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On L^2-boundedness of Fourier integral operators

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Abstract

Let $T_{a,\varphi}$ be a Fourier integral operator with symbol a and phase φ. In this paper, under the conditions $a(x,\xi) \in L^\infty S_\rho^{n(\rho-1)/2}(\omega)$ and $\varphi \in L^\infty \Phi^2$, the authors show that $T_{a,\varphi}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

MSC: Primary 42B20; secondary 42B37

Keywords: Pseudo-differential operator; Fourier integral operator; Phase function

1 Introduction and main results

Fourier integral operator on $\mathbb{R}^n$ has been studied extensively and is related to many areas in analysis and PDEs. In [1], Sogge considered the Cauchy problem of the hyperbolic equations via the L^p-estimates theory of the Fourier integral operators (also see, for the local smoothing estimates of wave equations, e.g., [2, 3] and the references therein for some recent developments). For the Fourier integral operators with smooth amplitude, the L^2-regularity theory is comparably more progress. In [4] and [5], Eskin and Hörmander found the local and global L^2-regularity theory for Fourier integral operators, respectively. There are also some results for the L^p boundedness of Fourier integral operators with classical symbol and phase (see Littman [6], Miyachi [7], Peral [8], and Beals [9]).

Let $\hat{f}$ be the Fourier transform of f. A Fourier integral operator T is a linear operator of the form

$$T_{a,\varphi}f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi)\hat{f}(\xi)\,d\xi \tag{1.1}$$

with symbol $a(x,\xi)$ and phase $\varphi(x,\xi)$, respectively. In particular, for $\varphi(x,\xi) = \langle x,\xi\rangle$, the operator T_a is a so-called pseudo-differential operator. In [10], Hörmander showed that T_a is bounded in $L^2(\mathbb{R}^n)$, when $a \in S^m_{\rho,\delta}$, $\delta < 1$ and $m \le n(\rho - \delta)/2$. For $a \in S^0_{1,1}$, Ching [11] proved that T_a is not bounded in $L^2(\mathbb{R}^n)$. Moveover, for $a \in S^m_{\rho,1}$, Rodino [12] showed that T_a is bounded in $L^2(\mathbb{R}^n)$ if and only if $m < n(\rho-1)/2$. However, the operator T_a is not always L^2-bounded for $a \in S^{n(\rho-1)/2}_{\rho,1}$; see, for example, [10–12]. The necessary and sufficient conditions of L^2-boundedness of T_a were obtained by Higuchi [13] as $m = n(\rho-1)/2$. It is natural to ask if the corresponding results hold for the Fourier integral operators. Recently, Kenig, David, Salvador, and Wolfgang [14–16] have studied the Fourier integral operators with rough symbol and rough phases, both of which behave in the spatial variable x like an L^∞-function. More precisely, the symbol belongs to the class $L^\infty S^m_\varrho$ whose constituent element a obeys

$$\left\| \partial_\xi^\alpha a(\cdot,\xi)\right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-\varrho|\alpha|}.$$

Under this condition, for $m = \min\{0, \frac{n}{2}(\rho-\delta)\}$, $0 \le \rho \le 1$, $0 \le \delta < 1$, and $a \in S^m_{\rho,\delta}$, Wolfgang [14] proved the global continuity on L^p-space with $p \in [1,\infty]$ of Fourier integral operators. A natural question is L^2-boundedness of Fourier integral operators for $\delta = 1$ and $m = n(\rho - 1)/2$. In this paper, we answer the question and prove the results for the Fourier integral operators.

Our main result could be stated as follows.

Theorem 1.1 *Let $T_{a,\varphi}$ be a Fourier integral operator given by (1.1) with symbol $a(x,\xi) \in L^\infty S^{n(\rho-1)/2}_\rho(\omega)$ and phase function $\varphi \in L^\infty\Phi^2$ satisfying the Lipschitz rough non-degeneracy condition. Then, for $0 \le \rho \le 1$, there exists a positive constant C such that*

$$\|T_{a,\varphi}u\|_{L^2} \le C\|u\|_{L^2}.$$

Here, the symbol class $L^\infty S^{n(\rho-1)/2}_\rho(\omega)$ is defined by Definition 2.2, the phase class $L^\infty\Phi^2$ is given by Definition 2.5, and the Lipschitz rough non-degeneracy condition is defined by Definition 2.6.

