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Some parameterized inequalities by means of fractional integrals with exponential kernels and their applications

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Abstract

We use the definition of a new class of fractional integral operators, recently introduced by Ahmad et al. in [J. Comput. Appl. Math. 353:120–129, 2019], to establish a fractional-type integral identity with one parameter. We derive some parameterized integral inequalities for convex mappings based on this identity, and provide two examples to illustrate the investigated results as well. Moreover, we present applications of our findings to special means of real numbers, and error estimations for the quadrature formula in numerical analysis.

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1 Introduction

Throughout this paper, let $I \subseteq \mathbb{R}$ be a real interval and I° be the interior of I.

Let $u : I \to \mathbb{R}$ be a convex mapping on the interval I, for any $a, b \in I$ with $a < b$. Then one has

$$u\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b u(\tau)\,\mathrm{d}\tau \leq \frac{u(a)+u(b)}{2}, \tag{1.1}$$

which is called a Hermite–Hadamard inequality. This well-known inequality gives estimates for the mean value of a continuous convex mapping $u : [a, b] \to \mathbb{R}$.

For recent results obtained in terms of inequality (1.1), we refer the reader to [7, 15, 18, 19, 22, 31, 32, 35] and the references therein.

Another classical inequality of equal significance, which is named Simpson's inequality, is expressed as follows:

$$\left|\frac{1}{6}\left[u(a) + 4u\left(\frac{a+b}{2}\right) + u(b)\right] - \frac{1}{b-a}\int_a^b u(\tau)\,\mathrm{d}\tau\right| \leq \frac{1}{2880}\left\|u^{(4)}\right\|_\infty (b-a)^4, \tag{1.2}$$

where $u : I \to \mathbb{R}$ is a four-order continuously differentiable mapping on I° with $\|u^{(4)}\|_\infty = \sup_{\tau \in I^\circ} |u^{(4)}(\tau)| < \infty$.

Many inequalities have been established in terms of inequality (1.2) via functions of different classes, such as convex functions [10], geometrically relative convex functions [24], extended (s, m)-convex functions [9], p-quasi-convex functions [12], preinvex functions [6], and h-convex functions [20].

In [27], the authors gave certain inequalities for twice differentiable convex mappings related to Hadamard's inequality. They used the following lemma to derive their results.

Lemma 1.1 *Let $u : I \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$. If $u'' \in L^1([a, b])$, then the following equality holds*:

$$\begin{aligned} &\frac{1}{b-a} \int_a^b u(\tau)\,\mathrm{d}\tau - u\left(\frac{a+b}{2}\right) \\ &\quad = \frac{(b-a)^2}{2} \int_0^1 h(t)\left[u''\big(ta + (1-t)b\big) + u''\big(tb + (1-t)a\big)\right]\mathrm{d}t, \end{aligned} \tag{1.3}$$

where

$$h(t) = \begin{cases} t^2, & t \in [0, \frac{1}{2}], \\ (1-t)^2, & t \in (\frac{1}{2}, 1]. \end{cases}$$

In [3], using mappings whose twice derivatives absolute values are quasi-convex, Alomari et al. presented some Hadamard inequalities based on the following lemma.

Lemma 1.2 *Let $u : I \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$. If $u'' \in L^1([a, b])$, then the following equality holds*:

$$\frac{u(a) + u(b)}{2} - \frac{1}{b-a} \int_a^b u(\tau)\,\mathrm{d}\tau = \frac{(b-a)^2}{2} \int_0^1 t(1-t)u''\big(ta + (1-t)b\big)\,\mathrm{d}t. \tag{1.4}$$

In [26], Sarikaya and Aktan gave the following general integral identity for twice differentiable mappings.

Lemma 1.3 *Let $u : I \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$. For $0 \le \xi \le 1$, if $u'' \in L^1([a, b])$, then the following equality holds*:

$$\begin{aligned} &(\xi - 1)u\left(\frac{a+b}{2}\right) - \xi \frac{u(a) + u(b)}{2} + \frac{1}{b-a} \int_a^b u(\tau)\,\mathrm{d}\tau \\ &\quad = \frac{(b-a)^2}{2} \int_0^1 D(t)u''\big(ta + (1-t)b\big)\,\mathrm{d}t, \end{aligned} \tag{1.5}$$

where

$$D(t) = \begin{cases} t(t - \xi), & t \in [0, \frac{1}{2}], \\ (1-t)(1 - \xi - t), & t \in (\frac{1}{2}, 1]. \end{cases}$$

Fractional calculus, as a very useful tool, has become a fascinating field of mathematics. This field has attracted many researchers to consider this issue. As a result, some well-known integral inequalities by the approach of fractional calculus have been carried out by many authors, including Chen [4] and Mohammed [23] in the study of the Hermite–Hadamard inequality, and Set et al. [29] in the Simpson type integral inequality for Riemann–Liouville fractional integrals, Chen and Katugampola [5] in the Hermite–Hadamard–Fejér type inequality for Katugampola fractional integrals, Wang et al. [33] in the Ostrowski type inequality for Hadamard fractional integrals, Du et al. [8] in the extensions of trapezium inequalities for k-fractional integrals, and Khan et al. [14] in the Hermite–Hadamard inequality for conformable fractional integrals. For more results related to the fractional integral operators, the interested reader is directed to [1, 11, 13, 16, 17, 21, 25, 28, 30] and the references cited therein.

In 2019, Ahmad et al. [2] proposed a new fractional integral operators with an exponential kernel as follows.

