


RESEARCH

Open Access



# Tenth order boundary value problem solution existence by fixed point theorem

Nicola Fabiano<sup>1,2</sup>, Nebojša Nikolić<sup>3\*</sup> , Thenmozhi Shanmugam<sup>4</sup>, Stojan Radenović<sup>1,2</sup> and Nada Čitaković<sup>5</sup>

\*Correspondence:

[nebojsa.nikolic@fon.bg.ac.rs](mailto:nebojsa.nikolic@fon.bg.ac.rs)

<sup>3</sup>Faculty of Organizational Sciences, University of Belgrade, Belgrade, Serbia

Full list of author information is available at the end of the article

## Abstract

In this paper we consider the Green function for a boundary value problem of generic order. For a specific case, the Leray–Schauder form of the fixed point theorem has been used to prove the existence of a solution for this particular equation. Our theoretical approach generalizes, extends, complements, and enriches several results in the existing literature.

**MSC:** Primary 34B10; secondary 34B15; 34K10

**Keywords:** Boundary value problem; Green function; Fixed point; Leray–Schauder

## 1 Introduction

Boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences. The higher-order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydro-magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. Such problems have been studied by many authors. For example, Oderinu [24] applied weighted residual via partition method to obtain a numerical solution for 10th and 12th order linear and nonlinear boundary value problems. Mohyud-Din and Yildirim [22] used modified variational iteration method for solving 9th and 10th order boundary value problems. Iqbal et al. [11] constructed a cubic spline algorithm to approximate 10th order boundary value problems. Noor et al. [23] applied variational iterative method to solve 10th order boundary value problems. Mai-Duy [20] have presented Chebyshev spectral collocation method to solve high-order ordinary differential equations. Islam et al. [13] solved 10th and 12th order linear and nonlinear boundary value problems numerically by the Galerkin weighted residual technique with two point boundary conditions. Akgül et al. [5] have given some reproducing kernel functions to find approximate solutions of 10th order boundary value problems. Jørgensen et al. [15] have presented a hierarchical basis of arbitrary order for integral equations solved with the method of moments (MoM). Akgül et al. [4] implemented reproducing kernel Hilbert space method to obtain approximate solutions to 10th order boundary value problems. Ramadan et al. [25] used homotopy analysis method to solve 7th, 8th, and 10th order

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

boundary value problems. Ramos and Singh [26] have presented a two-step hybrid block method for the numerical integration of ordinary differential initial value systems. Siddiqi and Twizell [30] developed an algorithm to approximate the solutions, and their higher-order derivatives, of differential equations. Siddiqi and Akram [28] found numerical solutions for 10th-order linear special case boundary value problems using an 11th degree spline. The same authors in [29] used a nonpolynomial spline to obtain numerical solutions for 10th order linear special case boundary value problems. Twizell et al. [31] have developed 2nd order finite-difference methods to obtain the numerical solutions for 8th, 10th, and 12th order eigenvalue problems. Wazwaz [32] proposed an algorithm for solving linear and nonlinear boundary value problems with two-point boundary conditions of 10th and 12th order. Ma [19] has given existence and uniqueness theorems based on the Leray–Schauder fixed point theorem for some 4th order nonlinear boundary value problems. Zvyagin and Baranovskii [33] have constructed a topological characteristic to investigate a class of controllable systems. Ahmad and Ntouyas [3] conferred some existence results based on some standard fixed point theorems and Leray–Schauder degree theory for a higher-order nonlinear differential equation with four-point nonlocal integral boundary conditions. In [14], a theorem of coupled fixed point on ordered sets has been proved, and its results have been used to obtain the existence and uniqueness of positive solution for a class of boundary value problems for fractional differential equations with singularities. Afsari et al. [2] introduced some new coupled fixed point theorems which have been used for finding a solution to a fractional differential equation of order  $\alpha \in (0, 1)$ . In [6], generalized  $\alpha$ - $\psi$ -contractive mappings have been introduced in metric-like spaces and some fixed point theorems have been proved. Such results are applied to a two-point boundary value problem for 2nd order differential equations. In [17], a new notion of Berinde type  $(\alpha, \psi)$  contraction and the existence and uniqueness of a fixed point for such mapping have been proved, and an application to nonlinear fractional differential equation was given. Aydi et al. [7] improved and extended a previous proof for existence and uniqueness for a differential problem with fixed point results by replacing  $\alpha$ -admissibility with orbital  $\alpha$ -admissibility. In the framework of extended b-metric spaces, Abdeljawad et al. [1] suggested fixed points results to a nonlinear Volterra–Fredholm integral equation and a Caputo fractional derivative differential equation. Karapinar et al. [16] unified existing fixed point results in the literature to show the existence of solutions for 2nd order nonlinear differential equations and Caputo fractional derivative boundary value problem of order  $\beta \in [1, 2]$ . More about the results related to the fixed point theory can be found in [10] and [21]. Motivated by these studies, we investigate the generic differential equation of order  $2n$  and hence seek results on the existence of solutions for 10th order boundary value problems.

