitly on the interval $[0,1]$ by

$$\psi_{m,n} = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} V_m^*(2^k t - \hat{n}), & \text{where } t \in [\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}], \\ \quad 1 \le \hat{n} \le 2^{k-1}, k = 1,2,\dots,\hat{n}, \hat{n} = 2n-1, 0 \le m \le M, \\ 0 \quad \text{otherwise}, \end{cases}$$

and

$$\psi_{m,n} = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} W_m^*(2^k t - \hat{n}), & \text{where } t \in [\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}], \\ \quad 1 \le \hat{n} \le 2^{k-1}, k = 1,2,\dots,\hat{n}, \hat{n} = 2n-1, 0 \le m \le M, \\ 0 \quad \text{otherwise}, \end{cases}$$

respectively.

2.4 Functional approximation

A function $f \in L^2(\mathbb{R})$ defined over $[0,1]$ is expanded in terms of Chebyshev wavelet series as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t), \tag{15}$$

where

$$\begin{aligned} c_{n,m} &= \langle f(t), \psi_{n,m}(t) \rangle_{w_i^*} \\ &= \int_0^1 w_i^* f(t) \psi_{n,m}(t)\, dt, \end{aligned} \tag{16}$$

with weights w_i^*, $i = 1,2$, defined in (13). If an infinite series in (15) is truncated, then it can be written as

$$S_{2^k,M}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) = C^T \Psi(t),$$

where C and $\Psi(t)$ are two $2^k M \times 1$ matrices given by

$$C = [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, \dots, c_{2,M-1}, \dots, c_{2^k,1}, \dots, c_{2^k,M-1}]$$

and

$$\Psi(t) = [\psi_{1,0}, \dots, \psi_{1,M-1}, \psi_{2,0}, \dots, \psi_{2,M-1}, \dots, \psi_{2^k,0}, \dots, \psi_{2^k,M-1}].$$

2.5 Multiresolution analysis

A sequence of closed subspaces V_j of $L^2(\mathbb{R})$, $j \in \mathbb{Z}$, is called a multiresolution in $L^2(\mathbb{R})$ if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$;

(ii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$;

(iii) $f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$;

(iv) $\bigcup_{-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$, and $\bigcap_{-\infty}^{\infty} V_j = 0$;

(v) There exists a function $\varphi \in V_0$ such that the collection $\{\varphi(x-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0 ([1]).

2.6 Projection $P_n(f)$

Let $P_n(f)$ be the orthogonal projection of $L^2(R)$ onto V_n. Then

$$P_n f = \sum_{-\infty}^{\infty} a_{n,k}\phi_{n,k}, \quad n = 1, 2, 3, \ldots,$$

$$a_{n,k} = \langle f, \phi_{n,k}\rangle.$$

Thus

$$P_n(f) = \sum_{-\infty}^{\infty} \langle f, \phi_{n,k}\rangle \phi_{n,k}, \quad n = 1, 2, 3, \ldots \text{ ([12])}.$$

2.7 Wavelet approximation

The wavelet approximation under the supremum norm is defined by

$$E_n(f) = \|f - P_n f\|_\infty = \sup_x \bigl\| \bigl(f(x) - P_n f(x)\bigr) \bigr\|_\infty \quad \text{([13])},$$

$$\|f\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \bigl|f(x)\bigr|^r \, dx \right\}^{\frac{1}{r}}, \quad 1 \le r < \infty.$$

The degree of wavelet approximation $E_n(f)$ of f by $P_n f$ under the norm $\|\cdot\|_r$ is given by

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_r \quad \text{([13])}.$$

Remark 2 If $E_n(f) \to 0$ as $n \to \infty$, then $E_n(f)$ is called the best approximation of f [13].

3 Main theorems

In this paper, we prove the following theorems.

