Liu *Journal of Inequalities and Applications* (2020) 2020:169
https://doi.org/10.1186/s13660-020-02431-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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Lyapunov-type inequalities for generalized one-dimensional Minkowski-curvature problems

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Abstract

In this paper, we consider some types of scalar equations and systems of generalized one-dimensional Minkowski-curvature problems. Using an inequality technique, we establish several new Lyapunov-type inequalities for the problems considered. Our results extend the existing work in the literature.

Keywords: Lyapunov-type inequality; Minkowski-curvature; p-Laplacian

1 Introduction

In [1], Russian mathematician Lyapunov proved the following result: If $y(t)$ is a solution of

$$y'' + q(t)y = 0 \tag{1}$$

satisfying $y(a) = y(b) = 0$ $(a < b)$ and $y(t) \neq 0$ for $t \in (a,b)$, then

$$\int_a^b |q(t)|\,\mathrm{d}t > \frac{4}{b-a}. \tag{2}$$

The above result is known as the Lyapunov inequality.

This result plays an important role in the study of various properties of solutions of Eq. (1) such as oscillation theory, disconjugacy and eigenvalue problems. After this seminal paper, the Lyapunov inequality and many of its generalizations have been studied by many researchers; see [2–42] and the references therein.

For example, Yang [43] obtained a Lyapunov-type inequality for the second order half-linear equation

$$\left(r(t)\left|y'(t)\right|^{p-1}y'(t)\right)' + q(t)\left|y(t)\right|^{p-1}y(t) = 0, \tag{3}$$

$$y(a) = (b) = 0, \qquad y(t) \neq 0, \quad t \in (a,b), \tag{4}$$

where $q, r \in C([a,b],\mathbb{R})$ such that $r(t) > 0$ for $t \in [a,b]$, and $p > 0$ is a constant.

Theorem 1.1 ([43]) *Assume Eq.* (3) *has a solution* $y(t)$, *then the inequality*

$$\int_a^b q_+(t)\,\mathrm{d}t \geq \frac{2^{p+1}}{(\int_a^b r^{-\frac{1}{p}}(t)\,\mathrm{d}t)^p} \tag{5}$$

holds, where $q_+(t) := \max\{q(t), 0\}$.

Recently, Yang et al. [44] investigated a Lyapunov-type inequality for one-dimensional Minkowski-curvature problem with singular weight

$$-\left(\frac{y'(t)}{\sqrt{1-|y'(t)|^2}}\right)' = k(t)y(t), \tag{6}$$

$$y(a) = y(b) = 0, \qquad y(t) \neq 0, \quad t \in (a,b). \tag{7}$$

They presented the following result.

Theorem 1.2 ([44]) *If the problem* (6) *has a positive solution, then one has*

$$\int_a^b (t-a)(b-t)k(t)\,\mathrm{d}t > b-a, \tag{8}$$

where $k(t) \geq 0$ *for all* $t \in (a,b)$, $k \not\equiv 0$ *in any compact subinterval of* $[a,b]$ *and* $k \in U = \{k \in L^1_{\mathrm{loc}}((a,b),[0,\infty)) : \int_a^b (t-a)(b-t)k(t)\,\mathrm{d}t < \infty\}$.

Motivated by this work, in this paper, we will establish Lyapunov-type inequalities for the generalized one-dimensional Minkowski-curvature problems with singular weight function

$$-\left(\frac{r(t)|y'(t)|^{p-2}y'(t)}{\sqrt{1-|y'(t)|^p}}\right)' = q(t)\big|y(t)\big|^{p-2}y(t), \tag{9}$$

$$y(a) = y(b) = 0, \qquad y(t) \neq 0, \quad t \in (a,b), \tag{10}$$

and

$$-\left(\frac{r(t)|y'(t)|^{p-2}y'(t)}{\sqrt{1-|y'(t)|^p}}\right)' = l(t)\big|y(t)\big|^{\alpha-2}y(t) - h(t)\big|y(t)\big|^{\beta-2}y(t), \tag{11}$$

$$y(a) = y(b) = 0, \qquad y(t) \neq 0, \quad t \in (a,b), \tag{12}$$

and the cycled systems of a generalized one-dimensional Minkowski-curvature problem with singular weight functions