## 2 The problem

We consider the generic differential equation of order  $2n$ :

$$y^{(2n)}(x) = \phi(x, y(x), y''(x)), \tag{1}$$

with boundary conditions

$$\begin{aligned} y(0) = y^{(1)}(0) = \dots = y^{(n)}(0) = 0, \\ y^{(n+1)}(1) = y^{(n+2)}(1) = \dots = y^{(2n-1)}(1) = 0. \end{aligned} \tag{2}$$

Such equations occur, for instance, when studying the problem of the onset of thermal instability in horizontal layers of fluid heated from below. The temperature gradient, maintained by heating the underside, is adverse since, on account of thermal expansion, the fluid at the bottom will be lighter than the fluid at the top [9].

Define the Green functions  $G_l, G_r \in C^{2n}$  of problem (1),  $G_l(x, s)$  for  $0 \leq x < s \leq 1$  and  $G_r(x, s)$  for  $0 \leq s < x \leq 1$ , such that they solve the following equation ( $G_{l,r}$  is  $G_l$  or  $G_r$ ):

$$\left(\frac{\partial}{\partial x}\right)^{2n} G_{l,r}(x, s) = \delta(s - x). \tag{3}$$

Given the inhomogeneous problem

$$y^{(2n)}(x) = f(x), \tag{4}$$

$x \in [0, 1], f \in C[0, 1]$ , together with boundary conditions

$$\begin{aligned} y(0) = y^{(1)}(0) = \dots = y^{(n)}(0) = 0, \\ y^{(n+1)}(1) = y^{(n+2)}(1) = \dots = y^{(2n-1)}(1) = 0, \end{aligned} \tag{5}$$

the Green functions provide solution to (4) in the integral form

$$y(x) = \int_0^1 G_{l,r}(x, s) f(s) ds. \tag{6}$$

Those functions are polynomials in  $x$  and  $s$  to be sought for in the form

$$G_l(x, s) = \frac{1}{(2n - 1)!} \sum_{i=0}^{2n-1} a_i x^i \tag{7}$$

and

$$G_r(x, s) = \frac{1}{(2n - 1)!} \sum_{i=0}^{2n-1} b_i (1 - x)^i, \tag{8}$$

where the coefficients  $a_i$  and  $b_i$  are polynomials in  $s$ .

Imposing boundary conditions (2) to Green functions respectively as

$$G_l(0, s) = \left(\frac{\partial}{\partial x}\right) G_l(x, s) \Big|_{x=0} = \dots = \left(\frac{\partial}{\partial x}\right)^n G_l(x, s) \Big|_{x=0} = 0 \tag{9}$$

and

$$\left(\frac{\partial}{\partial x}\right)^{n+1} G_r(x, s) \Big|_{x=1} = \dots = \left(\frac{\partial}{\partial x}\right)^{2n-1} G_r(x, s) \Big|_{x=1} = 0, \tag{10}$$

we conclude that many coefficients  $a_i$  and  $b_i$  do not contribute:

$$a_0 = a_1 = \dots = a_{n-1} = 0 \tag{11}$$

and

$$b_n = b_{n+1} = \dots = b_{2n-1} = 0. \tag{12}$$

Therefore, it is possible to write down equations (7) and (8) in a slightly simplified manner:

$$G_l(x, s) = \frac{1}{(2n-1)!} \sum_{i=n}^{2n-1} a_i x^i \tag{13}$$

and

$$G_r(x, s) = \frac{1}{(2n-1)!} \sum_{i=0}^{n-1} b_i (1-x)^i, \tag{14}$$

where for the coefficients  $a_i$  the maximal power of  $s$  involved is  $s^{n-1}$ , while for the coefficients  $b_i$  the minimal power of  $s$  is  $s^{n+1}$ .