Remark 1.1 Here we remark that, for $a \in S^{n(\rho-1)/2}_{\rho,1}$, Higuchi and Nagase [13] pointed out that the boundedness of the pseudo-differential operator T_a from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ is not always true. As the main result in this paper, we give an answer for this problem for the Fourier integral operator $T_{a,\varphi}$. The main idea of our approach is treating the symbol class $L^\infty S^m_\rho(\omega)$, where $m = n(\rho - 1)/2$. In particular, our results of $L^2(\mathbb{R}^n)$-boundedness for $T_{a,\varphi}$ are also the best as far as we know. We also remark that our methods are different from the previous methods; see, for example, [13].

Finally, we make some conventions on notation. Throughout this article, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We sometimes write $A \lesssim B$ as shorthand for $A \le CB$. Let $\mathbb{R}^n$ be an n-dimensional Euclidean space, $x = (x_1,\dots,x_n)$ be a point in $\mathbb{R}^n$, $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$, $\mathbb{N} = \{1,2,\dots\}$, $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, and $\mathbb{Z}^n_+ = (\mathbb{Z}_+)^n$. For any multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}^n_+$, we let

$$|\alpha| = \sum_{j=1}^n \alpha_j, \qquad \alpha+\beta = (\alpha_1+\beta_1,\dots,\alpha_n+\beta_n), \qquad \partial_x^\alpha = \frac{\partial^\alpha}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}},$$

and $\nabla_\xi = (\partial_{\xi_1},\dots,\partial_{\xi_n})$. Also, in the sequel we use the notation

$$|\xi| = \left(\sum_{j=1}^n \xi_j^2\right)^{1/2} \quad \text{and} \quad \langle\xi\rangle = \left(1+|\xi|^2\right)^{1/2}.$$

2 Definitions, notations, and preliminaries

The following definition is just [17].

Definition 2.1 Let $m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$. For any two multi-indices α and β, we assume that the function $a(x,\xi)$ satisfies the following condition:

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \tag{2.1}$$

where $C_{\alpha\beta}$ is a positive constant only dependent on α and β. Let the smooth amplitude $S^m_{\rho,\delta}$ be the set of all smooth functions $a(x,\xi)$ satisfying condition as in (2.1). Then the pseudo-differential operator T_a with the symbol $a(x,\xi) \in S^m_{\rho,\delta}$ is given formally by

$$(T_a f)(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\,d\xi.$$

The following definition for the class $L^\infty S^m_\rho(\omega)$ plays an important role in our setting.

Definition 2.2 Let m be a real number. A function $a(x,\xi)$, which is smooth in the frequency variable ξ and bounded measurable in the spatial variable x, belongs to the symbol class $L^\infty S^m_\rho(\omega)$ if, for all multi-indices α, it satisfies

$$\left\|\partial_\xi^\alpha a(x,\xi)\right\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha\langle\xi\rangle^{m-\rho|\alpha|}\omega\bigl(\langle\xi\rangle\bigr),$$

where $\omega(t)$ satisfies

$$\int_1^\infty \frac{\omega(t)^2}{t}\,dt < \infty, \tag{2.2}$$

and $\omega(t)$ is a nonnegative and decreasing function on $[1,\infty)$.

Remark 2.1 If $\omega(t)$ satisfies (2.1), then $\sum_{j=0}^\infty \omega^2(2^j) < \infty$.

David and Wolfgang [14] gave the class Φ^k as follows.