Definition 1.1 Let $g \in L^1([a,b])$. The fractional integrals $\mathcal{I}^{\alpha}_{a^+}g$ and $\mathcal{I}^{\alpha}_{b^-}g$ of order $\alpha \in (0,1)$ are, respectively, defined by

$$\mathcal{I}^{\alpha}_{a^+}g(x) = \frac{1}{\alpha}\int_a^x e^{(-\frac{1-\alpha}{\alpha}(x-\tau))}g(\tau)\,\mathrm{d}\tau, \quad x > a,$$

and

$$\mathcal{I}^{\alpha}_{b^-}g(x) = \frac{1}{\alpha}\int_x^b e^{(-\frac{1-\alpha}{\alpha}(\tau-x))}g(\tau)\,\mathrm{d}\tau, \quad x < b.$$

Note that

$$\lim_{\alpha\to 1}\mathcal{I}^{\alpha}_{a^+}g(x) = \int_a^x g(\tau)\,\mathrm{d}\tau, \qquad \lim_{\alpha\to 1}\mathcal{I}^{\alpha}_{b^-}g(x) = \int_x^b g(\tau)\,\mathrm{d}\tau.$$

In the same paper, they established a fractional version of Hermite–Hadamard type involving exponential kernels as follows.

Theorem 1.1 *Let $g : [a,b] \to \mathbb{R}$ be a positive convex mapping with $0 \le a < b$. If $g \in L^1([a,b])$, then the following inequality for fractional integrals with an exponential kernel holds*:

$$g\left(\frac{a+b}{2}\right) \le \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b) + \mathcal{I}^{\alpha}_{b^-}g(a)\right] \le \frac{g(a)+g(b)}{2}, \tag{1.6}$$

where

$$\rho = \frac{1-\alpha}{\alpha}(b-a).$$

In [34], Wu et al. obtained an inequality of Hermite–Hadamard type involving twice differentiable convex mappings. They used the following lemma to prove their result.

Lemma 1.4 *Let $g : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $g'' \in L^1([a,b])$, then the following identity holds*:

$$\begin{aligned} &\frac{1-\alpha}{2(1-e^{-\rho})}\big[\mathcal{I}_{a^+}^{\alpha} g(b) + \mathcal{I}_{b^-}^{\alpha} g(a)\big] - \frac{g(a)+g(b)}{2} \\ &\quad = \frac{(b-a)^2}{2\rho(1-e^{-\rho})}\int_0^1 \big(e^{-\rho t} + e^{-\rho(1-t)} - 1 - e^{-\rho}\big) g''\big(ta + (1-t)b\big)\,\mathrm{d}t. \end{aligned} \tag{1.7}$$

Using fractional integrals with an exponential kernel, another integral identity involving twice differentiable mapping was presented by Wu et al. [34] as follows.

Lemma 1.5 *Let $g : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $g'' \in L^1([a,b])$, then the following identity holds*:

$$\begin{aligned} &\frac{1-\alpha}{2(1-e^{-\rho})}\big[\mathcal{I}_{a^+}^{\alpha} g(b) + \mathcal{I}_{b^-}^{\alpha} g(a)\big] - g\left(\frac{a+b}{2}\right) \\ &\quad = \frac{(b-a)^2}{2}\int_0^1 m(t) g''\big(ta + (1-t)b\big)\,\mathrm{d}t, \end{aligned} \tag{1.8}$$

where

$$m(t) = \begin{cases} t - \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & t \in [0,\frac{1}{2}], \\ (1-t) - \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & t \in (\frac{1}{2},1]. \end{cases}$$

Motivated by the results mentioned above, especially the results developed in [2] and [34], we notice that it is possible to deal with these results uniformly via the fractional integrals with exponential kernels. For this purpose, we establish a general fractional-type integral identity for twice differentiable mappings. Using this integral identity, we derive certain parameterized fractional-type inequalities, which unifies Simpson's inequality, the averaged midpoint-trapezoid inequality, and the trapezoid inequality. This is the main contribution of this work.

2 Main results

To prove our primary theorems, we present the following lemma.

Lemma 2.1 *Let $g : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $g'' \in L^1([a,b])$ and $0 \le \lambda \le 1$, then the following identity for fractional integrals holds*:

$$\begin{aligned} &\frac{1-\alpha}{2(1-e^{-\rho})}\big[\mathcal{I}_{a^+}^{\alpha} g(b) + \mathcal{I}_{b^-}^{\alpha} g(a)\big] - (1-\lambda) g\left(\frac{a+b}{2}\right) - \lambda\frac{g(a)+g(b)}{2} \\ &\quad = \frac{(b-a)^2}{2}\int_0^1 w(t) g''\big(ta + (1-t)b\big)\,\mathrm{d}t, \end{aligned} \tag{2.1}$$

where

$$w(t) = \begin{cases} t(1-\lambda) - \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & t \in [0,\frac{1}{2}], \\ (1-t)(1-\lambda) - \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & t \in (\frac{1}{2},1]. \end{cases}$$

Proof Multiplying (1.7) by λ and (1.8) by $(1-\lambda)$ on both sides, respectively, and adding the resulting equalities obtained as a result, we get (2.1). Therefore, we deduce the desired result. □

By means of Lemma 2.1, we derive the following general integral inequalities.