The resulting Green functions  $G_{l,r}(x, s)$  and their  $x$  derivatives are continuous up to order  $2n - 2$  and present the discontinuity of  $-1$  at order  $2n - 1$  because of the Dirac  $\delta$  function. This concludes the proof of the following lemma:

**Lemma 2.1** *Let  $x \mapsto y(x), x \in [0, 1]$  be a function of class  $C^{2n}$  in  $\mathbb{R}$ , and  $(x, y, z) \mapsto \phi(x, y, z), x \in [0, 1], (y, z) \in \mathbb{R}^2$  be a function of class  $C$  in  $\mathbb{R}$ . Then the Green function of the problem defined in (4) and (5), obeying equation (3), is given by formulas (13) and (14).*

### 3 The case $n = 5$

Turning our attention to the particular case  $n = 5$ , we should solve the problem outlined in Sect. 2 for functions

$$G_l(x, s) = \frac{1}{9!} \sum_{i=5}^9 a_i x^i \tag{15}$$

and

$$G_r(x, s) = \frac{1}{9!} \sum_{i=0}^4 b_i (1-x)^i, \tag{16}$$

which should obey the system of equations

$$\begin{cases} G_l(s, s) = G_r(s, s), \\ (\frac{\partial}{\partial x})G_l(x, s)|_{x=s} = (\frac{\partial}{\partial x})G_r(x, s)|_{x=s}, \\ (\frac{\partial}{\partial x})^2 G_l(x, s)|_{x=s} = (\frac{\partial}{\partial x})^2 G_r(x, s)|_{x=s}, \\ \dots \\ (\frac{\partial}{\partial x})^8 G_l(x, s)|_{x=s} = (\frac{\partial}{\partial x})^8 G_r(x, s)|_{x=s}, \\ (\frac{\partial}{\partial x})^9 G_l(x, s)|_{x=s} = -1. \end{cases} \tag{17}$$

The solution to problem (17) is given by the following functions:

$$G_l(x, s) = -\frac{1}{9!} x^5 (x^4 - 9x^3 s + 36x^2 s^2 - 84xs^3 + 126s^4) \tag{18}$$

and

$$G_r(x, s) = -\frac{1}{9!} s^5 (s^4 - 9s^3x + 36s^2x^2 - 84sx^3 + 126x^4), \tag{19}$$

which could be rearranged as follows:

$$G_l(x, s) = -\frac{1}{9!} x^5 \left[ s^3 \left( \frac{776}{15} s - x \right) + 15s^2 \left( \frac{32}{15} s - x \right)^2 + (s-x)^4 + 5s(s-x)^3 \right],$$

$$G_r(x, s) = -\frac{1}{9!} s^5 \left[ x^3 \left( \frac{776}{15} x - s \right) + 15x^2 \left( \frac{32}{15} x - s \right)^2 + (x-s)^4 + 5x(x-s)^3 \right].$$

One could also expand (18) and (19) to obtain:

$$G_l(x, s) \cdot 9! = -x^9 + 9sx^8 - 36s^2x^7 + 84s^3x^6 - 126s^4x^5,$$

$$G_r(x, s) \cdot 9! = -126s^5x^4 + 84s^6x^3 - 36s^7x^2 + 9s^8x - s^9.$$

This concludes the proof of the following lemma:

**Lemma 3.1** *Let  $x \mapsto y(x), x \in [0, 1]$  be a function of class  $C^{10}$  in  $\mathbb{R}$ , and  $(x, y, z) \mapsto \phi(x, y, z), x \in [0, 1], (y, z) \in \mathbb{R}^2$  be a function of class  $C$  in  $\mathbb{R}$ . Then the Green function of the problem defined in (4) and (5) for  $n = 5$ , obeying equations (3) and (17), is given by formulas (18) and (19).*

#### 4 The kernel

In this section we will provide some results obtained by focusing on the particular case of the problem for  $n = 5$ .

Define the integral operator  $T$  as follows:

$$Ty(x) := \int_0^x G_r(x, s)f(s) ds + \int_x^1 G_l(x, s)f(s) ds. \tag{20}$$

According to Lemma 2.1, this operator provides a solution of problem defined in (4) and (5) for a generic order  $n$ , provided it has a fixed point in  $X$ .