Theorem 3.1 *If a continuous function $f \in L^2_{w_1^*}[0,1]$, $w_1^* = \sqrt{\frac{t}{1-t}}$, such that $|f''(t)| \le P < \infty$ is expanded as an infinite series of third kind Chebyshev wavelet series*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t), \quad \textit{where } c_{m,n} = \langle f, \psi_{m,n}\rangle_{w_1^*},$$

then the Chebyshev wavelet approximation $E_{2^{k-1},M}(t)$ *of* f *by* $(2^{k-2},M)$*th partial sums*

$$S_{2^{k-2},M}=\sum_{n=1}^{2^{k-2}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)$$

of its Chebyshev wavelet series in $L^2_{w_1^*}[0,1]$ *is given by*

$$\begin{aligned}E_{2^{k-1},M}(f)&=\|f-S_{2^{k-2},M}\|_2\\&=\left\|f-\sum_{n=1}^{2^{k-2}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right\|_2=O\left(\frac{1}{2^{2k}(M-1)^{\frac{3}{2}}}\right),\quad M>1.\end{aligned}$$

Theorem 3.2 *If a continuous function* $f\in L^2_{w_1^*}[0,1]$, $w_1^*=\sqrt{\frac{t}{1-t}}$, *is such that* $\sup_{t\in[0,1]}|f^M(t)|<\infty$, *then the Chebyshev wavelet approximation of* $E_{2^{k-1},M}(t)$ *of* f *by* $(2^{k-1},M)$*th partial sums*

$$S_{2^{k-2},M}=\sum_{n=1}^{2^{k-2}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)$$

of its Chebyshev wavelet series in $L^2_{w_1^*}[0,1]$ *is given by*

$$E_{2^{k-1},M}(f)=\|f-S_{2^{k-2},M}\|_2=\left\|f-\sum_{n=1}^{2^{k-2}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right\|_2=O\left(\frac{1}{M!2^{M(k+1)}}\right).$$

Theorem 3.3 *If a continuous function* $f\in L^2_{w_2^*}[0,1]$, $w_2^*=\sqrt{\frac{1-t}{t}}$, *is such that* $|f''(t)|\le P<\infty$ *can be expanded as an infinite series of fourth kind Chebyshev wavelet series*

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}(t),\quad \textit{where } c_{m,n}=\langle f,\psi_{m,n}\rangle_{w_2^*},$$

then the Chebyshev wavelet approximation $E_{2^{k-1},M}(t)$ *of* f *by* $(2^{k-2},M)$*th partial sums*

$$S_{2^{k-2},M}=\sum_{n=1}^{2^{k-2}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)$$

of its Chebyshev wavelet series in $L^2_{w_2^*}[0,1]$ *is given by*

$$\begin{aligned}E_{2^{k-1},M}(f)&=\left\|(f-S_{2^{k-2},M})\right\|_2\\&=\left\|f-\sum_{n=1}^{2^{k-2}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}(t)\right\|_2=O\left(\frac{1}{2^{2k}(M-1)^{\frac{3}{2}}}\right),\quad M>1.\end{aligned}$$

Theorem 3.4 *If a continuous function* $f\in L^2_{w_2^*}[0,1]$, $w_2^*=\sqrt{\frac{1-t}{t}}$, *is such that* $\sup_{t\in[0,1]}|f^M(t)|<\infty$, *then the Chebyshev wavelet approximation of* $E_{2^{k-1},M}(t)$ *of* f *by*

$(2^{k-1}, M)$th partial sums

$$S_{2^{k-2},M} = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)$$

of its Chebyshev wavelet series in $L^2_{w_2^*}[0,1]$ is given by

$$E_{2^{k-1},M}(f) = \|f - S_{2^{k-2},M}\|_2 = \left\| f - \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)\right\|_2 = O\left(\frac{1}{M!2^{M(k+1)}}\right).$$

4 Proof of the main theorems

Proof of Theorem 3.1 Chebyshev wavelet series $f \in L^2_{w_1^*}[0,1]$ is given by