$$\begin{cases} (\frac{r_1(t)|y_1'(t)|^{p-2}y_1'(t)}{\sqrt{1-|y_1'(t)|^p}})' + q_1(t)|y_2(t)|^{p-2}y_2(t) = 0, & t \in (a,b), \\ (\frac{r_2(t)|y_2'(t)|^{p-2}y_2'(t)}{\sqrt{1-|y_2'(t)|^p}})' + q_2(t)|y_3(t)|^{p-2}y_3(t) = 0, & t \in (a,b), \\ (\frac{r_3(t)|y_3'(t)|^{p-2}y_3'(t)}{\sqrt{1-|y_3'(t)|^p}})' + q_3(t)|y_4(t)|^{p-2}y_4(t) = 0, & t \in (a,b), \\ \cdots, \\ (\frac{r_n(t)|y_n'(t)|^{p-2}y_n'(t)}{\sqrt{1-|y_n'(t)|^p}})' + q_n(t)|y_1(t)|^{p-2}y_1(t) = 0, & t \in (a,b), \end{cases} \tag{13}$$

$$y_1(a) = \cdots = y_n(a) = 0 = y_1(b) = \cdots = y_n(b), \quad y_i(t) \not\equiv 0, t \in (a,b), \tag{14}$$

for $i = 1, 2, \ldots, n$,

and

$$\begin{cases} \left(\frac{r_1(t)|y_1'(t)|^{p-2}y_1'(t)}{\sqrt{1-|y_1'(t)|^p}}\right)' + l_1(t)|y_2(t)|^{\alpha-2}y_2(t) - h_1(t)|y_2(t)|^{\beta-2}y_2(t) = 0, & t \in (a,b), \\ \left(\frac{r_2(t)|y_2'(t)|^{p-2}y_2'(t)}{\sqrt{1-|y_2'(t)|^p}}\right)' + l_2(t)|y_3(t)|^{\alpha-2}y_3(t) - h_2(t)|y_3(t)|^{\beta-2}y_3(t) = 0, & t \in (a,b), \\ \left(\frac{r_3(t)|y_3'(t)|^{p-2}y_3'(t)}{\sqrt{1-|y_3'(t)|^p}}\right)' + l_3(t)|y_4(t)|^{\alpha-2}y_4(t) - h_3(t)|y_4(t)|^{\beta-2}y_4(t) = 0, & t \in (a,b), \\ \cdots, \\ \left(\frac{r_n(t)|y_n'(t)|^{p-2}y_n'(t)}{\sqrt{1-|y_n'(t)|^p}}\right)' + l_n(t)|y_1(t)|^{\alpha-2}y_1(t) - h_n(t)|y_1(t)|^{\beta-2}y_1(t) = 0, & t \in (a,b), \end{cases} \tag{15}$$

$$y_1(a) = \cdots = y_n(a) = 0 = y_1(b) = \cdots = y_n(b), \quad y_i(t) \not\equiv 0, t \in (a,b), \tag{16}$$

for $i = 1, 2, \ldots, n$,

where $p > 1$, $1 < p < \alpha < \beta$ or $1 < \beta < \alpha < p$, $r, r_i \in C([a,b],(0,+\infty))$, $q(t), q_i(t) \geq 0$ for all $t \in (a,b)$, $q, q_i \not\equiv 0$ in any compact subinterval of $[a,b]$ and

$$q, q_i \in \mathfrak{D} = \left\{ f \in L^1_{\mathrm{loc}}\big((a,b),[0,\infty)\big) : \int_a^b (t-a)^{p-1}(b-t)^{p-1} f(t)\,\mathrm{d}t < \infty \right\},$$

$i = 1, 2, \ldots, n$. $l(t), h(t), l_i(t), h_i(t) > 0$ for all $t \in (a,b)$ such that

$$A(t) = \frac{l(t)(\beta-\alpha)}{\beta-p}\left(\frac{(\beta-p)h(t)}{(\alpha-p)l(t)}\right)^{(\alpha-p)/(\alpha-\beta)},$$

$$A_i(t) = \frac{l_i(t)(\beta-\alpha)}{\beta-p}\left(\frac{(\beta-p)h_i(t)}{(\alpha-p)l_i(t)}\right)^{(\alpha-p)/(\alpha-\beta)},$$

satisfy $A, A_i \in \mathfrak{D}$, $i = 1, 2, \ldots, n$. Class $\mathfrak{D}$ admits rather stronger singular functions at the boundary. For example, $q(t) = t^{-(2p-1)/p} \in \mathfrak{D}$ with $a = 0$, $b = 1$ but not in $L^1(0,1)$.