Definition 2.3 ([14], Φ^k) A real-valued function $\varphi(x,\xi)$ belongs to the class Φ^k if $\varphi(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n_*)$ is positively homogeneous of degree 1 in the frequency variable ξ and satisfies the following condition: for any pair of multi-indices α and β, satisfying $|\alpha| + |\beta| \geq k$, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n_*} |\xi|^{-1+\alpha}\left|\partial_\xi^\alpha \partial_x^\beta \varphi(x,\xi)\right| \leq C_{\alpha,\beta}.$$

In connection to the problem of local boundedness of Fourier integral operators, one considers phase functions $\varphi(x,\xi)$ that are positively homogeneous of degree 1 in the frequency variable ξ for which

$$\left|\det \frac{\partial^2\varphi(x,\xi)}{\partial x_j \partial\xi_k}\right| \neq 0.$$

The latter is referred to as the non-degeneracy condition. However, for the purpose of proving global regularity results, we require a stronger condition than the non-degeneracy condition above.

Definition 2.4 ([14], The strong non-degeneracy condition) A real-valued function $\varphi(x,\xi) \in C^2(\mathbb{R}^n \times \mathbb{R}^n_*)$ satisfies strong non-degeneracy condition if there exists a positive constant c such that

$$\left|\det \frac{\partial^2 \varphi(x,\xi)}{\partial x_j \partial \xi_k}\right| \geq c$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_*$.

Remark 2.2 The phases in class Φ^2 satisfying the strong non-degeneracy condition arise naturally in the study of the equations of hyperbolic type, namely

$$\varphi(x,\xi) = |\xi| + \langle x,\xi\rangle$$

belongs to the class Φ^2 and satisfies the strong non-degeneracy condition.

In [14], they introduced the nonsmooth version of the class Φ^k which will be used in our setting.

Definition 2.5 ([14], $L^\infty\Phi^k$) A real-valued function $\varphi(x,\xi)$ belongs to the phase class $L^\infty\Phi^k$ if it is positively homogeneous of degree 1 and smooth on $\mathbb{R}^n_*$ in the frequency variable ξ, bounded measurable in the spatial variable x, and if for all multi-indices $|\alpha| \geq k$ it satisfies

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n_*} |\xi|^{-1+\alpha}\left|\partial_\xi^\alpha \varphi(x,\xi)\right| \leq C_\alpha.$$

Motivated by [14], we also need a Lipschitz rough non-degeneracy condition as follows.

Definition 2.6 (The Lipschitz rough non-degeneracy condition) A real-valued function satisfies Lipschitz rough non-degeneracy condition if it is C^∞ on $\mathbb{R}^n_*$ in the frequency variable ξ, bounded measurable in the spatial variable x, and there exist positive constants C_1 and C_2 such that, for all $x, y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n_*$,

$$\begin{aligned}
&\left|\partial_\xi \varphi(x,\xi) - \partial_\xi \varphi(y,\xi)\right| \geq C_1|x-y|,\\
&\left|\partial_\xi^\alpha \varphi(x,\xi) - \partial_\xi^\alpha \varphi(y,\xi)\right| \leq C_2|x-y| \quad \text{for } |\alpha| \geq 2.
\end{aligned}$$

3 Proof of the main result

In this section, we shall prove the main result, i.e., Theorem 1.1.

First we need a dyadic partition of unity. Let A be the annulus $A = \{\xi \in \mathbb{R}^n; \frac{1}{2} \leq |\xi| \leq 2\}$ and

$$\chi_0(\xi) + \sum_{j=1}^{\infty} \chi_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $\chi_0(\xi) \in C_0^\infty(B(0,2))$ and $\chi_j(\xi) = \chi(2^{-j}\xi)$ when $j \geq 1$ with $\chi(\xi) \in C_0^\infty(A)$. Now we decompose the operator $T_{a,\varphi}$ as follows:

$$T_{a,\varphi} = T_{\chi_0}(D) + \sum_{j=1}^{\infty} T_{\chi_j}(D) = T_0(D) + \sum_{j=1}^{\infty} T_j(D). \tag{3.1}$$