We shall make use of the following theorem of [8], [27], the Leray–Schauder form of the fixed point theorem [12], [18]:

**Theorem 4.1** *Let  $(E, \|\cdot\|)$  be a Banach space,  $U \subset E$  is an open bounded subset such that  $0 \in U$  and  $T : \overline{U} \rightarrow E$  is a completely continuous operator. Then either  $T$  has a fixed point,  $T\bar{x} = \bar{x}, \bar{x} \in \overline{U}$  or there exist an element  $x \in \partial\overline{U}$  and a real number  $\lambda > 1$  such that  $Tx = \lambda x$ .*

Therefore, in order to establish the existence of a solution, it is necessary to prove that our integral operator  $T$  has actually at least an eigenvalue larger than 1, i.e.,  $\lambda > 1$ . The following two theorems are devoted to this problem.

**Theorem 4.2** *Let  $x \mapsto y(x), x \in [0, 1]$  be a function of class  $C^{10}$  in  $\mathbb{R}$ ,  $(x, y, z) \mapsto \phi(x, y, z), x \in [0, 1], (y, z) \in \mathbb{R}^2$  be a function of class  $C$  in  $\mathbb{R}$  and  $|\phi(x, 0, 0)| \neq 0$ . Suppose that there exist three nonnegative functions  $x \mapsto u(x), v(x), w(x) \in L^1[0, 1]$  such that*

$$|\phi(x, y, z)| \leq u(x)|y| + v(x)|z| + w(x).$$

Define the kernel

$$\mathcal{K}(s) := \frac{1}{8!}(s^8 + 10s^6)$$

and suppose that

$$A := \int_0^1 \mathcal{K}(s)[u(s) + v(s)] ds < 1.$$

Then the problem defined in equations (1) and (2) for  $n = 5$  has at least one nontrivial solution  $x \mapsto \xi(x)$ ,  $x \in [0, 1]$  of class  $C^{10}$  in  $\mathbb{R}$ .

*Proof* Define the constant

$$B := \int_0^1 \mathcal{K}(s)w(s) ds.$$

By hypothesis,  $A < 1$  and  $w(s) \geq 0$ . Observe that  $\mathcal{K}(s) > 0$  for  $s \in [0, 1]$ . As  $|\phi(x, y, z)| \leq u(x)|y| + v(x)|z| + w(x)$  for all  $x \in [0, 1]$ ,  $(y, z) \in \mathbb{R}^2$  and  $\phi(x, 0, 0) \neq 0$  for all  $x \in [0, 1]$ , there exist an interval  $[a, b] \subset [0, 1]$  such that  $\max_{x \in [a, b]} |\phi(x, 0, 0)| > 0$ . Therefore,  $|\phi(x, 0, 0)| > 0$  and also  $w(x) > 0$  for some  $x \in [a, b] \subset [0, 1]$ . That implies the inequality  $\int_0^1 \mathcal{K}(s)w(s) ds \geq \int_a^b \mathcal{K}(s)w(s) ds > 0$ . We conclude that  $A < 1$  and  $B > 0$ .

Define  $L := B(1 - A)^{-1}$  which is positive by construction, and the set  $U = \{y \in E : \|y\| < L\}$ . Assume that  $y \in \partial U$  and  $\lambda > 1$ . As  $Ty = \lambda y$  per hypothesis, then  $\lambda L = \lambda \|y\| = \|Ty\| = \max_{x \in [0, 1]} |(Ty)(x)|$ . Adopting the simplified notation  $d\mu = |\phi(s, y(s), y''(s))| ds$ , we have