Our results not only extend the existing work in the literature, but also give necessary conditions for the existence of positive solutions for scalar equations and systems of generalized one-dimensional Minkowski-curvature problems with singular weight functions.

2 Preliminaries

In this section, we give some definitions and lemmas which are needed in the sequel.

Definition 2.1 We say y is a solution of problem (9)–(10) (or (11)–(12)) if $y \in C^1[a,b]$, $\|y'\|_\infty < 1$, and $\frac{r(\cdot)|y'(\cdot)|^{p-2}y'(\cdot)}{\sqrt{1-|y'(\cdot)|^p}}$ is absolutely continuous in any compact subinterval of (a,b), and y satisfies the equation and the boundary conditions in problem (9)–(10) (or (11)–(12)).

Definition 2.2 We say $(y_1, y_2, \ldots, y_n)$ is a solution of problem (13)–(14) (or (15)–(16)) if $y_i \in C^1[a,b]$, $\|y_i'\|_\infty < 1$, and $\frac{r(\cdot)|y_i'(\cdot)|^{p-2}y_i'(\cdot)}{\sqrt{1-|y_i'(\cdot)|^p}}$ is absolutely continuous in any compact subinterval of (a,b), and y_i satisfies the equations and the boundary conditions in problem (13)–(14) (or (15)–(16)).

Lemma 2.1 ([12]) *Suppose that $a, b \in \mathbb{R}$, $\gamma > 0$. Then*

$$\frac{1}{\widetilde{K}(\gamma)}\bigl(|a| + |b|\bigr)^{\gamma} \leq |a|^{\gamma} + |b|^{\gamma},$$

where

$$\widetilde{K}(\gamma) = \begin{cases} 1, & 0 < \gamma \leq 1, \\ 2^{\gamma-1}, & \gamma > 1. \end{cases}$$

Lemma 2.2 *If $y \in C^1[a,b]$, $y(a) = y(b) = 0$ and $p > 1$, then we have*

$$\bigl|y(t)\bigr|^p \leq K(p)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\left(\int_a^b \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right), \tag{17}$$

where

$$K(p) = \begin{cases} 1, & 1 < p \leq 2, \\ 2^{p-2}, & p > 2. \end{cases} \tag{18}$$

Proof From Hölder's inequality, we get $\forall t \in [a,b]$,

$$\bigl|y(t)\bigr| \leq \int_a^t \bigl|y'(s)\bigr| \,\mathrm{d}s \leq (t-a)^{1/p_*}\left(\int_a^t \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right)^{1/p},$$

where $p^* = \frac{p}{p-1}$. In view of $(b-t)/(b-a) \geq 0$, we obtain

$$\left(\frac{b-t}{b-a}\right)^{1/p_*}\bigl|y(t)\bigr| \leq \left(\frac{b-t}{b-a}\right)^{1/p_*}(t-a)^{1/p_*}\left(\int_a^t \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right)^{1/p}.$$

Thus

$$\left(\frac{b-t}{b-a}\right)^{p/p_*}\bigl|y(t)\bigr|^p \leq \left(\frac{b-t}{b-a}\right)^{p/p_*}(t-a)^{p/p_*}\left(\int_a^t \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right). \tag{19}$$

Similarly, from $(t-a)/(b-a) \geq 0$, and

$$\bigl|y(t)\bigr| \leq \int_t^b \bigl|y'(s)\bigr| \,\mathrm{d}s \leq (b-t)^{1/p_*}\left(\int_t^b \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right)^{1/p},$$

we obtain

$$\left(\frac{t-a}{b-a}\right)^{1/p_*}\bigl|y(t)\bigr| \leq \left(\frac{t-a}{b-a}\right)^{1/p_*}(b-t)^{1/p_*}\left(\int_t^b \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right)^{1/p}.$$

Thus

$$\left(\frac{t-a}{b-a}\right)^{p/p_*}\bigl|y(t)\bigr|^p \leq \left(\frac{t-a}{b-a}\right)^{p/p_*}(b-t)^{p/p_*}\left(\int_t^b \bigl|y'(s)\bigr|^p \,\mathrm{d}s\right). \tag{20}$$