The first term in (3.1) is bounded on $L^2(\mathbb{R}^n)$ from Theorem 1.1.8 in [14]. After a change of variables, we have

$$\begin{aligned} T_j(D) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} \chi_j(\xi) a(x,\xi) \hat{u}(\xi)\, d\xi \\ &= \frac{2^{j\varrho n}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\cdot 2^{j\varrho}\varphi(x,\xi)} \chi_j\big(2^{j\varrho}\xi\big) a\big(x, 2^{j\varrho}\xi\big) \hat{u}\big(2^{j\varrho}\xi\big)\, d\xi \\ &= \frac{2^{j\varrho n}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\cdot 2^{j\varrho}\varphi(x,\xi)} \chi_j\big(2^{j\varrho}\xi\big) a\big(x, 2^{j\varrho}\xi\big) \int_{\mathbb{R}^n} e^{-i2^{j\varrho}\xi\cdot y} u(y)\, dy\, d\xi \\ &= \frac{2^{j\varrho n}}{(2\pi)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\cdot 2^{j\varrho}(\varphi(x,\xi)-y\cdot\xi)} \chi_j\big(2^{j\varrho}\xi\big) a\big(x, 2^{j\varrho}\xi\big) u(y)\, d\xi\, dy. \end{aligned}$$

The kernel of the operator $T_j(D)$ is given by

$$T_j(x,y) = \frac{2^{j\varrho n}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\cdot 2^{j\varrho}(\varphi(x,\xi)-y\cdot\xi)} \chi_j\big(2^{j\varrho}\xi\big) a\big(x, 2^{j\varrho}\xi\big)\, d\xi.$$

Let

$$a_j(x,\xi) = \chi\big(2^{j(\varrho-1)}\xi\big) a\big(x, 2^{j\varrho}\xi\big).$$

Then

$$A_j = \operatorname*{Supp}_{\xi} a_j \subset \big\{\xi; 2^{-1}2^{j(1-\varrho)} < |\xi| < 2\cdot 2^{j(1-\varrho)}\big\}$$

and it satisfies

$$\big|\partial_\xi^\alpha a_j(x,\xi)\big| \leq C_\alpha \cdot 2^{jn(\rho-1)/2}. \tag{3.2}$$

We can confine ourselves to dealing with the high frequency component T_j of $T_{a,\varphi}$. Here we shall use a $S_j = T_j T_j^*$ argument, and therefore,

$$\begin{aligned} S_j u(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(\varphi(x,\xi)-\varphi(y,\xi))} \chi_j^2(\xi) a(x,\xi) \overline{a(y,\xi)} u(y)\, dy\, d\xi \\ &= \frac{2^{j\varrho n}}{(2\pi)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{i2^{j\varrho}(\varphi(x,\xi)-\varphi(y,\xi))} \\ &\quad + \chi_j^2\big(2^{j\varrho}\xi\big) a\big(x, 2^{j\varrho}\xi\big) \overline{a\big(y, 2^{j\varrho}\xi\big)} u(y)\, d\xi\, dy. \end{aligned}$$

The kernel of the operator $S_j = T_j T_j^*$ reads

$$S_j(x,y) = \frac{2^{j\varrho n}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i2^{j\varrho}(\varphi(x,\xi)-\varphi(y,\xi))} \chi_j^2\big(2^{j\varrho}\xi\big) a\big(x, 2^{j\varrho}\xi\big) \overline{a\big(y, 2^{j\varrho}\xi\big)}\, d\xi.$$

Let $b_j(x,y,\xi) = \chi_j^2(2^{j\varrho}\xi)a(x,2^{j\varrho}\xi)\overline{a(y,2^{j\varrho}\xi)}$. Then

$$\operatorname{Supp} b_j \subset \left\{\xi : \frac{2^{j(1-\varrho)}}{2} < |\xi| < 2\cdot 2^{j(1-\varrho)}\right\}.$$

We claim that

$$\left|\partial_\xi^\alpha b_j(x,y,\xi)\right| \le C_\alpha 2^{jn(\rho-1)}\omega^2\left(2^j\right).$$

