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Generalization of Szász–Mirakjan–Kantorovich operators using multiple Appell polynomials

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Abstract

The purpose of the present paper is to introduce and study a sequence of positive linear operators defined on suitable spaces of measurable functions on $[0,\infty)$ and continuous function spaces with polynomial weights. These operators are Kantorovich type generalization of Jakimovski–Leviatan operators based on multiple Appell polynomials. Using these operators, we approximate suitable measurable functions by knowing their mean values on a sequence of subintervals of $[0,\infty)$ that do not constitute a subdivision of it. We also discuss the rate of convergence of these operators using moduli of smoothness.

MSC: 41A36

Keywords: Szász operator; Multiple Appell polynomials; Moduli of smoothness

1 Introduction

A multiple polynomial system [10] $\{s_{n_1,n_2}(x)\}$ is called multiple Appell if it has a generating function of the form

$$H(t_1,t_2)e^{x(t_1+t_2)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(x)}{n_1!n_2!}t_1^{n_1}t_2^{n_2}, \tag{1.1}$$

where $H(t_1,t_2)$ has a series expansion

$$H(t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{a_{n_1,n_2}}{n_1!n_2!}t_1^{n_1}t_2^{n_2}, \tag{1.2}$$

with $H(0,0) = a_{0,0} \neq 0$ and $\frac{a_{n_1,n_2}}{H(1,1)} \geq 0$ for all $n_1,n_2 \in \mathbb{N}$. Also, (1.1) and (1.2) converge for $|t_1| \leq R_1$, $|t_2| \leq R_2$ $(R_1,R_2 > 1)$. s_{n_1,n_2} is a multiple polynomial system, and for every $n_1 + n_2 \geq 1$, this satisfies the following relationship:

$$s'_{n_1,n_2}(x) = n_1 s_{n_1-1,n_2}(x) + n_2 s_{n_1,n_2-1}(x).$$

Also, for the multiple Appell polynomial systems $s_{n_1,n_2}(x)$, there exists a sequence $\{a_{n_1,n_2}\}_{n_1,n_2=0}^{\infty}$ with $a_{0,0} \neq 0$ such that

$$s_{n_1,n_2}(x) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\binom{n_1}{k_1}\binom{n_2}{k_2}a_{n_1-k_1,n_2-k_2}x^{k_1+k_2}. \tag{1.3}$$

Therefore, $s_{n_1,n_2}(x)$ is a polynomial in x of degree $n_1 + n_2$.

Using these Appell polynomials, Varma [18] defined a generalization of Szász operators [17] as follows:

$$S_n(f;x) = \frac{e^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}f\left(\frac{n_1+n_2}{n}\right). \tag{1.4}$$

He also defined the Kantorovich type modification of these operators as follows:

$$K_n^*(f;x) = \frac{ne^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\int_{\frac{n_1+n_2}{n}}^{\frac{n_1+n_2+1}{n}}f(u)\,du, \tag{1.5}$$

and obtained the rate of convergence of these operators in terms of the classical modulus of continuity. Alternatively, the operator given by (1.5) may be expressed as

$$K_n^*(f;x) = \int_0^{\infty}K_n(x,u)f(u)\,du,$$

where $K_n(x,u) = \frac{ne^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\chi_{[\frac{n_1+n_2}{n},\frac{n_1+n_2+1}{n}]}(u)$, $\chi_{[\frac{n_1+n_2}{n},\frac{n_1+n_2+1}{n}]}(u)$ being the characteristic function of $[\frac{n_1+n_2}{n},\frac{n_1+n_2+1}{n}]$ on $[0,\infty)$.

The purpose of the present paper is to make a generalization of these operators that extends to the unbounded setting an idea given in [6] where the authors studied a modification of Kantorovich operators. We refer the reader to some of the related papers [1, 2, 7–9, 11–14, 16].

In this paper we will study the following sequence (P_n) of positive linear operators:

$$P_n(h;x) = \frac{ne^{-nx}}{(b_n-c_n)H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\int_{\frac{n_1+n_2+c_n}{n}}^{\frac{n_1+n_2+b_n}{n}}h(u)\,du \quad (n\geq 1, x\geq 0). \tag{1.6}$$

For every $h \in \Im[0,\infty)$ (the space of all Borel measurable locally integrable functions $g: [0,\infty)\to\mathbb{R}$ such that the antiderivative $G(x) = \int_0^x g(t)\,dt$ $(x\geq 0)$ belongs to $\ell([0,\infty))$, the space of all functions $b:[0,\infty)\to\mathbb{R}$ such that $|b(x)|\leq Me^{rx}$ $(x\geq 0)$ for some $M\geq 0$ and $r\in\mathbb{R}$).