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t) \\ &= \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) + \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}(t) \\ &\quad + \sum_{n=2^{k-2}+1}^{\infty} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) + \sum_{n=2^{k-2}+1}^{\infty} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}(t) \\ &= S_{2^{k-2},M} + \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}(t) \\ &\quad + \sum_{n=2^{k-2}+1}^{\infty} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) + \sum_{n=2^{k-2}+1}^{\infty} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}(t), \end{aligned} \tag{17}$$

where

$$\psi_{m,n} = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} V_m^*(2^k t - \hat{n}), & \text{where } t \in [\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}], \\ \quad 1 \le \hat{n} \le 2^{k-1}, 0 \le m \le M, \hat{n} = 2n-1, & \\ 0 \quad \text{otherwise.} & \end{cases} \tag{18}$$

From Chebyshev wavelet we have

$$\frac{\hat{n}-1}{2^{k-1}} \le t \le \frac{\hat{n}}{2^{k-1}} \quad \Rightarrow \quad \frac{2n-2}{2^{k-1}} \le t \le \frac{2n-1}{2^{k-1}} \quad \text{since } \hat{n} = 2n-1. \tag{19}$$

Let $n = 2^{k-2} + 1$. Then (19) becomes

$$\frac{2(2^{k-2}+1)-2}{2^{k-1}} \le t \le \frac{2(2^{k-2}+1)-1}{2^{k-1}} \quad \Rightarrow \quad 1 \le t < 1 + \frac{1}{2^{k-1}} \quad \text{for all } k.$$

Since $\psi_{n,m}$ vanishes outside the interval $[0,1]$, the third and fourth terms of (17) become 0. Thus (17) becomes

$$f = S_{2^{k-2},M} + \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}. \tag{20}$$

Now (20) can be written as

$$\begin{aligned}
&\|f - S_{2^{k-2},M}\|_2^2 \\
&\quad = \left\| \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m} \right\|_2^2 = \left\langle \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}, \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} c_{n,m}\psi_{n,m} \right\rangle \\
&\quad = \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} |c_{n,m}|^2 \|\psi_{n,m}\|_2^2 \\
&\quad\quad \text{(other terms vanish due to the orthogonality of } \psi_{n,m}\text{).} \qquad (21)
\end{aligned}$$

Now

$$\begin{aligned}
\|\psi_{n,m}\|_2^2 &= \int_{-\infty}^{\infty} \|\psi_{n,m}\|^2\,dt = \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}}{2^k}} \left(\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\right)^2 V_m^*\big(2^k t - \hat{n}\big)\overline{V_m^*\big(2^k t - \hat{n}\big)} w_1^*\,dt \\
&= \frac{2^{k+1}}{\pi} \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}}{2^k}} V_m^*\big(2^k t - \hat{n}\big)\overline{V_m^*\big(2^k t - \hat{n}\big)} w_1^*\,dt. \qquad (22)
\end{aligned}$$

Let $2^k t - \hat{n} = u$. Then (22) becomes

$$\|\psi_{n,m}\|_2^2 = \frac{2^{k+1}}{\pi} \int_0^1 \big|V_m^*(u)\big|^2 w_1^* \frac{du}{2^k}.$$

Using (7), we get

$$\|\psi_{n,m}\|_2^2 = \frac{2^{k+1}}{\pi} \times \frac{1}{2^k} \times \frac{\pi}{2} = 1. \qquad (23)$$

From (21) and (23) we get

$$\|f - S_{2^{k-2},M}\|_2^2 = \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} |c_{n,m}|^2. \qquad (24)$$

Now we have

$$c_{n,m} = \left(\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\right) \int_{\frac{\hat{n}-1}{2^{k-1}}}^{\frac{\hat{n}}{2^{k-1}}} f(t) V_m^*\big(2^k t - \hat{n}\big) w_1^*\big(2^k t - \hat{n}\big)\,dt. \qquad (25)$$