$$\begin{aligned} \lambda L &= \max_{x \in [0, 1]} |(Ty)(x)| \\ &\leq \left\{ \int_0^x G_r(x, s) d\mu + \int_x^1 G_l(x, s) d\mu \right\} \\ &= \frac{1}{9!} \left\{ - \int_0^x \left[ x^3 \left( \frac{776}{15}x - s \right) + 15x^2 \left( \frac{32}{15}x - s \right)^2 \right. \right. \\ &\quad \left. \left. + (x - s)^4 + 5x(x - s)^3 \right] s^5 d\mu - \int_x^1 \left[ s^3 \left( \frac{776}{15}s - x \right) \right. \right. \\ &\quad \left. \left. + 15s^2 \left( \frac{32}{15}s - x \right)^2 + (s - x)^4 + 5s(s - x)^3 \right] x^5 d\mu \right\} \\ &\leq \frac{1}{9!} \max_{x \in [0, 1]} \left\{ - \int_0^x \left[ x^3 \left( \frac{776}{15}x - s \right) + 15x^2 \left( \frac{32}{15}x - s \right)^2 \right. \right. \\ &\quad \left. \left. + (x - s)^4 + 5x(x - s)^3 \right] s^5 d\mu - \int_x^1 \left[ s^3 \left( \frac{776}{15}s - x \right) \right. \right. \\ &\quad \left. \left. + 15s^2 \left( \frac{32}{15}s - x \right)^2 + (s - x)^4 + 5s(s - x)^3 \right] x^5 d\mu \right\} \\ &\leq \frac{1}{9!} \left\{ - \int_0^1 \left[ \left( \frac{776}{15} - s \right) + 15 \left( \frac{32}{15} - s \right)^2 \right. \right. \\ &\quad \left. \left. + (1 - s)^4 + 5(1 - s)^3 \right] s^5 d\mu - \int_0^1 \left[ s^3 \left( \frac{776}{15} - s \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + 15s^2 \left( \frac{32}{15}s \right)^2 + (s)^4 + 5s(s)^3 \Big] s^5 \, d\mu \Big\} \\
 = & \frac{1}{9!} \left\{ - \int_0^1 [126 - 84s + 36s^2 - 9s^3 + s^4] s^5 \, d\mu \right. \\
 & \left. - \int_0^1 \left[ \frac{776}{15} + \frac{1024}{15} + 1 + 5 \right] s^9 \, d\mu \right\} \\
 = & \frac{1}{9!} \int_0^1 [-127s^9 + 9s^8 - 36s^7 + 84s^6 - 126s^5] \, d\mu \\
 \leq & \frac{1}{9!} \int_0^1 [9s^8 + 84s^6] \, d\mu \\
 \leq & \frac{1}{8!} \int_0^1 [s^8 + 10s^6] \, d\mu \\
 = & \int_0^1 \mathcal{K}(s) \, d\mu. \tag{21}
 \end{aligned}$$

By hypothesis,  $|\phi(x, 0, 0)|$  has an upper bound for  $x \in [0, 1]$ , so one has

$$\begin{aligned}
 \int_0^1 \mathcal{K}(s) |\phi(s, 0, 0)| \, ds & \leq \int_0^1 \mathcal{K}(s) [u(s)|y(s)| + v(s)|y''(s)| + w(s)] \, ds \\
 & \leq \int_0^1 \mathcal{K}(s) \left[ u(s) \max_{s \in [0,1]} |y(s)| \right. \\
 & \quad \left. + v(s) \max_{s \in [0,1]} |y''(s)| + w(s) \right] \, ds \\
 & \leq \int_0^1 \mathcal{K}(s) [u(s)|y|_\infty + v(s)|y''|_\infty + w(s)] \, ds \\
 & \leq \int_0^1 \mathcal{K}(s) [u(s)\|y\| + v(s)\|y\| + w(s)] \, ds \\
 & = \int_0^1 \mathcal{K}(s) [u(s) + v(s)] \|y\| \, ds + \int_0^1 \mathcal{K}(s) w(s) \, ds \\
 & = A \|y\| + B \\
 & = AL + B. \tag{22}
 \end{aligned}$$

Using equation (21), we obtain the bound

$$\lambda L \leq AL + B \quad \text{that implies} \quad \lambda \leq A + \frac{B}{L} = A + \frac{B}{B(1-A)^{-1}} = 1,$$

which contradicts the hypothesis for which  $\lambda < 1$ . Therefore, we conclude that there exists at least a nontrivial solution  $\xi(x)$  of problem defined in (1)–(2) for the case  $n = 5$ .  $\square$

**Theorem 4.3** *Let  $(x, y, z) \mapsto \phi(x, y, z)$ ,  $x \in [0, 1]$ ,  $(y, z) \in \mathbb{R}^2$  be a function of class  $C$  in  $\mathbb{R}$  and  $|\phi(x, 0, 0)| \neq 0$ . Suppose that there exist three nonnegative functions  $x \mapsto u(x), v(x), w(x) \in L^1[0, 1]$  such that*

$$|\phi(x, y, z)| \leq u(x)|y| + v(x)|z| + w(x).$$

Define

$$\mathcal{K}(s) := \frac{1}{8!}(s^8 + 10s^6)$$

and suppose that either of the following conditions is fulfilled:

1. There exists a constant  $k > -7$  such that

$$u(s) + v(s) < \frac{8!(k+7)(k+9)}{11k+97}s^k, \quad 0 \leq s \leq 1.$$

2. There exists a constant  $k' > -1$  such that

$$u(s) + v(s) < \frac{28 \prod_{i=1}^9 (k' + i)}{5k'^2 + 85k' + 388}(1-s)^{k'}, \quad 0 \leq s \leq 1.$$

3. There exists a constant  $a > 1$  such that

$$\left[ \int_0^1 (u(s) + v(s))^a \right]^{\frac{1}{a}} ds < \frac{1}{\frac{1}{8!} \left[ \left(\frac{1}{8b+1}\right)^{\frac{1}{b}} + 10 \left(\frac{1}{6b+1}\right)^{\frac{1}{b}} \right]},$$

where

$$\frac{1}{a} + \frac{1}{b} = 1 \quad \text{and} \quad \int_0^1 \mathcal{K}(s)[u(s) + v(s)] ds < 1.$$

Then the problem defined in equations (1) and (2) for  $n = 5$  has at least one nontrivial solution  $x \mapsto \xi(x)$ ,  $x \in [0, 1]$  of class  $C^{10}$  in  $\mathbb{R}$ .

*Proof* In order to prove the claim of this theorem in each case, one has to show that the integral operator (20) has  $A < 1$ , with  $A$  defined in Theorem 4.2.

To prove the claim in case 1, we proceed as follows:

$$\begin{aligned} \int_0^1 \mathcal{K}(s)[u(s) + v(s)] ds &< \frac{8!(k+7)(k+9)}{11k+97} \int_0^1 \mathcal{K}(s)s^k ds \\ &= \frac{8!(k+7)(k+9)}{11k+97} \cdot \frac{1}{8!} \int_0^1 [s^8 + 10s^6]s^k ds \\ &= \frac{8!(k+7)(k+9)}{11k+97} \cdot \frac{1}{8!} \frac{11k+97}{k^2 + 16k + 63} \\ &= 1, \end{aligned} \tag{23}$$

and when  $k > -7$  one has that  $\frac{8!(k+7)(k+9)}{11k+97} > 0$ .



For case 2, we have

$$\begin{aligned} \int_0^1 \mathcal{K}(s)[u(s) + v(s)] \, ds &< \frac{28 \prod_{i=1}^9 (k' + i)}{5k'^2 + 85k' + 388} \int_0^1 \mathcal{K}(s)(1 - s)^{k'} \, ds \\ &= \frac{28 \prod_{i=1}^9 (k' + i)}{5k'^2 + 85k' + 388} \cdot \frac{1}{8!} \int_0^1 [s^8 + 10s^6](1 - s)^{k'} \, ds \\ &= \frac{28 \prod_{i=1}^9 (k' + i)}{5k'^2 + 85k' + 388} \cdot \frac{1}{8!} \frac{1440(5k'^2 + 85k' + 388)}{\prod_{i=1}^9 (k' + i)} \\ &= 1, \end{aligned} \tag{24}$$

and when  $k' > -1$  one has that  $\frac{28 \prod_{i=1}^9 (k' + i)}{5k'^2 + 85k' + 388} > 0$ .

In the case 3, we make use of Hölder inequality

$$\int_S |f(s)g(s)| \, ds \leq \left( \int_S |f(s)|^a \, ds \right)^{1/a} \left( \int_S |g(s)|^b \, ds \right)^{1/b},$$

whenever  $f$  and  $g$  are measurable functions on the domain  $S$  and  $1/a + 1/b = 1$ . We have

$$\begin{aligned} \int_0^1 \mathcal{K}(s)[u(s) + v(s)] \, ds &\leq \left[ \int_0^1 (u(s) + v(s))^a \, ds \right]^{\frac{1}{a}} \cdot \frac{1}{8!} \left\{ \left[ \int_0^1 (s^8)^b \, ds \right]^{\frac{1}{b}} \right. \\ &\quad \left. + 10 \left[ \int_0^1 (s^6)^b \, ds \right]^{\frac{1}{b}} \right\} \\ &= \left[ \int_0^1 (u(s) + v(s))^a \, ds \right]^{\frac{1}{a}} \cdot \frac{1}{8!} \left\{ \left[ \frac{1}{8b + 1} \right]^{\frac{1}{b}} \right. \\ &\quad \left. + 10 \left[ \frac{1}{6b + 1} \right]^{\frac{1}{b}} \right\} \\ &< \frac{1}{\frac{1}{8!} \left[ \left( \frac{1}{8b + 1} \right)^{\frac{1}{b}} + 10 \left( \frac{1}{6b + 1} \right)^{\frac{1}{b}} \right]} \cdot \frac{1}{8!} \left\{ \left[ \frac{1}{8b + 1} \right]^{\frac{1}{b}} \right. \\ &\quad \left. + 10 \left[ \frac{1}{6b + 1} \right]^{\frac{1}{b}} \right\} \\ &= 1. \end{aligned} \tag{25}$$

□

### 5 Conclusion

Generic-order boundary value problem and its 10th order solution existence by means of Leary–Schauder fixed point theorem is the main purpose of the presented paper.