Here $(c_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real numbers satisfying $0\leq c_n < b_n\leq 1$ for every $n\geq 1$. If $c_n = 0$ and $b_n = 1$ for all $n\geq 1$, then the P_n's (operators in (1.6) turn into (1.5). By using P_n's we can reconstruct some suitable continuous or integrable functions by knowing their mean values on subinterval of $[0,\infty)$ which do not necessarily constitute a subdivision of $[0,\infty)$.

We will study the approximation properties of (P_n) for every $n\geq 1$ on several continuous and weighted continuous function spaces as well as on Lebesgue spaces. We also discuss the rate of convergence of these operators by using appropriate moduli of smoothness.

2 Generalizing Jakimovski–Leviatan operators

Throughout this paper the following notations are used:

$\wp[0,\infty)$: The space of all continuous real-valued functions on $[0,\infty)$.

$\wp_b[0,\infty)$: The subspace of all functions in $\wp[0,\infty)$ which are bounded. This space endowed with the sup-norm and the natural pointwise ordering is a Banach lattice.

$\wp_*[0,\infty)$: The space of all continuous functions converging at infinity. This space is a Banach sublattice of $\wp_b([0,\infty))$.

$\wp_0[0,\infty)$: subspace of $\wp_*[0,\infty)$, consisting of all those functions that vanish at infinity.

Moreover, for every $m \geq 1$, we set $r_m(y) = (1+y^m)^{-1}$ $(y \geq 0)$ and

$$G_m := \Big\{ h \in \wp[0,\infty) \Big| \sup_{y\geq 0} r_m(y)\big|h(y)\big| \in \mathbb{R} \Big\};$$

G_m is a Banach lattice endowed with the pointwise ordering and the weighted norm

$$\|h\|_m := \sup_{y\geq 0} r_m(y)\big|h(y)\big| \quad (h \in G_m).$$

The following space is the Banach sublattices of G_m:

$$G_m^* := \Big\{ h \in G_m \Big| \lim_{y\to\infty} r_m(y)h(y) \in \mathbb{R} \Big\}.$$

For a given $g \in \Im([0,\infty))$ and $G(x) = \int_0^x g(t)\,dt$, the antiderivative of $g(x)$, operators given in (1.6) may be expressed as

$$\begin{aligned} P_n(g;x) &= \frac{ne^{-nx}}{(b_n - c_n)H(1,1)} \\ &\quad \times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\biggl[G\biggl(\frac{n_1+n_2+b_n}{n}\biggr) - G\biggl(\frac{n_1+n_2+c_n}{n}\biggr)\biggr] && (2.1) \\ &= \frac{n}{(b_n - c_n)} S_n\bigl(\tau_n(G)\bigr)(x), && (2.2) \end{aligned}$$

where S_n is given by (1.4) and the mapping τ_n is defined as

$$\tau_n(G)(x) := G\biggl(\frac{n_1+n_2+b_n}{n}\biggr) - G\biggl(\frac{n_1+n_2+c_n}{n}\biggr) \quad (x \geq 0),$$

$P_n(g)$ can also be written as

$$P_n(g)(x) = \int_0^{+\infty} g\, d\mu_{n,x} \quad (n \geq 1, x \geq 0), \tag{2.3}$$

where

$$\mu_{n,x} = \frac{ne^{-nx}}{(b_n - c_n)H(1,1)} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\mu_{n,k},$$

and each $\mu_{n,k}$ denotes the Borel measure on $[0,\infty)$ having density, the characteristic function of $[\frac{n_1+n_2+c_n}{n}, \frac{n_1+n_2+b_n}{n}]$ w.r.t. the Borel–Lebesgue measure on $[0,\infty)$. Throughout the paper the symbol c_m denotes the function x^m by setting $c_m(x) = x^m$ for every $m \geq 0$ and for every $x \geq 0$. Particularly $c_0 = \mathbf{1}$, where $\mathbf{1}$ denotes the constant function on $[0,\infty)$ of constant value 1. Finally, we shall set $\rho_x(y) = y - x$ ($y \geq 0$) and x is a fixed nonnegative real number.