Adding (19) and (20), we have

$$\left(\left(\frac{b-t}{b-a}\right)^{p/p_*}+\left(\frac{t-a}{b-a}\right)^{p/p_*}\right)\left|y(t)\right|^p \le \left(\frac{(t-a)(b-t)}{b-a}\right)^{p/p_*}\left(\int_a^b \left|y'(s)\right|^p \,\mathrm{d}s\right).$$

According to $\frac{p}{p^*}=p-1$, we get

$$\left(\left(\frac{b-t}{b-a}\right)^{p-1}+\left(\frac{t-a}{b-a}\right)^{p-1}\right)\left|y(t)\right|^p \le \left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\left(\int_a^b \left|y'(s)\right|^p \,\mathrm{d}s\right). \tag{21}$$

On the other hand, from Lemma 2.1 we obtain

$$\left(\frac{b-t}{b-a}\right)^{p-1}+\left(\frac{t-a}{b-a}\right)^{p-1} \ge \frac{1}{K(p)}\left(\frac{b-t}{b-a}+\frac{t-a}{b-a}\right)^{p-1}=\frac{1}{K(p)}. \tag{22}$$

Therefore, by (21) and (22), we get

$$\left|y(t)\right|^p \le K(p)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\left(\int_a^b \left|y'(s)\right|^p \,\mathrm{d}s\right). \tag{23}$$

The proof is complete. □

Lemma 2.3 ([45]) *Let m, n, p, α and β be positive constants, then, for each $x \ge 0$,*

$$mx^\alpha - nx^\beta \le \frac{m(\beta-\alpha)}{\beta-p}\left(\frac{(\beta-p)n}{(\alpha-p)m}\right)^{(\alpha-p)/(\alpha-\beta)} x^p \tag{24}$$

holds for the cases when $0<p<\alpha<\beta$ or $0<\beta<\alpha<p$.

3 Main results

Theorem 3.1 *If $y(t)$ is a positive solution of problem (9)–(10), then*

$$\int_a^b q(t)(t-a)^{p-1}(b-t)^{p-1}\,\mathrm{d}t > \frac{(b-a)^{p-1}}{K(p)}\min_{a\le b}\left\{r(t)\right\}, \tag{25}$$

where $K(p)$ is defined as in (18).

Proof Multiplying (9) by $y(t)$ and integrating from a to b by parts yield

$$\int_a^b \frac{r(t)|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t = \int_a^b q(t)\left|y(t)\right|^p \,\mathrm{d}t. \tag{26}$$

By Lemma 2.2, we get

$$\begin{aligned}\int_a^b q(t)\left|y(t)\right|^p \,\mathrm{d}t &\le \int_a^b q(t)K(p)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\left(\int_a^b \left|y'(t)\right|^p \,\mathrm{d}t\right)\mathrm{d}t\\ &= K(p)\int_a^b \left|y'(t)\right|^p \,\mathrm{d}t\int_a^b q(t)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\mathrm{d}t.\end{aligned} \tag{27}$$

On the other hand,

$$\int_a^b \frac{r(t)|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t \geq \min_{a\leq b}\{r(t)\}\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t. \tag{28}$$

It follows from (26)–(28) and $\|y'\|_\infty < 1$ that

$$\begin{aligned}
&\min_{a\leq b}\{r(t)\}\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t \\
&\quad\leq K(p)\int_a^b \big|y'(t)\big|^p\,\mathrm{d}t\int_a^b q(t)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\mathrm{d}t \\
&\quad< K(p)\left(\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t\right)\int_a^b q(t)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\mathrm{d}t.
\end{aligned} \tag{29}$$

Now, we claim that

$$\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t > 0.$$

In fact, if the above inequality is not true, then we have

$$\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t = 0.$$

Then $y'(t) = 0$ for $t \in [a,b]$. By condition (10), we obtain $y(t) = 0$ for $t \in [a,b]$, which contradicts to $y(t) \not\equiv 0$, $t \in [a,b]$. Thus dividing both sides of (29) by

$$\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t,$$

we obtain

$$K(p)\int_a^b q(t)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\mathrm{d}t > \min_{a\leq b}\{r(t)\}, \tag{30}$$

from which (25) is obtained. The proof is complete. □

Remark 3.1 If we take $p = 2$ and $r(t) \equiv 1$, then Theorem 3.1 reduces to [44, Theorem 2.1].