#### Acknowledgements

The authors would like to thank the anonymous referees for their encouraging reports and comments which helped improve this manuscript.

#### Funding

This research was funded by the Ministry of Education, Science and Technological Development of the Republic of Serbia, OI-174010, OI-174009 and OI-174002. The APC was funded by University of Belgrade, Faculty of Organizational Sciences.

**Abbreviations**

Not applicable.

**Availability of data and materials**

Not applicable.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the different parts of the manuscript. All authors read and approved the final manuscript.

**Author details**

<sup>1</sup>Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam. <sup>2</sup>Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. <sup>3</sup>Faculty of Organizational Sciences, University of Belgrade, Belgrade, Serbia. <sup>4</sup>Department of Mathematics, Bharathidasan University, Tamilnadu, India. <sup>5</sup>Military Academy, Belgrade, Serbia.

**Publisher's Note**

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 February 2020 Accepted: 1 June 2020 Published online: 15 June 2020

**References**

1. Abdeljawad, T., Agarwal, R.P., Karapinar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. *Symmetry* **11**(5), Article ID 686 (2019). <https://doi.org/10.3390/sym11050686>
2. Afshari, H., Kalantari, S., Karapinar, E.: Solution of fractional differential equations via coupled fixed point. *Electron. J. Differ. Equ.* **2015**, Article ID 286 (2015)
3. Ahmad, B., Ntouyas, S.K.: A study of higher-order nonlinear ordinary differential equations with four-point nonlocal integral boundary conditions. *J. Appl. Math. Comput.* **39**, 97–108 (2012)
4. Akgül, A., Akgül, E.K., Baleanu, D., Inc, M.: New numerical method for solving tenth order boundary value problems. *Mathematics* **6**, Article ID 245 (2018). <https://doi.org/10.3390/math6110245>
5. Akgül, A., Akgül, E.K., Orcan, B., Inc, M.: Reproducing kernel functions for linear tenth-order boundary value problems. *ITM Web Conf.* **22**, Article ID 01027 (2018). <https://doi.org/10.1051/itmconf/20182201027>
6. Aydi, H., Karapinar, E.: Fixed point results for generalized  $\alpha$ - $\psi$ -contractions in metric-like spaces and applications. *Electron. J. Differ. Equ.* **2015**, Article ID 133 (2015)
7. Aydi, H., Karapinar, E., Mitrović, Z.D., Rashid, T.: A remark on "Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results F-metric spaces". *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.* **113**, 3197–3206 (2019). <https://doi.org/10.1007/s13398-019-00690-9>
8. Bekri, Z., Benaicha, S.: Nontrivial solution of a nonlinear sixth-order boundary value problem. *Waves Wavelets Fractals* **4**(1), 10–18 (2018)
9. Chandrasekhar, S.: *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford (1961)
10. Ćirić, Lj.: *Some Recent Results in Metrical Fixed Point Theory*. C-print, Belgrade (2003)
11. Iqbal, M.J., Rehman, S., Pervaiz, A., Hakeem, A.: Approximations for linear tenth-order boundary value problems through polynomial and non-polynomial cubic spline techniques. *Proc. Pak. Acad. Sci.* **52**(4), 389–396 (2015)
12. Isac, G.: *Leray–Schauder Type Alternatives, Complementarity Problems and Variational Inequalities*. Springer, Berlin (2006)
13. Islam, M.S., Hossain, M.B., Rahman, M.A.: Numerical approaches for tenth and twelfth order linear and nonlinear differential equations. *Br. J. Math. Comput. Sci.* **5**(5), 637–653 (2015)
14. Jleli, M., Karapinar, E., Samet, B.: Positive solutions for multipoint boundary value problems for singular fractional differential equations. *J. Appl. Math.* **24**, Article ID 596123 (2014). <https://doi.org/10.1155/2014/596123>
15. Jørgensen, E., Volakis, J.L., Meincke, P., Breinbjerg, O.: Higher order hierarchical Legendre basis functions for electromagnetic modeling. *IEEE Trans. Antennas Propag.* **52**(11), 2985–2995 (2004). <https://doi.org/10.1109/TAP.2004.835279>
16. Karapinar, E., Abdeljawad, T., Jarad, F.: Applying new fixed point theorems on fractional and ordinary differential equations. *Adv. Differ. Equ.* **2019**, Article ID 421 (2019). <https://doi.org/10.1186/s13662-019-2354-3>
17. Karapinar, E., Fulga, A., Rashid, M., Shahid, L., Aydi, H.: Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations. *Mathematics* **7**(5), Article ID 444 (2019). <https://doi.org/10.3390/math7050444>
18. Klaus, D.: *Nonlinear Functional Analysis*. Springer, Berlin (1985)
19. Ma, R.: Existence and uniqueness theorems for some fourth-order nonlinear boundary value problems. *Int. J. Appl. Math. Comput.* **23**, 783–788 (2000)
20. Mai-Duy, N.: An effective spectral collocation method for the direct solution of high-order ODEs. *Commun. Numer. Methods Eng.* **22**, 627–642 (2006)
21. Malkowsky, E., Rakočević, V.: *Advanced Functional Analysis*. CRC Press, Boca Raton (2019)
22. Mohyud-Din, S.T., Yildirim, A.: Solutions of tenth and ninth-order boundary value problems by modified variational iteration method. *Appl. Appl. Math.* **5**(1), 11–25 (2010)
23. Noor, M.A., Al-Said, E., Mohyud-Din, S.T.: A reliable algorithm for solving tenth-order boundary value problems. *Appl. Math. Inf. Sci.* **6**(1), 103–107 (2012)