3 Preliminaries

Lemma 1 *For operators* (1.6), *the estimates of moments are as follows* [18]:

$$\text{(i)}\quad P_n(1;y) = 1;$$

$$\text{(ii)}\quad P_n(t;y) = y + \frac{1}{2}\frac{b_n + c_n}{n} + \frac{H_{t_1}(1,1) + H_{t_2}(1,1)}{nH(1,1)};$$

$$\begin{aligned}\text{(iii)}\quad P_n\left(t^2;y\right) &= y^2 + \frac{y}{n}\left(1 + \frac{b_n + c_n}{n} + 2\frac{(H_{t_1}(1,1) + H_{t_2}(1,1))}{H(1,1)}\right) + \frac{b_n^2 + c_n^2 + b_n c_n}{3n^2} \\ &\quad \times \frac{1}{n^2 H(1,1)}\bigl\{H_{t_1}(1,1)(1 + b_n + c_n) + H_{t_2}(1,1)(1 + b_n + c_n) \\ &\quad + H_{t_1 t_1}(1,1) + H_{t_2 t_2}(1,1) + 2H_{t_1 t_2}(1,1)\bigr\};\end{aligned}$$

$$\text{(iv)}\quad P_n\left(\rho_y(t);y\right) = \frac{1}{2}\frac{b_n + c_n}{n} + \frac{H_{t_1}(1,1) + H_{t_2}(1,1)}{nH(1,1)};$$

$$\begin{aligned}\text{(v)}\quad P_n\left(\rho_y^2(t);y\right) &= \frac{y}{n} + \frac{1}{n^2 H(1,1)}\bigl\{H_{t_1}(1,1) + H_{t_2}(1,1) + H_{t_1 t_1}(1,1) \\ &\quad + H_{t_2 t_2}(1,1) + 2H_{t_1 t_2}(1,1)\bigr\} \\ &\quad + \frac{b_n^2 + c_n^2 + b_n c_n}{n^2} + \frac{1}{n}(b_n + c_n)\left(\frac{H_{t_1}(1,1) + H_{t_2}(1,1)}{nH(1,1)}\right).\end{aligned}$$

Proposition 1 *For every* $\rho > 0$, *let* $f_\rho(x) = e^{-\rho x}$ *where* $x \geq 0$. *Then*

$$S_n(f_\rho)(x) = \frac{H(e^{\frac{-\rho}{n}}, e^{\frac{-\rho}{n}})}{H(1,1)} \exp\bigl(nx\bigl(e^{\frac{-\rho}{n}} - 1\bigr)\bigr). \tag{3.1}$$

Proof

$$S_n(f_\rho)(x) = \frac{e^{-nx}}{H(1,1)} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1! n_2!} e^{-\rho(\frac{n_1+n_2}{n})}. \tag{3.2}$$

Putting $x = \frac{nx}{2}$ and $t_1 = t_2 = e^{(\frac{-\rho}{n})}$ in (1.1), we get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1! n_2!} e^{-\rho(\frac{n_1+n_2}{n})} = H\bigl(e^{\frac{-\rho}{n}}, e^{\frac{-\rho}{n}}\bigr) e^{\frac{nx}{2}(e^{\frac{-\rho}{n}} + e^{\frac{-\rho}{n}})}. \tag{3.3}$$

Using (3.3) in (3.2), we have the result. □

Proposition 2

$$P_n(f_\rho) = \frac{n}{\rho(b_n - c_n)}\bigl(e^{\frac{-\rho c_n}{n}} - e^{\frac{-\rho b_n}{n}}\bigr) S_n(f_\rho) \tag{3.4}$$

for every $n \geq 1$.

