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A new Hermite–Hadamard type inequality for coordinate convex function

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Abstract
In the article, we establish a new Hermite–Hadamard type inequality for the coordinate convex function by constructing two monotonic sequences. The given result is the generalization and improvement of some previously obtained results.

MSC: 26D15

Keywords: Hermite–Hadamard's inequality; Convex function; Coordinates

1 Introduction

Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex (concave) if the inequality

$$f\big(ta + (1-t)b\big) \le (\ge) tf(a) + (1-t)f(b)$$

holds for all $a, b \in I$ and $t \in [0,1]$. Recently, the generalizations, extensions, variants and applications of convexity have attracted the attention of many researchers (e.g., [4, 20–22]). In particular, many inequalities can be found in the literature (e.g., [13, 15, 17]) via the convexity theory.

The well known Hermite–Hadamard inequality for convex function is formulated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I = [a,b]$ with $a < b$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

In recent years, more and more refinements of the Hermite–Hadamard inequality for convex functions have been extensively investigated by a number of authors (e.g., [1–3, 5, 6, 8–10, 12, 14, 16, 18, 23]).

In [11], A.E. Farissi improved the Hermite–Hadamard inequality as follows:

Theorem 1.1 ([11]) *Let $f : I \to \mathbb{R}$ be a convex function on $I = [a, b]$ with $a < b$. Then for all $\lambda \in [0, 1]$,*

$$f\left(\frac{a+b}{2}\right) \le l(\lambda) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le L(\lambda) \le \frac{f(a)+f(b)}{2}, \tag{2}$$

where

$$l(\lambda) = \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}\bigl(f\bigl(\lambda b + (1-\lambda)a\bigr) + \lambda f(a) + (1-\lambda) f(b)\bigr).$$

Consider the two-dimensional interval $\Delta := [a, b] \times [c, d]$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is said to be coordinate convex on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$, are convex for all $y \in [c, d]$ and $x \in [a, b]$.

In [7], S.S. Dragomir established the following Hadamard-type inequalities for coordinate convex functions in a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2 ([7]) *Let $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ be a coordinate convex function on Δ. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y)\,dy\,dx \\ &\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b \bigl[f(x, c) + f(x, d)\bigr] dx + \frac{1}{d-c}\int_c^d \bigl[f(a, y) + f(b, y)\bigr] dy\right] \\ &\le \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned} \tag{3}$$

In [19], M.E. Özdemir defined a new mapping associated with coordinate convexity and proved the following inequalities based on the properties of this mapping.

Theorem 1.3 ([19]) *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a coordinate convex function on $\Delta = [a, b] \times [c, d]$. Then*

$$\begin{aligned} &\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y)\,dy\,dx \\ &\quad \le \frac{1}{4}\left[\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}\right. \\ &\qquad \left. + \frac{f(\frac{a+b}{2}, c) + f(\frac{a+b}{2}, d) + f(a, \frac{c+d}{2}) + f(b, \frac{c+d}{2})}{2} + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right]. \end{aligned} \tag{4}$$

In this paper, we present some new Hermite–Hadamard inequalities for coordinate convex function by defining two sequences $F(x, y; n)$ and $H(x, y; n)$, which also are generalizations of some existing results. Moreover, we also discuss the monotonicity of the sequences $F(x,y;n)$ and $H(x,y;n)$.

2 Main results

In this section, a refinement of the Hermite–Hadamard inequality by defining two sequences $F(x,y;n)$ and $H(x,y;n)$ is presented.