24. Oderinu, R.A.: On the numerical solution of tenth and twelfth order boundary value problems using weighted residual method (WRM). *Gen. Math. Notes* **24**(1), 17–24 (2014)
25. Ramadan, M.A., Lashien, I.F., Zahra, W.K.: High order accuracy nonpolynomial spline solutions for  $2\mu$ th order two point boundary value problems. *Appl. Math. Comput.* **204**, 920–927 (2008). <https://doi.org/10.1016/j.amc.2008.07.038>
26. Ramos, H., Singh, G.: A tenth order A-stable two-step hybrid block method for solving initial value problems of ODEs. *Appl. Math. Comput.* **310**, 75–88 (2017). <https://doi.org/10.1016/j.amc.2017.04.020>
27. Shanmugam, T., Muthiah, M., Radenović, S.: Existence of positive solution for the eighth-order boundary value problem using classical version of Leray–Schauder alternative fixed point theorem. *Axioms* **8**, Article ID 129 (2019). <https://doi.org/10.3390/axioms8040129>
28. Siddiqi, S.S., Akram, G.: Solutions of tenth-order boundary value problems using eleventh degree spline. *Appl. Math. Comput.* **185**, 115–127 (2007). <https://doi.org/10.1016/j.amc.2006.07.013>
29. Siddiqi, S.S., Akram, G.: Solution of 10th-order boundary value problems using non-polynomial spline technique. *Appl. Math. Comput.* **190**, 641–651 (2007). <https://doi.org/10.1016/j.amc.2007.01.075>
30. Siddiqi, S.S., Twizell, E.H.: Spline solutions of linear tenth-order boundary-value problems. *Int. J. Comput. Math.* **68**, 345–362 (1998). <https://doi.org/10.1080/00207169808804701>
31. Twizell, E.H., Boutayeb, A., Djidjeli, K.: Numerical methods for eighth-, tenth- and twelfth-order eigenvalue problems arising in thermal instability. *Adv. Comput. Math.* **2**, 407–436 (1994)
32. Wazwaz, A.-M.: The modified Adomian decomposition method for solving linear and nonlinear boundary value problems of tenth-order and twelfth-order. *Int. J. Nonlinear Sci. Numer. Simul.* **1**, 17–24 (2000). <https://doi.org/10.1515/IJNSNS.2000.1.1.17>
33. Zvyagin, V.G., Baranovskii, E.S.: Topological degree of condensing multi-valued perturbations of the  $(S)_+$ -class maps and its applications. *J. Math. Sci.* **170**, 405–422 (2010)

Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

---

Submit your next manuscript at ► [springeropen.com](https://www.springeropen.com)

---