Moreover, for every $n \geq 1$ and $\rho > 0$,

$$P_n(f_\rho) \leq S_n(f_\rho). \tag{3.5}$$

Proof (3.4) holds after a straightforward computation, and the proof of (3.5) is as follows:

$$\frac{n}{\rho(b_n - c_n)}\left(e^{\frac{-\rho c_n}{n}} - e^{\frac{-\rho b_n}{n}}\right) \leq \frac{n}{\rho(b_n - c_n)}\left(1 - e^{-(\frac{\rho b_n}{n} - \frac{\rho c_n}{n})}\right) \leq 1$$

since $1 - e^{-x} \leq x$ $(x \geq 0)$. □

Theorem 1 *The operator P_n for every $n \geq 1$ defined by (1.6) has the following properties:*

(i) *P_n is a positive and continuous linear operator from $\wp_b([0,\infty))$ to $\wp_b([0,\infty))$ and $\|P_n\|_{\wp_b([0,\infty))} = 1$.*

(ii) *$P_n(\wp_0([0,\infty))) \subset \wp_0([0,\infty))$.*

Proof (i) For any $f \in \wp_b([0,\infty))$, there exists an M_f depending on f such that $|f| \leq M_f$. Therefore, for every $n \geq 1$,

$$\begin{aligned}\left|P_n(f)\right| &\leq \frac{ne^{-nx}}{(b_n - c_n)H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}\int_{\frac{n_1+n_2+c_n}{n}}^{\frac{n_1+n_2+b_n}{n}}\left|f(u)\right|du \\ &\leq M_f,\end{aligned}$$

and $P_n(1) = 1$. Hence $\|P_n\|_{\wp_b([0,\infty))} = 1$.

(ii) For fixed $h \in \wp_0([0,\infty))$ and $\epsilon \geq 0$, there exists $u \geq 0$ such that $|h(x)| \leq \frac{\epsilon}{4}$ for any $x \geq u - 1$. Now, since $T_m(x)e^{-nx} \to 0$ as $x \to \infty$, where $T_m(x)$ is any polynomial of degree $m \in \mathbb{N}$ in x for every m, therefore there exists $v > u$ such that, for every $x \geq v$,

$$T_m(x)e^{-nx} \leq \frac{\epsilon}{4\|h\|_\infty(n[u]+1)^2}$$

for any $m = 0, 1, \ldots, 2n[u]$, where $[u]$ denotes the integer part of u.

Using (1.3), we have $\frac{s_{n_1,n_2}(\frac{nx}{2})}{H(1,1)}$ is a polynomial of degree $n_1 + n_2$. Therefore, for every $x \geq v$, we have

$$\begin{aligned}\left|P_n(h;x)\right| \leq{}& \frac{n}{b_n - c_n}\sum_{n_1=0}^{n[u]}\sum_{n_2=0}^{n[u]}e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)}\int_{\frac{n_1+n_2+c_n}{n}}^{\frac{n_1+n_2+b_n}{n}}\left|h(t)\right|dt \\ &+ \frac{n}{b_n - c_n}\sum_{n_1=n[u]+1}^{\infty}\sum_{n_2=0}^{n[u]}e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)}\int_{\frac{n_1+n_2+c_n}{n}}^{\frac{n_1+n_2+b_n}{n}}\left|h(t)\right|dt \\ &+ \frac{n}{b_n - c_n}\sum_{n_1=0}^{n[u]}\sum_{n_2=n[u]+1}^{\infty}e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)}\int_{\frac{n_1+n_2+c_n}{n}}^{\frac{n_1+n_2+b_n}{n}}\left|h(t)\right|dt \\ &+ \frac{n}{b_n - c_n}\sum_{n_1=n[u]+1}^{\infty}\sum_{n_2=n[u]+1}^{\infty}e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)}\int_{\frac{n_1+n_2+c_n}{n}}^{\frac{n_1+n_2+b_n}{n}}\left|h(t)\right|dt \\ \leq{}& \frac{\epsilon}{4} + \frac{\epsilon}{4}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\epsilon}{4}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)} \\
&+ \frac{\epsilon}{4}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} e^{-nx}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!H(1,1)} \\
= {}& \epsilon. \qquad \square
\end{aligned}
$$

Theorem 2 *If $f \in \wp_*([0,\infty))$, then $\lim_{n\to\infty} P_n(f) = f$ uniformly on $[0,\infty)$.*

Moreover, if $f \in \wp_b([0,\infty))$, then $\lim_{n\to\infty} P_n(f) = f$ uniformly on every compact subset of $[0,\infty)$.