Theorem 3.2 *If $y(t)$ is a positive solution of problem* (11)–(12), *then*

$$\int_a^b A(t)\left(\frac{(t-a)(b-t)}{b-a}\right)^{p-1}\mathrm{d}t > \frac{1}{K(p)}\min_{a\leq b}\{r(t)\}, \tag{31}$$

where

$$A(t) = \frac{l(t)(\beta-\alpha)}{\beta-p}\left(\frac{(\beta-p)h(t)}{(\alpha-p)l(t)}\right)^{(\alpha-p)/(\alpha-\beta)},$$

and $K(p)$ is defined as in (18).

Proof Multiplying (11) by $y(t)$ and integrating from a to b by parts yield

$$\int_a^b \frac{r(t)|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t = \int_a^b \bigl(l(t)\bigl|y(t)\bigr|^\alpha - h(t)\bigl|y(t)\bigr|^\beta\bigr)\,\mathrm{d}t. \tag{32}$$

By Lemma 2.3, the right side of (32) satisfies

$$\int_a^b \bigl(l(t)\bigl|y(t)\bigr|^\alpha - h(t)\bigl|y(t)\bigr|^\beta\bigr)\,\mathrm{d}t \le \int_a^b A(t)\bigl|y(t)\bigr|^p\,\mathrm{d}t. \tag{33}$$

From Lemma 2.2, we have

$$\begin{aligned}\int_a^b A(t)\bigl|y(t)\bigr|^p\,\mathrm{d}t &\le \int_a^b A(t)K(p)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\biggl(\int_a^b \bigl|y'(s)\bigr|^p\,\mathrm{d}s\biggr)\,\mathrm{d}t \\ &= K(p)\int_a^b \bigl|y'(t)\bigr|^p\,\mathrm{d}t\int_a^b A(t)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\,\mathrm{d}t. \end{aligned} \tag{34}$$

On the other hand,

$$\int_a^b \frac{r(t)|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t \ge \min_{a\le b}\bigl\{r(t)\bigr\}\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t. \tag{35}$$

It follows from (32)–(35) and $\|y'\|_\infty < 1$ that

$$\begin{aligned}&\min_{a\le b}\bigl\{r(t)\bigr\}\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t \\ &\quad\le K(p)\int_a^b \bigl|y'(t)\bigr|^p\,\mathrm{d}t\int_a^b A(t)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\,\mathrm{d}t \\ &\quad< K(p)\biggl(\int_a^b \frac{|y'(t)|^p}{\sqrt{1-|y'(t)|^p}}\,\mathrm{d}t\biggr)\int_a^b A(t)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\,\mathrm{d}t. \end{aligned} \tag{36}$$

The rest of the proof is similar to that of Theorem 3.1, and therefore is omitted. The proof is complete. □

Theorem 3.3 *If* $(y_1(t), y_2(t), \dots, y_n(t))$ *is a positive solution of problem* (13)–(14), *then*

$$\prod_{i=1}^n\biggl(\int_a^b q_i(t)(t-a)^{p-1}(b-t)^{p-1}\,\mathrm{d}t\biggr) > \frac{(b-a)^{n(p-1)}}{[K(p)]^n}\prod_{i=1}^n \min_{a\le b}\bigl\{r_i(t)\bigr\}, \tag{37}$$

where $K(p)$ *is defined as in* (18).

Proof From Lemma 2.2, we get

$$\bigl|y_i(t)\bigr|^p \le K(p)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\biggl(\int_a^b \bigl|y_i'(t)\bigr|^p\,\mathrm{d}t\biggr),\quad i=1,2,\dots,n, \tag{38}$$

i.e.,

$$\bigl|y_i(t)\bigr| \le \bigl[K(p)\bigr]^{1/p}\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{(p-1)/p}\biggl(\int_a^b \bigl|y_i'(t)\bigr|^p\,\mathrm{d}t\biggr)^{1/p},\quad i=1,2,\dots,n. \tag{39}$$

Multiplying (13) by $y_1(t)$ and integrating from a to b by parts, we have

$$\int_a^b \frac{r_1(t)|y_1'(t)|^p}{\sqrt{1-|y_1'(t)|^p}}\,\mathrm{d}t = \int_a^b q_1(t)\big|y_1(t)\big|\big|y_2(t)\big|^{p-1}\,\mathrm{d}t. \tag{40}$$