In fact,

$$\begin{aligned}
\left|\partial_\xi^\alpha b_j(x,y,\xi)\right| &= \left|\partial_\xi^\alpha\left[\chi_j^2\left(2^{j\varrho}\xi\right)a\left(x,2^{j\varrho}\xi\right)\overline{a\left(y,2^{j\varrho}\xi\right)}\right]\right| \\
&= \sum_{\alpha_1+\alpha_2=\alpha}\left|\partial_\xi^{\alpha_1}\left[a\left(x,2^{j\varrho}\xi\right)\overline{a\left(y,2^{j\varrho}\xi\right)}\right]\right|\left|\partial_\xi^{\alpha_2}\chi^2\left(2^{-j(1-\varrho)}\xi\right)\right| \\
&\lesssim \sum_{\alpha_1+\alpha_2=\alpha}\left(2^{j\varrho}\right)^{|\alpha_1|}\left|\left(\partial_\xi^{\alpha_1}(a\cdot\bar{a})\right)\left(x,2^{j\varrho}\xi\right)\right|\omega^2\left(2^j\right) \\
&\quad\times 2^{-j(1-\varrho)|\alpha_2|}\left|\left(\partial_\xi^{\alpha_2}\chi\right)\left(2^{-j(1-\varrho)}\xi\right)\right| \\
&\lesssim \sum_{\alpha_1+\alpha_2=\alpha} 2^{j\varrho|\alpha_1|}\left\langle 2^{j\varrho}\xi\right\rangle^{n(\rho-1)-\varrho|\alpha_1|}2^{-j(1-\varrho)|\alpha_2|}\omega^2\left(2^j\right) \\
&\lesssim \sum_{\alpha_1+\alpha_2=\alpha} 2^{j\varrho|\alpha_1|}2^{j(n(\rho-1)-\varrho|\alpha_1|)}2^{-j(1-\varrho)|\alpha_2|}\omega^2\left(2^j\right) \\
&= \sum_{\alpha_1+\alpha_2=\alpha} 2^{jn(\rho-1)-j(1-\varrho)|\alpha_2|}\omega^2\left(2^j\right) \\
&= 2^{jn(\rho-1)}\sum_{\alpha_2}2^{-j(1-\varrho)|\alpha_2|}\omega^2\left(2^j\right) \\
&\lesssim 2^{jn(\rho-1)}\omega^2\left(2^j\right).
\end{aligned}\tag{3.3}$$

Next we consider the following differential operators for $j \in \mathbb{N}$:

$$L_j(x,y,D) = \frac{\nabla_\xi \Phi \nabla_\xi}{i2^{j\varrho}|\nabla_\xi \Phi|^2},\tag{3.4}$$

where $\Phi(x,y,\xi) = \varphi(x,\xi) - \varphi(y,\xi)$. So $L_j^N(x,y,D)e^{i2^{j\varrho}\Phi} = e^{i2^{j\varrho}\Phi}$ and

$$L_j^*(x,y,D) = -\nabla_\xi \frac{\nabla_\xi \Phi}{i2^{j\varrho}|\nabla_\xi \Phi|^2}.\tag{3.5}$$

From this and (3.4), it follows that

$$\begin{aligned}
S_j(x,y) &= \frac{2^{j\varrho n}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i2^{j\varrho}(\varphi(x,\xi)-\varphi(y,\xi))}\chi_j^2\left(2^{j\varrho}\xi\right)a\left(x,2^{j\varrho}\xi\right)\overline{a\left(y,2^{j\varrho}\xi\right)}\,d\xi \\
&= \frac{2^{j\varrho n}}{(2\pi)^n}\int_{\mathbb{R}^n}\left(L_j^N e^{i2^{j\varrho}\Phi}\right)b_j(x,y\xi)\,d\xi \\
&= \frac{2^{j\varrho n}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i2^{j\varrho}\Phi}\left(L_j^*\right)^N b_j\,d\xi.
\end{aligned}$$