Theorem 2.1 *Let $g:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) with $a<b$ satisfying $g''\in L^1([a,b])$ and $0\le\lambda\le 1$. If $|g''|$ is convex on $[a,b]$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}+\frac{1-\lambda}{8}\right)\left(\left|g''(a)\right|+\left|g''(b)\right|\right).\end{aligned}\tag{2.2}$$

Proof Using Lemma 2.1 and the definition of $w(t)$, we have

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{2}\left[\int_0^{\frac12}\left|t(1-\lambda)-\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\right.\\&\qquad\left.+\int_{\frac12}^1\left|(1-t)(1-\lambda)-\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\right]\\&\quad\le\frac{(b-a)^2}{2}\left[\int_0^{\frac12}t(1-\lambda)\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\right.\\&\qquad+\int_{\frac12}^1(1-t)(1-\lambda)\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\\&\qquad\left.+\int_0^1\left|\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\right].\end{aligned}\tag{2.3}$$

Since $2e^{-\frac{\rho}{2}}\le e^{-\rho t}+e^{-\rho(1-t)}\le 1+e^{-\rho}$ for any $t\in[0,1]$ and $|g''|$ is convex on $[a,b]$, we obtain

$$\begin{aligned}&\int_0^1\left|\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\\&\quad\le\int_0^1\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\left(t\left|g''(a)\right|+(1-t)\left|g''(b)\right|\right)\mathrm{d}t\\&\quad=\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}\left(\left|g''(a)\right|+\left|g''(b)\right|\right).\end{aligned}\tag{2.4}$$

On the other hand,

$$\begin{aligned}&\int_0^{\frac12}t(1-\lambda)\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t+\int_{\frac12}^1(1-t)(1-\lambda)\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\\&\quad\le\frac{(1-\lambda)(|g''(a)|+|g''(b)|)}{8}.\end{aligned}\tag{2.5}$$

Using (2.4) and (2.5) in (2.3), we get the desired result in (2.2). Thus, the proof is completed. □

Corollary 2.1 *Under all assumptions of Theorem 2.1, if $|g''(x)| \leq M$ on $[a,b]$, then we have*

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\leq(b-a)^2M\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}+\frac{1-\lambda}{8}\right).\end{aligned}$$

Corollary 2.2 *Consider Theorem 2.1.*

(1) *For $\lambda = 0$, we have Theorem 3 established by Wu et al. in [34].*

(2) *For $\lambda = \frac{1}{3}$, we have the following Simpson inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]\right|\\&\quad\leq\frac{(b-a)^2}{2}\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}+\frac{1}{12}\right)\left(\left|g''(a)\right|+\left|g''(b)\right|\right).\end{aligned}$$

(3) *For $\lambda = \frac{1}{2}$, we have the following averaged midpoint-trapezoid integral inequality:*

$$\begin{aligned}&\left|\frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]\right|\\&\quad\leq\frac{(b-a)^2}{2}\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}+\frac{1}{16}\right)\left(\left|g''(a)\right|+\left|g''(b)\right|\right).\end{aligned}$$

(4) *For $\lambda = 1$, we have Theorem 2 established by Wu et al. in [34].*

Remark 2.1 In (2.2) of Theorem 2.1, if we take $\alpha \to 1$, i.e. $\rho = \frac{1-\alpha}{\alpha}(b-a) \to 0$, then we have

$$\lim_{\alpha\to1}\frac{1-\alpha}{2(1-e^{-\rho})}=\frac{1}{2(b-a)} \tag{2.6}$$

and

$$\lim_{\alpha\to1}\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}=\frac{1}{12}. \tag{2.7}$$

Thus, Theorem 2.1 is transformed to

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\leq\frac{(b-a)^2}{2}\left(\frac{1}{12}+\frac{1-\lambda}{8}\right)\left(\left|g''(a)\right|+\left|g''(b)\right|\right).\end{aligned} \tag{2.8}$$

Specially, putting $\lambda = 1$, we have Proposition 2 established by Sarikaya and Aktan in [26].

Remark 2.2 For $\lambda = \frac{1}{3}$ and $\alpha \to 1$, we have the following Simpson inequality:

$$\begin{aligned} &\left|\frac{1}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right| \\ &\quad\leq \frac{(b-a)^2}{12}\left(\left|g''(a)\right| + \left|g''(b)\right|\right). \end{aligned}$$

Remark 2.3 For $\lambda = \frac{1}{2}$ and $\alpha \to 1$, we have the averaged midpoint-trapezoid integral inequality:

$$\begin{aligned} &\left|\frac{1}{4}\left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right] - \frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right| \\ &\quad\leq \frac{7(b-a)^2}{96}\left(\left|g''(a)\right| + \left|g''(b)\right|\right). \end{aligned}$$

Before giving the following results, we recall that hyperbolic tangent function is defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Theorem 2.2 *Let $g : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$ satisfying $g'' \in L^1([a,b])$ and $0 \leq \lambda \leq 1$. For $q > 1$ with $p^{-1} + q^{-1} = 1$, if $|g''|^q$ is convex on $[a,b]$, then the following inequalities for fractional integrals hold*:

(1) *For $0 \leq \lambda < 1$, we have*

$$\begin{aligned} &\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b) + \mathcal{I}^{\alpha}_{b^-}g(a)\right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda\frac{g(a)+g(b)}{2}\right| \\ &\quad\leq \frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}} \\ &\qquad\times\left(\frac{|g''(a)|^q + |g''(b)|^q}{2}\right)^{\frac{1}{q}}. \end{aligned} \tag{2.9}$$

(2) *For $\lambda = 1$, we have*

$$\begin{aligned} &\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b) + \mathcal{I}^{\alpha}_{b^-}g(a)\right] - \frac{g(a)+g(b)}{2}\right| \\ &\quad\leq \frac{(b-a)^2\tanh(\frac{\rho}{4})}{2\rho}\left(\frac{|g''(a)|^q + |g''(b)|^q}{2}\right)^{\frac{1}{q}}. \end{aligned} \tag{2.10}$$