Considering $2^k t - \hat{n} = \cos\theta$, we get

$$\begin{aligned}
c_{n,m} &= \left(\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}2^k}\right) \int_0^{\pi} f\left(\frac{\cos\theta + n}{2^k}\right) \frac{\cos(m+\frac{1}{2})\theta}{\cos(\frac{\theta}{2})} \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \times \sin\theta\,d\theta \\
&= \left(\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}2^k}\right) \int_0^{\pi} f\left(\frac{\cos\theta + n}{2^k}\right) \frac{\cos(m+\frac{1}{2})\theta}{\cos(\frac{\theta}{2})} \sqrt{\frac{2\cos^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}} \\
&\quad \times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta
\end{aligned}$$

$$= \left(\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}2^k}\right)\int_0^\pi f\left(\frac{\cos\theta+n}{2^k}\right)2\cos\left(m+\frac{1}{2}\right)\theta\cos\left(\frac{\theta}{2}\right)d\theta$$

$$= \left(\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\right)\int_0^\pi f\left(\frac{\cos\theta+n}{2^k}\right)\left\{\cos(m+1)\theta+\cos(m\theta)\right\}d\theta. \tag{26}$$

Integrating (26) by parts, we get

$$\begin{aligned}
&= \left(\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\right)\left[f\left(\frac{\cos\theta+n}{2^k}\right)\left(\frac{\sin(m+1)\theta}{m+1}+\frac{\sin m\theta}{m}\right)\right]_0^\pi \\
&\quad - \left(\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}}\right)\int_0^\pi f'\left(\frac{\cos\theta+n}{2^k}\right)\left(\frac{-\sin\theta}{2^k}\right)\left(\frac{\sin(m+1)\theta}{m+1}+\frac{\sin m\theta}{m}\right)d\theta \\
&= \left(\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}2^k}\right)\left[\int_0^\pi f'\left(\frac{\cos\theta+n}{2^k}\right)\left\{\frac{2\sin\theta\sin(m+1)\theta}{2(m+1)}+\frac{2\sin\theta\sin m\theta}{2m}\right\}d\theta\right] \\
&= \left(\frac{1}{\sqrt{\pi}\times 2^{\frac{3k+1}{2}}}\right)\int_0^\pi\left[f'\left(\frac{\cos\theta+n}{2^k}\right)\left\{\frac{\cos(m\theta)-\cos(m+2)\theta}{(m+1)}\right.\right. \\
&\quad \left.\left.+\frac{\cos(m-1)\theta-\cos(m+1)\theta}{m}\right\}\right]d\theta.
\end{aligned} \tag{27}$$

Now integrating (27) by parts, we get

$$\begin{aligned}
c_{m,n} &= \left(\frac{1}{\sqrt{\pi}\times 2^{\frac{5k+1}{2}}}\right)\int_0^\pi\left[f''\left(\frac{\cos\theta+n}{2^k}\right)(-\sin\theta)\left\{\frac{1}{m+1}\left(\frac{\sin(m)\theta}{m}-\frac{\sin(m+2)\theta}{m+2}\right)\right.\right. \\
&\quad \left.\left.+\frac{1}{m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)\right\}\right]d\theta.
\end{aligned} \tag{28}$$