Theorem 2.1 *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a coordinate convex function on $\Delta=[a,b]\times[c,d]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le H(x,y;n)\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\le F(x,y;n)\le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4} \end{aligned} \tag{5}$$

for all $x\in[a,b]$, $y\in[c,d]$ and $n\in\mathbb{N}$, where

$$\begin{aligned} H(x,y;n) = \frac{1}{2^{n+1}}\sum_{i=1}^{2^n}\biggl[&\frac{1}{b-a}\int_a^b f\left(x, c+i\frac{d-c}{2^n}-\frac{d-c}{2^{n+1}}\right)dx \\ &+\frac{1}{d-c}\int_c^d f\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}},y\right)dy\biggr] \end{aligned}$$

and

$$\begin{aligned} &F(x,y;n) \\ &\quad=\frac{1}{2^{n+2}}\sum_{i=1}^{2^n}\biggl[\frac{1}{b-a}\int_a^b\left[f\left(x,\left(1-\frac{i}{2^n}\right)c+\frac{i}{2^n}d\right)+f\left(x,\left(1-\frac{i-1}{2^n}\right)c+\frac{i-1}{2^n}d\right)\right]dx \\ &\qquad+\frac{1}{d-c}\int_c^d\left[f\left(\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b,y\right)+f\left(\left(1-\frac{i-1}{2^n}\right)a+\frac{i-1}{2^n}b,y\right)\right]dy\biggr]. \end{aligned}$$

Proof Since f is coordinate convex on $\Delta=[a,b]\times[c,d]$, its partial mapping $g_x(y)=f(x,y)$ is convex on $[c,d]$ for all $x\in[a,b]$, and so, applying (1) to $g_x(y)$,

$$g_x\left(\frac{c+d}{2}\right)\le\frac{1}{d-c}\int_c^d g_x(y)\,dy\le\frac{g_x(c)+g_x(d)}{2}. \tag{6}$$

On the one hand, by (6), we have

$$\begin{aligned} \frac{1}{d-c}\int_c^d g_x(y)\,dy &= \frac{1}{d-c}\sum_{i=1}^{2^n}\int_{c+(i-1)\frac{d-c}{2^n}}^{c+i\frac{d-c}{2^n}} g_x(y)\,dy \\ &\le \frac{1}{2^{n+1}}\sum_{i=1}^{2^n}\left[g_x\left(\left(1-\frac{i}{2^n}\right)c+\frac{i}{2^n}d\right)+g_x\left(\left(1-\frac{i-1}{2^n}\right)c+\frac{i-1}{2^n}d\right)\right] \\ &= y(x;n). \end{aligned} \tag{7}$$

On the other hand, by the convexity of $g_x(y)$, we obtain

$$\begin{aligned} y(x;n) &\leq \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} \left[\left(1 - \frac{i}{2^n}\right) g_x(c) + \frac{i}{2^n} g_x(d) + \left(1 - \frac{i-1}{2^n}\right) g_x(c) + \frac{i-1}{2^n} g_x(d) \right] \\ &= \frac{1}{2^{n+1}} \left[g_x(c) \sum_{i=1}^{2^n} \left(2 - \frac{i}{2^{n-1}} + \frac{1}{2^n}\right) + g_x(d) \sum_{i=1}^{2^n} \left(\frac{i}{2^{n-1}} - \frac{1}{2^n}\right) \right] \\ &= \frac{g_x(c) + g_x(d)}{2}. \end{aligned} \tag{8}$$

By (7) and (8), we have

$$\frac{1}{d-c} \int_c^d g_x(y)\,dy \leq y(x;n) \leq \frac{g_x(c) + g_x(d)}{2}. \tag{9}$$

Integrating both sides of (9) with respect to x on $[a,b]$, we have

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\quad\leq \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} \left[\frac{1}{b-a} \int_a^b f\left(x, \left(1 - \frac{i}{2^n}\right)c + \frac{i}{2^n} d\right) dx \right. \\ &\qquad \left. + \frac{1}{b-a} \int_a^b f\left(x, \left(1 - \frac{i-1}{2^n}\right)c + \frac{i-1}{2^n} d\right) dx \right] \\ &\quad\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(x,c)\,dx + \frac{1}{b-a} \int_a^b f(x,d)\,dx \right]. \end{aligned} \tag{10}$$