Proof First, suppose that $0 \leq \lambda < 1$. Utilizing Lemma 2.1, the definition of $w(t)$, and the Hölder inequality, we obtain

$$\begin{aligned} &\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b) + \mathcal{I}^{\alpha}_{b^-}g(a)\right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda\frac{g(a)+g(b)}{2}\right| \\ &\quad\leq \frac{(b-a)^2}{2}\int_0^1 \left|w(t)\right|\left|g''\left(ta + (1-t)b\right)\right|\,\mathrm{d}t \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(b-a)^2}{2}\left(\int_0^1 \left|w(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 \left|g''\big(ta+(1-t)b\big)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^2}{2}\left(\int_0^{\frac{1}{2}} \left|w_1(t)\right|^p \mathrm{d}t + \int_{\frac{1}{2}}^1 \left|w_2(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^1 \left|g''\big(ta+(1-t)b\big)\right|^q \mathrm{d}t\right)^{\frac{1}{q}},
\end{aligned} \tag{2.11}$$

where

$$w_1(t) = t(1-\lambda) - \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, \quad t \in \left[0, \frac{1}{2}\right],$$

and

$$w_2(t) = (1-t)(1-\lambda) - \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, \quad t \in \left(\frac{1}{2}, 1\right].$$

Owing to $2e^{-\frac{\rho}{2}} \leq e^{-\rho t} + e^{-\rho(1-t)} \leq 1 + e^{-\rho}$ for any $t \in [0,1]$, we have

$$\begin{aligned}
\int_{\frac{1}{2}}^1 \left|w_2(t)\right|^p \mathrm{d}t &= \int_0^{\frac{1}{2}} \left|w_1(t)\right|^p \mathrm{d}t \\
&\leq \int_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)^p \mathrm{d}t \\
&\leq \int_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1+e^{-\rho}-2e^{-\frac{\rho}{2}}}{\rho(1-e^{-\rho})}\right)^p \mathrm{d}t \\
&= (1-\lambda)^p \int_0^{\frac{1}{2}} \left(t + \frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})(1-\lambda)}\right)^p \mathrm{d}t \\
&= (1-\lambda)^p \frac{1}{p+1}\left[\left(\frac{1}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}\right].
\end{aligned}$$

As a result,

$$\int_0^1 \left|w(t)\right|^p \mathrm{d}t \leq (1-\lambda)^p \frac{2}{p+1}\left[\left(\frac{1}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}\right]. \tag{2.12}$$

Since $|g''|^q$ is convex on $[a,b]$, we get

$$\int_0^1 \left|g''\big(ta+(1-t)b\big)\right|^q \mathrm{d}t \leq \frac{|g''(a)|^q + |g''(b)|^q}{2}. \tag{2.13}$$

Using (2.12) and (2.13) in (2.11), we obtain the desired result in (2.9). Thus, this ends the proof for this case.

Now, suppose that $\lambda = 1$. The remainder of the argument is analogous to that of part one in Theorem 2.2 and we omit the details. Thus, the proof of Theorem 2.2 is completed. □

Corollary 2.3 *Under all assumptions of Theorem 2.2, if $|g''(x)| \leq M$ on $[a,b]$, then we obtain*

$$\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right| \\
&\quad\leq\begin{cases}\frac{M(b-a)^2(1-\lambda)}{2}(\frac{2}{p+1})^{\frac{1}{p}}[(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)})^{p+1}-(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)})^{p+1}]^{\frac{1}{p}}, & 0\leq\lambda<1,\\ \frac{M(b-a)^2\tanh(\frac{\rho}{4})}{2\rho}, & \lambda=1.\end{cases}
\end{aligned}$$

Corollary 2.4 *Consider Theorem 2.2.*

(1) *For $\lambda = 0$, we have the following midpoint inequality*:

$$\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-g\left(\frac{a+b}{2}\right)\right| \\
&\quad\leq\frac{(b-a)^2}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho}\right)^{p+1}-\left(\frac{\tanh(\frac{\rho}{4})}{\rho}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\quad\quad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

(2) *For $\lambda = \frac{1}{3}$, we have the following Simpson inequality*:

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]\right| \\
&\quad\leq\frac{(b-a)^2}{3}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{3\tanh(\frac{\rho}{4})}{2\rho}\right)^{p+1}-\left(\frac{3\tanh(\frac{\rho}{4})}{2\rho}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\quad\quad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

(3) *For $\lambda = \frac{1}{2}$, we have the following averaged midpoint-trapezoid integral inequality*:

$$\begin{aligned}
&\left|\frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]\right| \\
&\quad\leq\frac{(b-a)^2}{4}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{2\tanh(\frac{\rho}{4})}{\rho}\right)^{p+1}-\left(\frac{2\tanh(\frac{\rho}{4})}{\rho}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\quad\quad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 2.4 In (2.9) of Theorem 2.2, if we take $\alpha \to 1$, i.e. $\rho = \frac{1-\alpha}{\alpha}(b-a) \to 0$, then we have

$$\lim_{\alpha\to1}\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})}=\frac{1}{4}. \tag{2.14}$$

Using (2.6) and (2.14) in (2.9), Theorem 2.2 is transformed to

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\
&\quad\le\frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\\
&\quad\quad\times\left[\left(\frac{1}{2}+\frac{1}{4(1-\lambda)}\right)^{p+1}-\left(\frac{1}{4(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}}\\
&\quad\quad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.15}$$

Theorem 2.3 *Let $g:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) with $a<b$ satisfying $g''\in L^1([a,b])$ and $0\le\lambda<1$. If $|g''|^q$ is convex on $[a,b]$ with $q>1$, then the following inequality holds*:

$$\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\
&\quad\le\frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1}-\left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1}\right]^{\frac{1}{q}}\\
&\quad\quad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.16}$$