Proof To prove the first part, it suffices to show that $P_n(f) \to f$ for every $f \in \wp_0([0,\infty))$ or, in fact, for each function f_ρ defined in Proposition 1 since the subspace generated by them is dense in $\wp_0([0,\infty))$ and the sequence $(P_n)_{n\geq 1}$ is equibounded on $\wp_0([0,\infty))$. Now, by using (3.1), for every $x \geq 0$ and $n \geq 1$, we get

$$
\begin{aligned}
&\left|P_n(f_\rho)(x) - f_\rho(x)\right| \\
&\quad \leq \left|\frac{n}{\rho(b_n - c_n)}\left(e^{-\rho\frac{c_n}{n}} - e^{-\rho\frac{b_n}{n}}\right) - 1\right| S_n(f_\rho)(x) + \left|S_n(f_\rho)(x) - f_\rho(x)\right|.
\end{aligned}
$$

Now since

$$
\begin{aligned}
S_n(f_\rho)(x) &= \frac{e^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!}e^{-\rho(\frac{n_1+n_2}{n})} \\
&\leq \frac{e^{-nx}}{H(1,1)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{s_{n_1,n_2}(\frac{nx}{2})}{n_1!n_2!} \\
&= 1.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\left|P_n(f_\rho)(x) - f_\rho(x)\right| \\
&\quad \leq \left|\frac{n}{\rho(b_n - c_n)}\left(e^{-\rho\frac{c_n}{n}} - e^{-\rho\frac{b_n}{n}}\right) - 1\right| + \left|S_n(f_\rho)(x) - f_\rho(x)\right|,
\end{aligned}
$$

and by using the inequality $\frac{n}{\rho(b_n-c_n)}(e^{-\rho\frac{c_n}{n}} - e^{-\rho\frac{b_n}{n}}) \leq 1$ (proved in Proposition 2), we get

$$
\begin{aligned}
&\left|P_n(f_\rho)(x) - f_\rho(x)\right| \\
&\quad \leq \left(1 - \frac{n}{\rho(b_n - c_n)}\left(e^{-\rho\frac{c_n}{n}} - e^{-\rho\frac{b_n}{n}}\right)\right) + \left|S_n(f_\rho)(x) - f_\rho(x)\right| \\
&\quad \leq \left(1 - \frac{n}{\rho(b_n - c_n)}\left(e^{-\rho\frac{c_n}{n}} - e^{-\rho\frac{b_n}{n}}\right)\right) + \left\|S_n(f_\rho) - f_\rho\right\|_\infty.
\end{aligned}
$$

Now from [[5], p. 845], we have

$$
\left(1 - \frac{n}{\rho(b_n - c_n)}\left(e^{-\rho\frac{c_n}{n}} - e^{-\rho\frac{b_n}{n}}\right)\right) \leq \frac{\rho}{n}.
$$

Therefore

$$\left|P_n(f_\rho)(x)-f_\rho(x)\right| \le \frac{\rho}{n} + \left\|S_n(f_\rho)-f_\rho\right\|_\infty.$$

Now since the sequence $(S_n(f_\rho))_{n\ge 1}$ converges uniformly to f_ρ by Proposition 1, the result is achieved.

For the second part of the theorem, we see that, from Lemma 1, $\lim_{n\to\infty} P_n(g) = g$ uniformly on compact subsets of $[0,\infty)$ for every $g \in \{\mathbf{1}, c_1, c_2\} \subset G_2^*$, the result holds from [[3], Theorem 3.5]. □

4 Estimating the rate of convergence

We now present some estimates of the rate of convergence of $(P_n(h))_{n\ge 1}$ to h by using the moduli of smoothness of first and second order $\omega(h,\gamma)$ and $\omega_2(h,\gamma)$. For the definitions of $\omega(h,\gamma)$ and $\omega_2(h,\gamma)$, we refer the reader to [[4], Sect. 5.1].

Theorem 3 *Let $h \in \wp_b([0,\infty))$, $n \ge 1$, and $y \ge 0$. Then*

$$\begin{aligned}
&\left|P_n(h)(y)-h(y)\right| \\
&\quad\le \left(\frac{b_n+c_n}{2\sqrt{n}} + \frac{H_{t_1}(1,1)+H_{t_2}(1,1)}{\sqrt{n}H(1,1)}\right)\omega\left(h,\frac{1}{\sqrt{n}}\right) \\
&\qquad + \left[1+\frac{1}{2}\left(y+\frac{1}{nH(1,1)}\left\{H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)\right.\right.\right. \\
&\qquad \left.+ H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\right\} \\
&\qquad \left.\left.+ \frac{b_n^2+c_n^2+b_nc_n}{n} + (b_n+c_n)\left(\frac{H_{t_1}(1,1)+H_{t_2}(1,1)}{H(1,1)}\right)\right)\right]\omega_2\left(h,\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Proof Since (2.3) holds, by [[15], Theorem 2.2.1] and Lemma 1, for every $\gamma > 0$,