Together with (39), we obtain

$$\begin{aligned}\min_{a\le t\le b}\big\{r_1(t)\big\}\int_a^b \big|y_1'(t)\big|^p\,\mathrm{d}t &< \int_a^b \frac{r_1(t)|y_1'(t)|^p}{\sqrt{1-|y_1'(t)|^p}}\,\mathrm{d}t \\ &= \int_a^b q_1(t)\big|y_1(t)\big|\big|y_2(t)\big|^{p-1}\,\mathrm{d}t \\ &\le K(p)\biggl(\int_a^b \big|y_1'(t)\big|^p\,\mathrm{d}t\biggr)^{1/p}\biggl(\int_a^b \big|y_2'(t)\big|^p\,\mathrm{d}t\biggr)^{(p-1)/p} \\ &\quad\times \int_a^b q_1(t)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\,\mathrm{d}t. \end{aligned} \tag{41}$$

Repeating this procedure to each equation in problem (13)–(14), for $i = 2, 3, \dots, n$, we have

$$\begin{aligned}&\min_{a\le t\le b}\big\{r_i(t)\big\}\int_a^b \big|y_i'(t)\big|^p\,\mathrm{d}t \\ &\quad< K(p)\biggl(\int_a^b \big|y_i'(t)\big|^p\,\mathrm{d}t\biggr)^{1/p}\biggl(\int_a^b \big|y_{i+1}'(t)\big|^p\,\mathrm{d}t\biggr)^{(p-1)/p} \\ &\quad\quad\times \int_a^b q_i(t)\biggl(\frac{(t-a)(b-t)}{b-a}\biggr)^{p-1}\,\mathrm{d}t, \end{aligned} \tag{42}$$

where $y_{n+1}(t) = y_1(t)$. Multiplying all inequalities, and from the fact $\int_a^b |y_i'(t)|^p\,\mathrm{d}t > 0$, $i = 1, 2, \dots, n$, we obtain (37). The proof is complete. □

Remark 3.2 If we take $p = 2$ and $r_i(t) \equiv 1$, $i = 1, 2, \dots, n$, then Theorem 3.3 reduces to [44, Theorem 4.1].

Theorem 3.4 *If* $(y_1(t), y_2(t), \dots, y_n(t))$ *is a positive solution of problem* (15)–(16), *then*

$$\prod_{i=1}^n\biggl(\int_a^b A_i(t)(t-a)^{p-1}(b-t)^{p-1}\,\mathrm{d}t\biggr) > \frac{(b-a)^{n(p-1)}}{[K(p)]^n}\prod_{i=1}^n \min_{a\le b}\big\{r_i(t)\big\}, \tag{43}$$

where

$$A_i(t) = \frac{l_i(t)(\beta-\alpha)}{\beta-p}\biggl(\frac{(\beta-p)h_i(t)}{(\alpha-p)l_i(t)}\biggr)^{(\alpha-p)/(\alpha-\beta)}, \quad i = 1, 2, \dots, n,$$

and $K(p)$ *is defined as in* (18).

Proof Multiplying (15) by $y_1(t)$ and integrating from a to b by parts, we have

$$\begin{aligned}\int_a^b \frac{r_1(t)|y_1'(t)|^p}{\sqrt{1-|y_1'(t)|^p}}\,\mathrm{d}t &= \int_a^b \bigl(l_1(t)\bigl|y_2(t)\bigr|^{\alpha-1} - h_1(t)\bigl|y_2(t)\bigr|^{\beta-1}\bigr)y_1(t)\,\mathrm{d}t \\ &\le \int_a^b A_1(t)\bigl|y_2(t)\bigr|^{p-1}\bigl|y_1(t)\bigr|\,\mathrm{d}t. \end{aligned} \tag{44}$$

The rest of the proof is similar to that of Theorem 3.3, and therefore is omitted. The proof is complete. □

Acknowledgements
The author is indebted to the anonymous referees for their valuable suggestions and helpful comments which helped improve the paper significantly.

Funding
This research was supported by the Natural Science Foundation of Shandong Province (China) (No.: ZR2018MA018), and the National Natural Science Foundation of China (No.: 61873144).

Availability of data and materials
Not applicable.

Competing interests
The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
The author carried out the results, and read and approved the current version of the manuscript.

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Received: 27 November 2019 Accepted: 5 June 2020 Published online: 16 June 2020

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