Proof Utilizing Lemma 2.1, the definition of $w(t)$, and the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\
&\quad\le\frac{(b-a)^2}{2}\left(\int_0^1 1\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|w(t)g''\big(ta+(1-t)b\big)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\le\frac{(b-a)^2}{2}\left(\left|g''(a)\right|^q\int_0^1 t\left|w(t)\right|^q\mathrm{d}t+\left|g''(b)\right|^q\int_0^1(1-t)\left|w(t)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}\tag{2.17}$$

Using the properties of integration, we get

$$\int_0^1 t\left|w(t)\right|^q\mathrm{d}t=\int_0^{\frac{1}{2}}t\left|w_1(t)\right|^q\mathrm{d}t+\int_{\frac{1}{2}}^1 t\left|w_2(t)\right|^q\mathrm{d}t$$

with

$$\begin{aligned}
\int_0^{\frac{1}{2}}t\left|w_1(t)\right|^q\mathrm{d}t&\le\int_0^{\frac{1}{2}}t\left(t(1-\lambda)+\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)^q\mathrm{d}t\\
&\le\int_0^{\frac{1}{2}}t\left(t(1-\lambda)+\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})}\right)^q\mathrm{d}t\\
&=(1-\lambda)^q\int_0^{\frac{1}{2}}t\left(t+\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})(1-\lambda)}\right)^q\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}\int_{\frac{1}{2}}^{1} t\left|w_2(t)\right|^q \,\mathrm{d}t &\le \int_{\frac{1}{2}}^{1} t\left((1-t)(1-\lambda)+\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)^q \mathrm{d}t \\ &\le \int_{\frac{1}{2}}^{1} t\left((1-t)(1-\lambda)+\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})}\right)^q \mathrm{d}t \\ &= (1-\lambda)^q \int_{\frac{1}{2}}^{1} t\left((1-t)+\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})(1-\lambda)}\right)^q \mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}&\int_{0}^{\frac{1}{2}} t\left(t+\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})(1-\lambda)}\right)^q \mathrm{d}t \\ &\quad= \frac{1}{2(q+1)}\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} - \frac{1}{(q+1)(q+2)}\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+2} \\ &\qquad+ \frac{1}{(q+1)(q+2)}\left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+2}\end{aligned}$$

and

$$\begin{aligned}&\int_{\frac{1}{2}}^{1} t\left((1-t)+\frac{(1-e^{-\frac{\rho}{2}})^2}{\rho(1-e^{-\rho})(1-\lambda)}\right)^q \mathrm{d}t \\ &\quad= -\frac{1}{(q+1)}\left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} + \frac{1}{2(q+1)}\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} \\ &\qquad- \frac{1}{(q+1)(q+2)}\left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+2} + \frac{1}{(q+1)(q+2)}\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+2}.\end{aligned}$$

Thus,

$$\int_{0}^{1} t\left|w(t)\right|^q \,\mathrm{d}t \le (1-\lambda)^q \frac{1}{q+1}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1}\right]. \tag{2.18}$$

Analogously,

$$\int_{0}^{1} (1-t)\left|w(t)\right|^q \,\mathrm{d}t \le (1-\lambda)^q \frac{1}{q+1}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1}\right]. \tag{2.19}$$

Using (2.18) and (2.19) in (2.17), we deduce the desired result in (2.16). Thus, the proof is completed. □

Corollary 2.5 *Under all assumptions of Theorem* 2.3, *if* $|g''(x)| \le M$ *on* $[a,b]$, *then the following inequality is true*:

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha} g(b)+\mathcal{I}_{b^-}^{\alpha} g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right| \\ &\quad\le \frac{M(b-a)^2(1-\lambda)}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1}\right]^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.6 *Consider Theorem 2.3.*

(1) *For $\lambda = 0$, we have the following midpoint inequality*:

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-g\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho}\right)^{q+1}-\left(\frac{\tanh(\frac{\rho}{4})}{\rho}\right)^{q+1}\right]^{\frac{1}{q}}\\&\qquad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

(2) *For $\lambda = \frac{1}{3}$, we have the following Simpson inequality*:

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]\right|\\&\quad\le\frac{(b-a)^2}{3}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{3\tanh(\frac{\rho}{4})}{2\rho}\right)^{q+1}-\left(\frac{3\tanh(\frac{\rho}{4})}{2\rho}\right)^{q+1}\right]^{\frac{1}{q}}\\&\qquad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

(3) *For $\lambda = \frac{1}{2}$, we have the averaged midpoint-trapezoid integral inequality*:

$$\begin{aligned}&\left|\frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]\right|\\&\quad\le\frac{(b-a)^2}{4}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{2\tanh(\frac{\rho}{4})}{\rho}\right)^{q+1}-\left(\frac{2\tanh(\frac{\rho}{4})}{\rho}\right)^{q+1}\right]^{\frac{1}{q}}\\&\qquad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.4 *Let $g:[a,b]\to\mathbb{R}$ be a twice differentiable mapping on (a,b) with $a<b$ satisfying $g''\in L^1([a,b])$ and $0\le\lambda\le1$. If $|g''|^q$ is convex on $[a,b]$ with $q>1$, then the following inequality holds*:

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1-\lambda}{4}+\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}\right)\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}\tag{2.20}$$

Proof Utilizing Lemma 2.1, the definition of $w(t)$, and the power-mean integral inequality, we have

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{2}\int_0^1\left|w(t)\right|\left|g''\big(ta+(1-t)b\big)\right|\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\le \frac{(b-a)^2}{2}\left(\int_0^1 \left|w(t)\right| \mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|w(t)\right|\left|g''\left(ta+(1-t)b\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \frac{(b-a)^2}{2}\left(\int_0^1 \left|w(t)\right| \mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\left|g''(a)\right|^q \int_0^1 t\left|w(t)\right|\mathrm{d}t + \left|g''(b)\right|^q \int_0^1 (1-t)\left|w(t)\right| \mathrm{d}t\right)^{\frac{1}{q}}. \qquad (2.21)
\end{aligned}
$$