Moreover, by (3.5), we see that

$$\begin{aligned}\partial_{\xi_{\mu_1}}\left[\frac{\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2}b_j\right] &= \left(\partial_{\xi_{\mu_1}}\left[\frac{\nabla_\xi \Phi b_j}{|\nabla_\xi \Phi|^2}\right]\right)\\ &= \left[\frac{\partial_{\xi_1}\Phi}{|\nabla_\xi \Phi|^2}\right]\partial_{\xi_{\mu_1}}b_j + \partial_{\xi_{\mu_1}}\left[\frac{\partial_{\xi_1}\Phi}{|\nabla_\xi \Phi|^2}\right]b_j, \dots, \left[\frac{\partial_{\xi_n}\Phi}{|\nabla_\xi \Phi|^2}\right]\partial_{\xi_{\mu_1}}b_j\\ &\quad + \partial_{\xi_{\mu_1}}\left[\frac{\partial_{\xi_n}\Phi}{|\nabla_\xi \Phi|^2}\right]b_j,\end{aligned}$$

which implies that

$$\begin{aligned}L_j^* b_j &= \nabla_\xi\left[\frac{\nabla_\xi \Phi}{-i2^{j\varrho}|\nabla_\xi \Phi|^2}b_j\right]\\ &= \frac{1}{-i2^{j\varrho}}\sum_{l=1}^{n}\left\{\left[\frac{\partial_{\xi_l}\Phi}{|\nabla_\xi \Phi|^2}\right]\partial_{\xi_l}b_j + \partial_{\xi_l}\left[\frac{\partial_{\xi_l}\Phi}{|\nabla_\xi \Phi|^2}\right]b_j\right\}.\end{aligned}$$

Thus

$$\begin{aligned}\left(L_j^*\right)^N b_j &= \frac{1}{(-i2^{j\varrho})^N}\nabla_\xi\underbrace{\left\{\frac{\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2}\cdots\left[\nabla_\xi\left(\frac{\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2}b_j\right)\right]\right\}}_{N}\\ &= \frac{1}{(-i2^{j\varrho})^N}\sum_{\alpha_1+\cdots+\alpha_N+\beta=N}\partial^{\alpha_1}\left(\frac{\Phi_{\mu_1}}{|\nabla\Phi|^2}\right)\cdots\partial^{\alpha_N}\left(\frac{\Phi_{\mu_N}}{|\nabla\Phi|^2}\right)\partial^\beta b_j,\end{aligned}\tag{3.6}$$

where $\Phi_{\mu_k} = \partial_{\xi_{\mu_k}}\Phi$. Because of the following equation

$$\partial_\xi^k\left(\frac{\Phi_\mu}{|\nabla\Phi|^2}\right) = \sum_{k_0+k_1+\cdots+k_j=k}\frac{C_{k_0,\dots,k_j}\partial_\xi^{k_0}\Phi_\mu\partial_\xi^{k_1}|\nabla\Phi|^2\cdot\partial_\xi^{k_j}|\nabla\Phi|^2}{|\nabla\Phi|^{2+2j}},$$

by Definition 2.6, we see that

$$\left|\partial_\xi^{k_0}\Phi_\mu\right| \lesssim |x-y| \quad\text{and}\quad \left|\partial_\xi^k|\nabla\Phi|^2\right| \lesssim |x-y|^2.$$

From this, together with (3.3) and (3.6), we further obtain

$$\begin{aligned}\left|\left(L_j^*\right)^N b_j\right| &\lesssim \frac{1}{2^{j\varrho N}}\cdot\frac{1}{|x-y|^N}2^{jn(\rho-1)}\omega^2\left(2^j\right).\\ &= \frac{1}{[2^{j\varrho}|x-y|]^N}2^{jn(\rho-1)}\omega^2\left(2^j\right).\end{aligned}$$