Applying the given condition $f''(x) \le P$ in (28), we get

$$\begin{aligned}
|c_{m,n}| &\le \left|\left(\frac{P}{\sqrt{\pi}\times 2^{\frac{5k+1}{2}}}\right)\int_0^\pi\left[\sin\theta\left\{\frac{1}{m+1}\left(\frac{\sin m\theta}{m}-\frac{\sin(m+2)\theta}{m+2}\right)\right.\right.\right. \\
&\quad \left.\left.\left.+\frac{1}{m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)\right\}\right]d\theta\right| \\
&\le \left(\frac{P}{\sqrt{\pi}\times 2^{\frac{5k+1}{2}}}\right)\left|\int_0^\pi\left[\sin\theta\left\{\frac{1}{m+1}\left(\frac{\sin m\theta}{m}-\frac{\sin(m+2)\theta}{m+2}\right)\right.\right.\right. \\
&\quad \left.\left.\left.+\frac{1}{m}\left(\frac{\sin(m-1)\theta}{m-1}-\frac{\sin(m+1)\theta}{m+1}\right)\right\}\right]d\theta\right| \\
&\le \frac{P\pi}{\sqrt{\pi}\times 2^{\frac{5k+1}{2}}}\left\{\frac{1}{m+1}\left(\frac{1}{m}+\frac{1}{m+2}\right)+\frac{1}{m}\left(\frac{1}{m-1}+\frac{1}{m+1}\right)\right\} \\
&\le \frac{P\sqrt{2\pi}}{2^{\frac{5k}{2}}}\left\{\frac{1}{m(m+2)}+\frac{1}{(m-1)(m+1)}\right\} \\
&\le \frac{P\sqrt{2\pi}}{2^{\frac{5k}{2}}}\left\{\frac{4}{(m-1)(m+1)}\right\} \\
&\le \frac{P\sqrt{2\pi}}{2^{\frac{5k}{2}}}\left\{\frac{4}{m(m-1)}\right\};
\end{aligned}$$

$$
\begin{aligned}
|c_{n,m}|^2 &\le \left(\frac{4P\sqrt{2\pi}}{2^{\frac{5k}{2}}m(m-1)}\right)^2 \\
&= \frac{32P^2\pi}{2^{5k}m^2(m-1)^2} \\
&\le \frac{32P^2\pi}{2^{5k}(m-1)^4}, \quad m > 1.
\end{aligned}
\tag{29}
$$

From (24) and (29) we get

$$
\begin{aligned}
\|(f - S_{2^{k-2},M})\|_2^2 &\le \sum_{n=1}^{2^{k-2}} \sum_{m=M}^{\infty} \frac{32P^2\pi}{2^{5k}(m-1)^4} \\
&\le \frac{32P^2\pi}{2^{4k+1}(M-1)^3}, \quad M > 1.
\end{aligned}
$$

Hence

$$
\|(f - S_{2^{k-2},M})\|_2 = O\left(\frac{1}{2^{2k}(M-1)^{\frac{3}{2}}}\right), \quad M > 1.
$$

This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2 Since a function f is M times differentiable, by Taylor's expansion we have

$$
\begin{aligned}
&f(a+h) = f_{M+1} = f(a) + \frac{h}{1!}f'(a) + \cdots + \frac{h^{M-1}}{(M-1)!}f^{M-1}(a) + \frac{h^M}{M!}f^M(a\theta + h), \\
&f_{M+1} = f_M + \frac{h^M}{M!}f^M(a + \theta h), \quad \text{where } 0 < \theta < 1,
\end{aligned}
$$

where

$$
f_M = f(a) + \frac{h}{1!}f'(a) + \cdots + \frac{h^{M-1}}{(M-1)!}f^{M-1}(a).
$$

Now we write

$$
f_{M+1} - f_M = \frac{h^M}{M!}f^M(a + \theta h), \quad \text{where } 0 < \theta < 1. \tag{30}
$$

Using (30) and dividing the interval $[0,1]$ into subintervals $[\frac{l}{2^{k-1}}, \frac{l+1}{2^{k-1}}]$, we get

$$
\begin{aligned}
\|f - S_{2^{k-2},M}\|_2^2 &= \int_0^1 \left| f(x) - \sum_{l=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m} \right|^2 dx \\
&= \sum_{l=0}^{2^{k-1}-1} \int_{\frac{l}{2^{k-1}}}^{\frac{l+1}{2^{k-1}}} \left| f(x) - \sum_{l=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m} \right|^2 dx
\end{aligned}
$$

$$\leq \int_{\frac{l}{2^{k-1}}}^{\frac{l+1}{2^{k-1}}} \left(\frac{1}{M!} \left(\frac{1}{2^{k-1}} \right)^M \sup_{x\in[0,1]} \left| f^M(x) \right| \right)^2 dx$$

$$= \int_0^1 \left(\frac{1}{M!} \right)^2 \left(\frac{1}{2^{M(k-1)}} \right)^2 \sup_{x\in[0,1]} \left| f^M(x) \right|^2 dx.$$