Using the properties of the modulus and direct computation, we obtain

$$
\begin{aligned}
\int_0^1 \left|w(t)\right|\mathrm{d}t &= \int_0^{\frac{1}{2}} \left|w_1(t)\right| \mathrm{d}t + \int_{\frac{1}{2}}^1 \left|w_2(t)\right|\mathrm{d}t \\
&\le \int_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1 \left((1-t)(1-\lambda) + \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)\mathrm{d}t \\
&= \frac{1-\lambda}{4} + \frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})} \qquad (2.22)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1 (1-t)\left|w(t)\right|\mathrm{d}t = \int_0^1 t\left|w(t)\right|\mathrm{d}t
&= \int_0^{\frac{1}{2}} t\left|w_1(t)\right|\mathrm{d}t + \int_{\frac{1}{2}}^1 t\left|w_2(t)\right|\mathrm{d}t \\
&\le \int_0^{\frac{1}{2}} t\left(t(1-\lambda) + \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1 t\left((1-t)(1-\lambda) + \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)\mathrm{d}t \\
&= \frac{1-\lambda}{8} + \frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}. \qquad (2.23)
\end{aligned}
$$

Using (2.22) and (2.23) in (2.21), we obtain the desired result in (2.20). Thus, the proof is completed. □

Corollary 2.7 *Under all assumptions of Theorem 2.4, if $|g''(x)| \le M$ on $[a,b]$, then the following inequality is true:*

$$
\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha} g(b) + \mathcal{I}_{b^-}^{\alpha} g(a)\right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda\frac{g(a)+g(b)}{2}\right| \\
&\quad \le \frac{(b-a)^2 M}{2}\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})} + \frac{1-\lambda}{4}\right).
\end{aligned}
$$

Corollary 2.8 *Consider Theorem 2.4.*

(1) *For* $\lambda = 0$, *we have the following midpoint inequality*:

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-g\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1}{4}+\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}\right)\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

(2) *For* $\lambda = \frac{1}{3}$, *we have the following Simpson inequality*:

$$\begin{aligned}&\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1}{6}+\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}\right)\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

(3) *For* $\lambda = \frac{1}{2}$, *we have the averaged midpoint-trapezoid integral inequality*:

$$\begin{aligned}&\left|\frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1}{8}+\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}\right)\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

(4) *For* $\lambda = 1$, *we have the trapezoid inequality*:

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-\frac{g(a)+g(b)}{2}\right|\\&\quad\le\frac{(b-a)^2(\rho+\rho e^{-\rho}+2e^{-\rho}-2)}{2\rho^2(1-e^{-\rho})}\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 2.5 Using (2.6) and (2.7) in (2.20), Theorem 2.4 is transformed to

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\le\frac{(b-a)^2}{2}\left(\frac{1-\lambda}{4}+\frac{1}{6}\right)\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}\tag{2.24}$$

Corollary 2.9 *Under all assumptions of Theorems 2.1–2.4 with* $0 \le \lambda < 1$, *we have*

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\le\min\{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4\},\end{aligned}$$

where

$$\mathcal{L}_1=(b-a)^2M\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}+\frac{1-\lambda}{8}\right),$$

$$\mathcal{L}_2 = \frac{M(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}} \times \left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}},$$

$$\mathcal{L}_3 = \frac{M(b-a)^2(1-\lambda)}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}} \times \left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{q+1}\right]^{\frac{1}{q}}$$

and

$$\mathcal{L}_4 = \frac{(b-a)^2M}{2}\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}+\frac{1-\lambda}{4}\right).$$

3 Examples

In this section, we provide two examples to illustrate our main results.

Example 3.1 Let $g(x) = x^2$, for $x \in (-\infty,\infty)$. Then $|g''|$ is convex on $(-\infty,\infty)$. If we take $a = 0$, $b = 1$, $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{4}$, then all assumptions in Theorem 2.1 are satisfied.

Clearly, $\rho = \frac{1-\alpha}{\alpha}(b-a) = 1$. The left-hand side term of (2.2) is

$$\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right| \\
&\quad=\left|\frac{1}{2(1-e^{-1})}\left(\int_0^1 e^{s-1}s^2\,\mathrm{d}s+\int_0^1 e^{-s}s^2\,\mathrm{d}s\right)-\frac{5}{16}\right| \\
&\quad=\left|\frac{1}{2(1-e^{-1})}\left(\left(1-2e^{-1}\right)+\left(2-5e^{-1}\right)\right)-\frac{5}{16}\right|\approx 0.0235.
\end{aligned}$$

The right-hand side term of (2.2) is

$$\begin{aligned}
&\frac{(b-a)^2}{2}\left(\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{2\rho^2(1-e^{-\rho})}+\frac{1-\lambda}{8}\right)\left(\left|g''(a)\right|+\left|g''(b)\right|\right) \\
&\quad=\frac{3e^{-1}-1}{(1-e^{-1})}+\frac{3}{16}\approx 0.3515.
\end{aligned}$$

It is clear that $0.0235 < 0.3515$, which demonstrates the result described in Theorem 2.1.

Example 3.2 Let $g(x) = e^x$, for $x \in (-\infty,\infty)$. Then $|g''|^q$ is convex on $(-\infty,\infty)$. If we take $a = 0$, $b = 1$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$ and $p = 2 = q$, then all assumptions in Theorem 2.2 are satisfied.