By a similar process, we can obtain

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\quad\leq \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} \left[\frac{1}{d-c} \int_c^d f\left(\left(1 - \frac{i}{2^n}\right)a + \frac{i}{2^n} b, y\right) dy \right. \\ &\qquad \left. + \frac{1}{d-c} \int_c^d f\left(\left(1 - \frac{i-1}{2^n}\right)a + \frac{i-1}{2^n} b, y\right) dy \right] \\ &\quad\leq \frac{1}{2} \left[\frac{1}{d-c} \int_c^d f(a,y)\,dy + \frac{1}{d-c} \int_c^d f(b,y)\,dy \right]. \end{aligned} \tag{11}$$

By (10) and (11), we have

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\quad\leq \frac{1}{2^{n+2}} \sum_{i=1}^{2^n} \left[\frac{1}{b-a} \int_a^b f\left(x, \left(1 - \frac{i}{2^n}\right)c + \frac{i}{2^n} d\right) dx \right. \\ &\qquad + \frac{1}{b-a} \int_a^b f\left(x, \left(1 - \frac{i-1}{2^n}\right)c + \frac{i-1}{2^n} d\right) dx \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{d-c}\int_c^d f\left(\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b, y\right)dy \\
&+ \frac{1}{d-c}\int_c^d f\left(\left(1-\frac{i-1}{2^n}\right)a+\frac{i-1}{2^n}b, y\right)dy\Bigg] \\
&= F(x,y;n) \\
&\le \frac{1}{4}\Bigg[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \\
&+ \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\Bigg].
\end{aligned}
$$

Furthermore, by the convexity of $f(x,y)$, we have

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x,c)\,dx \le \frac{f(a,c)+f(b,c)}{2}, \\
&\frac{1}{b-a}\int_a^b f(x,d)\,dx \le \frac{f(a,d)+f(b,d)}{2}, \\
&\frac{1}{d-c}\int_c^d f(a,y)\,dy \le \frac{f(a,c)+f(a,d)}{2}, \\
&\frac{1}{d-c}\int_c^d f(b,y)\,dy \le \frac{f(b,c)+f(b,d)}{2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&\quad\le F(x,y;n) \le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}.
\end{aligned}
\tag{12}
$$

Moreover, by (1), we have

$$
\begin{aligned}
\frac{1}{d-c}\int_c^d g_x(y)\,dy &= \frac{1}{d-c}\sum_{i=1}^{2^n}\int_{c+(i-1)\frac{d-c}{2^n}}^{c+i\frac{d-c}{2^n}} g_x(y)\,dy \\
&\ge \frac{1}{2^n}\sum_{i=1}^{2^n} g_x\left(c+i\frac{d-c}{2^n}-\frac{d-c}{2^{n+1}}\right) \\
&= x(x;n).
\end{aligned}
\tag{13}
$$

By the convexity of $g_x(y)$ and Jensen's inequality, we obtain

$$
x(x;n) \ge g_x\left[\frac{1}{2^n}\sum_{i=1}^{2^n}\left(c+i\frac{d-c}{2^n}-\frac{d-c}{2^{n+1}}\right)\right] = g_x\left(\frac{c+d}{2}\right).
\tag{14}
$$

It follows from (13) and (14) that

$$
\frac{1}{d-c}\int_c^d g_x(y)\,dy \ge x(x;n) \ge g_x\left(\frac{c+d}{2}\right).
\tag{15}
$$

Integrating both sides of (15) with respect to x on $[a,b]$, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&\quad\geq \frac{1}{2^n}\sum_{i=1}^{2^n}\left[\frac{1}{b-a}\int_a^b f\left(x, c+i\frac{d-c}{2^n}-\frac{d-c}{2^{n+1}}\right)dx\right] \\
&\quad\geq \frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx. \qquad (16)
\end{aligned}$$