Integration by parts yields

$$\begin{aligned}S_j(x,y) &= \frac{2^{j\varrho n}}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i2^{j\varrho}\Phi}\left(L_j^*\right)^N b_j\,d\xi\\ &\lesssim \frac{2^{j\varrho n}}{[2^{j\varrho}|x-y|]^N}2^{jn(\rho-1)}\omega^2\left(2^j\right)2^{j(1-\varrho)n}\\ &= \frac{2^{j\varrho n}}{[2^{j\varrho}|x-y|]^N}\omega^2\left(2^j\right).\end{aligned}$$

Thus,

$$\left|S_j(x,y)\right| \sum_{l=0}^{N} \left(2^{j\varrho}|x-y|\right)^l \lesssim 2^{j\varrho n}\omega^2\left(2^j\right),$$

which implies that

$$\left|S_j(x,y)\right| \lesssim \frac{2^{j\varrho n}}{(1+2^{j\varrho}|x-y|)^N}\omega^2\left(2^j\right).$$

This further gives

$$\begin{aligned}\sup_x \int_{\mathbb{R}^n} \left|S_j(x,y)\right| dy &\lesssim 2^{j\varrho n}\omega^2\left(2^j\right)\int_{\mathbb{R}^n} \frac{1}{(1+2^{j\varrho}|x-y|)^N}\,dy \\ &\lesssim \omega^2\left(2^j\right)\int_{\mathbb{R}^n} \frac{1}{(1+|z|)^N}\,dz \lesssim \omega^2\left(2^j\right).\end{aligned}$$

By Young's inequality, we obtain

$$\left\|S_j u(x)\right\|_{L^2} \lesssim \omega^2\left(2^j\right)\left\|u(x)\right\|_{L^2}.$$

Therefore, we have

$$\begin{aligned}\left\|T_j^* u\right\|_{L^2}^2 &= \left\langle T_j^* u, T_j^* u\right\rangle = \left\langle u, T_j T_j^* u\right\rangle \\ &\le \|u\|_{L^2}\|S_j u\|_{L^2} \\ &\le C\omega^2\left(2^j\right)\|u\|_{L^2}^2.\end{aligned}$$

Namely,

$$\|T_j u\|_{L^2} \le C\omega\left(2^j\right)\|u\|_{L^2}. \tag{3.7}$$

Next we need a Littlewood–Paley decomposition. Let $\psi_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth radial function which is equal to one on the unit ball centric at the origin and supported on its concentric double. Set $\psi(\xi) = \psi_0(\xi) - \psi_0(2\xi)$ and $\psi_k(\xi) = \psi(2^{-k}\xi)$. Then

$$\psi_0(\xi) + \sum_{k=1}^{\infty} \psi_k(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

and $\operatorname{supp}\psi_k(\xi) \subset \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$. And we further have

$$\hat{u}(\xi) = \sum_{k=0}^{\infty} \hat{u}(\xi)\psi_k(\xi) = \sum_{k=0}^{\infty} \hat{u}_k(\xi).$$

Then

$$T_{a,\varphi}u = \sum_{j=1}^{\infty} T_j u = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a_j(x,\xi)\hat{u}_k(\xi)\,d\xi.$$

For simplicity of notation, we write

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a_j(x,\xi)\hat{u}_k(\xi)\,d\xi = \sum_{j=1}^{\infty} T_j u_j,$$

where

$$u_j = \sum_{k=0}^{\infty} a_j(x,\xi)\hat{u}_k(\xi).$$

From this, (3.7), Cauchy–Schwartz's inequality, and Remark 2.1, it follows that

$$\begin{aligned}\|T_{a,\varphi}u\|_{L^2} &= \left\|\sum_{j=0}^{\infty} T_j u_j\right\|_{L^2}\\ &\lesssim \sum_{j=0}^{\infty}\omega\bigl(2^j\bigr)\|u_j\|_{L^2}\\ &\lesssim \left(\sum_{j=0}^{\infty}\omega^2\bigl(2^j\bigr)\right)^{1/2}\left(\sum_{j=0}^{\infty}\|u_j\|_{L^2}^2\right)^{1/2}\\ &\lesssim \|u\|_{L^2}.\end{aligned}$$

This finishes the proof of Theorem 1.1.

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