Clearly, $\rho = \frac{1-\alpha}{\alpha}(b-a) = 1$. The left-hand side term of (2.9) is

$$\begin{aligned}
&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right| \\
&\quad=\left|\frac{1}{2(1-e^{-1})}\left(\int_0^1 e^{s-1}e^s\,\mathrm{d}s+\int_0^1 e^{-s}e^s\,\mathrm{d}s\right)-\frac{1}{2}e^{\frac{1}{2}}-\frac{1+e}{4}\right| \\
&\quad=\left|\frac{1}{2(1-e^{-1})}\left(\frac{e-e^{-1}}{2}+1\right)-\frac{1}{2}e^{\frac{1}{2}}-\frac{1+e}{4}\right|\approx 0.0334.
\end{aligned}$$

The right-hand side term of (2.9) is

$$\begin{aligned}
&\frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}-\left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\quad\times\left(\frac{|g''(a)|^q+|g''(b)|^q}{2}\right)^{\frac{1}{q}} \\
&\quad=\frac{1}{4}\left(\frac{1+e^2}{3}\right)^{\frac{1}{2}}\left[\left(\frac{1}{2}+2\tanh\left(\frac{1}{4}\right)\right)^3-\left(2\tanh\left(\frac{1}{4}\right)\right)^3\right]^{\frac{1}{2}}\approx 0.3859.
\end{aligned}$$

It is clear that $0.0334 < 0.3859$, which demonstrates the result described in Theorem 2.2.

Remark 3.1 Theorems 2.1–2.4 provide an upper bound for the approximation of the fractional integrals $\frac{1-\alpha}{2(1-e^{-\rho})}[\mathcal{I}^{\alpha}_{a^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g(a)]$. There exist certain integral functions that cannot be expressed by elementary functions. So Theorems 2.1–2.4 are of importance to deal with such integral functions. For example, let $g(x)=e^{-x^2+x}$, for $x\in[2,\infty)$. Then $|g''|^q$ for $q\geq 1$ is convex on $[2,\infty)$. If we take $a=2$, $b=3$, $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{2}$, then all assumptions in Theorem 2.3 are satisfied.

Clearly, $\rho=\frac{1-\alpha}{\alpha}(b-a)=1$. The left-hand side term of (2.16) is

$$\left|\frac{1}{2(1-e^{-1})}\left(e^{-2}\int_2^3 e^{-(s-1)^2}\,\mathrm{d}s+e^2\int_2^3 e^{-s^2}\,\mathrm{d}s\right)-\frac{1}{2}e^{-\frac{15}{4}}-\frac{1}{2}\frac{e^{-2}+e^{-6}}{2}\right|. \tag{3.1}$$

Obviously, the term $\int_2^3 e^{-(s-1)^2}\,\mathrm{d}s$ and $\int_2^3 e^{-s^2}\,\mathrm{d}s$ cannot be solved directly due to the fact that $\int e^{-s^2}\,\mathrm{d}s$ cannot be expressed by elementary functions. However, applying Theorem 2.3 with $q=2$, we obtain an upper bound for (3.1), i.e.

$$\begin{aligned}
&\frac{1}{4}\left(\frac{2}{3}\right)^{\frac{1}{2}}\left[\left(\frac{1}{2}+2\tanh\left(\frac{1}{4}\right)\right)^3-\left(2\tanh\left(\frac{1}{4}\right)\right)^3\right]^{\frac{1}{2}}\left(\frac{(7e^{-2})^2+(23e^{-6})^2}{2}\right)^{\frac{1}{2}} \\
&\quad\approx 0.1265.
\end{aligned} \tag{3.2}$$

4 Application to special means

We consider the following means for arbitrary real numbers m, n ($m\neq n$).

(a) The arithmetic mean:

$$A(m,n)=\frac{m+n}{2}.$$

(b) The geometric mean:

$$G(m,n)=\sqrt{mn},\quad mn\geq 0.$$

(c) The harmonic mean:

$$H(m,n)=\frac{2}{\frac{1}{m}+\frac{1}{n}},\quad m,n\in\mathbb{R}\setminus\{0\}, m\neq -n.$$

(d) The logarithmic mean:

$$L(m,n) = \frac{m-n}{\ln|m| - \ln|n|}, \quad |m| \neq |n|, mn \neq 0.$$

(e) The generalized logarithmic mean:

$$L_r(m,n) = \left[\frac{n^{r+1} - m^{r+1}}{(n-m)(r+1)}\right]^{\frac{1}{r}}, \quad r \in \mathbb{Z} \setminus \{-1,0\}, m \neq n.$$

(f) The identric mean:

$$I(m,n) = \begin{cases} m, & m = n, \\ \frac{1}{e}(\frac{n^n}{m^m})^{\frac{1}{n-m}}, & m \neq n, \end{cases} \quad m,n > 0.$$

We have the following results.

Proposition 4.1 *Let $m,n \in \mathbb{R}$, $m < n$, $0 \leq \lambda \leq 1$ and $r \in \mathbb{Z}$, $|r| \geq 2$. Then*

$$\begin{aligned} &\left|L_r^r(m,n) - (1-\lambda)A^r(m,n) - \lambda A\left(m^r,n^r\right)\right| \\ &\quad \leq (n-m)^2\left(\frac{1}{12} + \frac{1-\lambda}{8}\right)r(r-1)A\left(|m|^{r-2},|n|^{r-2}\right). \end{aligned}$$

Proof Applying the mapping $g(x) = x^r$, $x \in \mathbb{R}$, $|r| \geq 2$ to Remark 2.1, we obtain the required result. □