By a similar process, we can obtain

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \qquad (17)\\
&\quad\geq \frac{1}{2^n}\sum_{i=1}^{2^n}\left[\frac{1}{d-c}\int_c^d f\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}},y\right)dy\right] \qquad (18)\\
&\quad\geq \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy. \qquad (19)
\end{aligned}$$

By (16) and (17), we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&\quad\geq \frac{1}{2^{n+1}}\sum_{i=1}^{2^n}\left[\frac{1}{b-a}\int_a^b f\left(x, c+i\frac{d-c}{2^n}-\frac{d-c}{2^{n+1}}\right)dx\right. \\
&\qquad\left.+\frac{1}{d-c}\int_c^d f\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}},y\right)dy\right] \\
&\quad= H(x,y;n) \\
&\quad\geq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right].
\end{aligned}$$

Moreover, by the convexity of $f(x,y)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx \geq f\left(\frac{a+b}{2},\frac{c+d}{2}\right), \\
&\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy \geq f\left(\frac{a+b}{2},\frac{c+d}{2}\right).
\end{aligned}$$

Therefore,

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \geq H(x,y;n) \geq f\left(\frac{a+b}{2},\frac{c+d}{2}\right). \qquad (20)$$

By (12) and (20), we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le H(x,y;n) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx$$
$$\le F(x,y;n) \le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}. \qquad \square$$

Remark 2.1 Let $n=0$. Then inequality (5) reduces to (3). Therefore, our Theorem 1.2 is a generalization of Theorem 1.2 of [7].

In the following, we discuss the monotonicity of $F(x;y;n)$ and $H(x;y;n)$ which are defined as in Theorem 2.1.

Theorem 2.2 *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a coordinate convex function on $\Delta=[a,b]\times[c,d]$. Then $F(x,y;n)$ decreasing, $H(x,y;n)$ is increasing and*

$$\lim_{n\to\infty}F(x,y;n)=\lim_{n\to\infty}H(x,y;n)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx.$$

Proof On the one hand, we have

$$\begin{aligned} x(x;n) &= \frac{1}{2^n}\sum_{i=1}^{2^n} g_x\left(c+i\frac{d-c}{2^n}-\frac{d-c}{2^{n+1}}\right) \\ &= \frac{1}{2^n}\sum_{i=1}^{2^n} g_x\left(\frac{1}{2}\frac{(2^{n+2}-4i+3)c+(4i-3)d+(2^{n+2}-4i+1)c+(4i-1)d}{2^{n+2}}\right) \\ &\le \frac{1}{2^{n+1}}\sum_{i=1}^{2^n} g_x\left(\frac{(2^{n+2}-4i+3)c+(4i-3)d}{2^{n+2}}\right) \\ &\quad + \frac{1}{2^{n+1}}\sum_{i=1}^{2^n} g_x\left(\frac{(2^{n+2}-4i+1)c+(4i-1)d}{2^{n+2}}\right). \end{aligned}$$

Setting $A=\{1,3,\ldots,2^{n+1}-1\}$ and $B=\{2,4,\ldots,2^{n+1}\}$, thus we obtain

$$\sum_{i=1}^{2^n} g_x\left(\frac{(2^{n+2}-4i+3)c+(4i-3)d}{2^{n+2}}\right)=\sum_A g_x\left(\frac{(2^{n+2}-2i+1)c+(2i-1)d}{2^{n+2}}\right),$$
$$\sum_{i=1}^{2^n} g_x\left(\frac{(2^{n+2}-4i+1)c+(4i-1)d}{2^{n+2}}\right)=\sum_B g_x\left(\frac{(2^{n+2}-2i+1)c+(2i-1)d}{2^{n+2}}\right),$$

which implies that

$$x(x;n)\le\frac{1}{2^{n+1}}\sum_{A\cup B} g_x\left(\frac{(2^{n+2}-2i+1)c+(2i-1)d}{2^{n+2}}\right)=x(x;n+1).$$

Since integration is sign-preserving, we know

$$H(x,y;n)\le H(x,y;n+1).$$

So $H(x,y;n)$ is increasing.