Now

$$\|f - S_{2^{k-2},M}\|_2^2 = \left(\frac{1}{M!} \right)^2 \left(\frac{1}{2^{M(k-1)}} \right)^2 \sup_{x\in[0,1]} \left| f^M(x) \right|^2.$$

Hence

$$\|f - S_{2^{k-2},M}\|_2 \leq \left(\frac{1}{M!} \right) \left(\frac{1}{2^{M(k-1)}} \right) \sup_{x\in[0,1]} \left| f^M(x) \right|.$$

Thus

$$\begin{aligned} E_{2^{k-1},M}(f) &= \|f - S_{2^{k-2},M}\|_2 \\ &\leq \left(\frac{1}{M!} \right) \left(\frac{1}{2^{M(k-1)}} \right) \sup_{x\in[0,1]} \left| f^M(x) \right| \\ &= O\left(\frac{1}{M! 2^{M(k-1)}} \right). \end{aligned}$$

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3 Theorem 3.3 can be proved along the lines of the proof of Theorem 3.1. □

Proof of Theorem 3.4 Theorem 3.4 can be proved along the lines of the proof of Theorem 3.2. □

5 Corollaries

Corollary 5.1 *If a continuous function $f \in L^2_{w_2^*}[0,1]$, $w_2^* = \sqrt{\frac{1-t}{t}}$, such that $|f''(t)| \leq P < \infty$ can be expanded as an infinite series of fourth kind Chebyshev wavelet series*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t), \quad \text{where } c_{m,n} = \langle f, \psi_{m,n} \rangle_{w_2^*},$$

then the Chebyshev wavelet approximation $E_{2^{k-1},M}(t)$ of f by $(2^{k-2}, M)$th partial sums

$$S_{2^{k-2},M} = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t)$$

of its Chebyshev wavelet series in $L^2_{w_2^*}[0,1]$ is given by

$$\begin{aligned} E_{2^{k-1},M}(f) &= \left\| (f - S_{2^{k-2},M}) \right\|_2 \\ &= \left\| f - \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t) \right\|_2 = O\left(\frac{1}{2^{2k} (M-1)^{\frac{3}{2}}} \right), \quad M > 1. \end{aligned}$$

Proof Replacing V_n^* by W_n^* and ω_1^* by ω_2^* in Theorem 3.1, we obtain Corollary 5.1. □

Corollary 5.2 *If a continuous function* $f \in L^2_{w_2^*}[0,1]$, $w_2^* = \sqrt{\frac{1-t}{t}}$, *is such that* $\sup_{t\in[0,1]} |f^M(t)| < \infty$, *then the Chebyshev wavelet approximation of* $E_{2^{k-1},M}(t)$ *of* f *by* $(2^{k-1}, M)$*th partial sums*

$$S_{2^{k-2},M} = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)$$

of its Chebyshev wavelet series in $L^2_{w_2^*}[0,1]$ *is given by*

$$E_{2^{k-1},M}(f) = \|f - S_{2^{k-2},M}\|_2 = \left\| f - \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \right\|_2 = O\left(\frac{1}{M!2^{M(k+1)}}\right).$$

Proof Replacing V_n^* by W_n^* and ω_1^* by ω_2^* in Theorem 3.2, we obtain Corollary 5.2. □

6 Conclusion

1. In our results, the estimate of wavelet approximation of a function having more bounded derivatives is sharper than the estimate of wavelet approximation of a function having less bounded derivatives.
2. In view of Remark 2, our results are best possible.

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Author details
[1]Department of Mathematics, Central University of South Bihar, Panchanpur, Gaya, Bihar 824236, India. [2]Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA.

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