Proposition 4.2 *Let $m,n \in \mathbb{R}$, $0 < m < n$ and $0 \leq \lambda \leq 1$. Then*

$$\begin{aligned} &\left|L^{-1}(m,n) - (1-\lambda)A^{-1}(m,n) - \lambda H^{-1}(m,n)\right| \\ &\quad \leq (n-m)^2(1-\lambda)\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2} + \frac{1}{4(1-\lambda)}\right)^{p+1} - \left(\frac{1}{4(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}} \\ &\qquad \times A^{\frac{1}{q}}\left(m^{-3q},n^{-3q}\right). \end{aligned}$$

Proof Applying the mapping $g(x) = \frac{1}{x}$, for $x > 0$ to Remark 2.4, we obtain the required result. □

Proposition 4.3 *Let $m,n \in \mathbb{R}$, $0 < m < n$ and $0 \leq \lambda \leq 1$. Then*

$$\begin{aligned} &\left|-\ln I(m,n) + (1-\lambda)\ln A(m,n) + \lambda \ln G(m,n)\right| \\ &\quad \leq \frac{(n-m)^2}{2}\left(\frac{1-\lambda}{4} + \frac{1}{6}\right)A^{\frac{1}{q}}\left(m^{-2q},n^{-2q}\right). \end{aligned}$$

Proof Applying the mapping $g(x) = -\ln x$, for $x > 0$ to Remark 2.5, we obtain the required result. □

Next, we give an application using trapezoid formula and midpoint formula. Let $X : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$. We consider the following quadrature formula:

$$\int_a^b g(x)\,\mathrm{d}x = T_i(g,X) + E_i(g,X), \quad i = 1,2, \tag{4.1}$$

where

$$T_1(g,X) = \sum_{i=0}^{n-1} \frac{g(x_i)+g(x_{i+1})}{2}(x_{i+1}-x_i) \tag{4.2}$$

is the trapezoid version, and

$$T_2(g,X) = \sum_{i=0}^{n-1} g\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i), \tag{4.3}$$

is the midpoint version. The related approximation error is denoted by $E_i(g,X)$, $i = 1,2$.

Now, we derive an error estimate related to trapezoid formula and midpoint formula.

Proposition 4.4 *Let $g : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $g'' \in L^1([a,b])$ and $|g''|$ is convex on $[a,b]$ with $0 \le \lambda \le 1$, for every division X of $[a,b]$, the following inequality holds*:

$$\left|\lambda E_1(g,X) + (1-\lambda)E_2(g,X)\right| \le \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^3}{2}\left(\frac{1}{12}+\frac{1-\lambda}{8}\right)\left(\left|g''(x_i)\right| + \left|g''(x_{i+1})\right|\right).$$

Proof Using Eqs. (4.1), (4.2) and (4.3), we have

$$\lambda E_1(g,X) = \lambda \int_a^b g(x)\,\mathrm{d}x - \lambda T_1(g,X)$$

and

$$(1-\lambda)E_2(g,X) = (1-\lambda)\int_a^b g(x)\,\mathrm{d}x - (1-\lambda)T_2(g,X).$$

Applying Remark 2.1 on the subinterval $[x_i, x_{i+1}]$ $(i = 0,1,\dots,n-1)$ of the division X, we deduce

$$\begin{aligned}
&\left|\int_{x_i}^{x_{i+1}} g(x)\,\mathrm{d}x - (1-\lambda)g\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i) - \lambda\frac{g(x_i)+g(x_{i+1})}{2}(x_{i+1}-x_i)\right| \\
&\quad \le \frac{(x_{i+1}-x_i)^3}{2}\left(\frac{1}{12}+\frac{1-\lambda}{8}\right)\left(\left|g''(x_i)\right| + \left|g''(x_{i+1})\right|\right).
\end{aligned}$$

Summing over from 0 to $n-1$ and utilizing the convexity of $|g''|$, we have

$$\begin{aligned}
&\left|\lambda E_1(g,u) + (1-\lambda)E_2(g,u)\right| \\
&\quad = \left|\int_a^b g(x)\,\mathrm{d}x - (1-\lambda)T_2(g,X) - \lambda T_1(g,X)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{i=0}^{n-1} \left[\int_{x_i}^{x_{i+1}} g(x)\,dx - (1-\lambda) g\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) - \lambda \frac{g(x_i) + g(x_{i+1})}{2}(x_{i+1} - x_i) \right] \right| \\
&\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} g(x)\,dx - (1-\lambda) g\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) - \lambda \frac{g(x_i) + g(x_{i+1})}{2}(x_{i+1} - x_i) \right| \\
&\leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{2} \left(\frac{1}{12} + \frac{1-\lambda}{8} \right) \left(\left| g''(x_i) \right| + \left| g''(x_{i+1}) \right| \right).
\end{aligned}
$$

Thus, the proof is completed. □

Remark 4.1 For $\lambda = 0$, we have

$$
\left| E_2(g, X) \right| \leq \sum_{i=0}^{n-1} \frac{5(x_{i+1} - x_i)^3}{48} \left(\left| g''(x_i) \right| + \left| g''(x_{i+1}) \right| \right),
$$

which is given by Wu et al. in [34], Proposition 4.

Remark 4.2 For $\lambda = 1$, we have

$$
\left| E_1(g, X) \right| \leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{24} \left(\left| g''(x_i) \right| + \left| g''(x_{i+1}) \right| \right).
$$

5 Conclusion

Using the fractional integrals with exponential kernels, certain inequalities related to the Hermite–Hadamard and Simpson inequalities for convex mappings are established. The inequalities are parameterized by the parameter $0 \leq \lambda \leq 1$. These inequalities generalize and extend parts of the results provided by Wu et al. in [34]. Some applications of the obtained results to special means and quadrature formula are also presented. With these contributions, we hope to motivate the interested researcher to further explore this enchanting field of the fractional integral inequalities based on these techniques and the ideas developed in the present paper.

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