On the other hand, we have

$$\begin{aligned} y(x;n+1) &= \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i=1}^{2^{n+1}-1} f\left[\left(1-\frac{i}{2^{n+1}}\right)a+\frac{i}{2^{n+1}}b\right]\right] \\ &= \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i=1}^{2^{n+1}-1} f\left(\frac{(2^{n+1}-i)a+ib}{2^{n+1}}\right)\right]. \end{aligned}$$

Setting $C=\{2,4,6,\dots,2^{n+1}-2\}$, we obtain

$$\begin{aligned} y(x;n+1) &= \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i\in C} f\left(\frac{(2^{n+1}-i)a+ib}{2^{n+1}}\right)+2\sum_{i\in A} f\left(\frac{(2^{n+1}-i)a+ib}{2^{n+1}}\right)\right] \\ &= \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i=1}^{2^n-1} f\left(\frac{(2^n-i)a+ib}{2^n}\right)\right. \\ &\quad \left.+2\sum_{i=1}^{2^n} f\left(\frac{1}{2}\frac{(2^n-i)a+ib+(2^n-i+1)a+(i-1)b}{2^n}\right)\right] \\ &\le \frac{1}{2^{n+2}}\left[f(a)+f(b)+2\sum_{i=1}^{2^n-1} f\left(\frac{(2^n-i)a+ib}{2^n}\right)+\sum_{i=1}^{2^n} f\left(\frac{(2^n-i)a+ib}{2^n}\right)\right. \\ &\quad \left.+\sum_{i=1}^{2^n} f\left(\frac{(2^n-i+1)a+(i-1)b}{2^n}\right)\right] \\ &= \frac{1}{2^{n+1}}\left[f(a)+f(b)+2\sum_{i=1}^{2^n-1} f\left(\frac{(2^n-i)a+ib}{2^n}\right)\right] \\ &= y(x;n). \end{aligned}$$

So $y(x;n)$ is decreasing.

Since integration is sign-preserving,we know

$$F(x,y;n)\ge F(x,y;n+1).$$

For the proof of the last assertions, since $f(x,y)$ is continuous on $[a,b]\times[c,d]$, we use the following well known equalities:

$$\begin{aligned} \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\frac{b-a}{n},y\right) &= \int_a^b f(x,y)\,dx, \\ \lim_{n\to\infty}\frac{d-c}{n}\sum_{i=1}^{n} f\left(x,c+i\frac{d-c}{n}\right) &= \int_c^d f(x,y)\,dy. \end{aligned}$$

So we obtain

$$\lim_{n\to\infty}F(x,y;n)=\lim_{n\to\infty}H(x,y;n)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx. \qquad \square$$

By the above theorems, the following corollary can be easily obtained:

Corollary 2.1 *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be a coordinate convex on Δ. Then*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad\le H(x,y;0)=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\quad\le H(x,y;1)\le\cdots\le H(x,y;n)\le\cdots\\
&\quad\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\\
&\quad\le\cdots\le F(x,y;n)\le\cdots\le F(x,y;1)\\
&\quad\le F(x,y;0)=\frac{1}{4}\left[\frac{1}{b-a}\int_a^b\big[f(x,c)+f(x,d)\big]dx+\frac{1}{d-c}\int_c^d\big[f(a,y)+f(b,y)\big]dy\right]\\
&\quad\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}\tag{21}$$

Remark 2.2 Corollary 2.1 shows that inequalities (21) are better than (3) and (4).

3 Conclusions

In this paper, we present some new Hermite–Hadamard inequalities for coordinate convex functions by defining two sequences $F(x,y;n)$ and $H(x,y;n)$,

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le H(x,y;n)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx\\
&\le F(x,y;n)\le\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4},
\end{aligned}$$

which also are generalizations of some existing results. Moreover, we show the monotonicity of the sequences $F(x,y;n)$ and $H(x,y;n)$ in Theorem 2.2.

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