$$\begin{aligned}
&\left|P_n(h)(y)-h(y)\right| \\
&\quad\le \left|P_n(1)(y)-1\right|\left|h(y)\right| + \frac{1}{\gamma}\left|P_n(\rho_y)(y)\right|\omega(h,\gamma) + \left[P_n(1)(y)+\frac{1}{2\gamma^2}P_n\left(\rho_y^2\right)y)\right]\omega_2(h,\gamma) \\
&\quad= \frac{1}{\gamma}\left(\frac{b_n+c_n}{2n} + \frac{H_{t_1}(1,1)+H_{t_2}(1,1)}{nH(1,1)}\right)\omega(h,\gamma) \\
&\qquad + \left[1+\frac{1}{2\gamma^2}\left(\frac{y}{n}+\frac{1}{n^2H(1,1)}\left\{H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)\right.\right.\right. \\
&\qquad \left.+ H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\right\} \\
&\qquad \left.\left.+ \frac{b_n^2+c_n^2+b_nc_n}{n^2} + \frac{1}{n}(b_n+c_n)\left(\frac{H_{t_1}(1,1)+H_{t_2}(1,1)}{nH(1,1)}\right)\right)\right]\omega_2(h,\gamma).
\end{aligned}$$

Putting $\gamma = \frac{1}{\sqrt{n}}$, we get the result. □

5 Quantitative estimates

Lemma 2 *Let* $0 \leq c_n \leq b_n \leq 1$ $(n \geq 1)$, $h \in \wp_b([0,\infty))$, *and* $G(x) = \int_0^x g(t)\,dt$ (≥ 0). *Then, for every* $x \geq 0$ *and* $n \geq 1$,

$$\left|\frac{n}{c_n - a_n}\tau_n(G)(x) - g(x)\right| \leq \omega\left(g, \frac{b_n - c_n}{n}\right). \tag{5.1}$$

Moreover, for every $\gamma > 0$,

$$\omega\big(\tau_n(G), \gamma\big) \leq \frac{b_n - c_n}{n}\omega\left(g, \gamma + \frac{b_n - c_n}{n}\right). \tag{5.2}$$

Proof Let $x \geq 0$ be fixed and $n \geq 1$; then by applying Lagrange's theorem to the function G in the interval $[x + \frac{c_n}{n}, x + \frac{b_n}{n}]$, we have

$$\frac{n}{b_n - c_n}\tau_n(G)(x) = g(\xi_{n,x}),$$

where $\xi_{n,x}$ is a point in $[x + \frac{c_n}{n}, x + \frac{b_n}{n}]$.

Now,

$$\left|\frac{n}{b_n - c_n}\tau_n(G)(x) - g(x)\right| = \big|g(\xi_{n,x}) - g(x)\big| \leq \omega\big(g, |\xi_{n,x} - x|\big) \leq \omega\left(g, \frac{b_n - c_n}{n}\right).$$

Now, for $x, y \geq 0$ such that $|x - y| \leq \gamma$ where $\gamma \geq 0$ is a fixed number, again by Lagrange's theorem,

$$\big|\tau_n(G)(x) - \tau_n(G)(y)\big| = \frac{b_n - c_n}{n}\big|g(\xi_{n,x}) - g(\zeta_{n,y})\big| \leq \frac{b_n - c_n}{n}\omega\big(g, |\xi_{n,x} - \zeta_{n,y}|\big),$$

where $\zeta_{n,y}$ is some element in the interval $[y + \frac{c_n}{n}, y + \frac{b_n}{n}]$, and hence the result since

$$|\xi_{n,x} - \zeta_{n,y}| \leq |x - y| + \frac{b_n - c_n}{n} \leq \gamma + \frac{b_n - c_n}{n}. \qquad \square$$

Theorem 4 *Consider* $g \in \wp_b([0,\infty))$, $n \geq 1$, *and* $x \geq 0$. *Then*

$$\begin{aligned}&\big|P_n(g)(x) - g(x)\big| \\ &\quad\leq \left(2 + \sqrt{x + \frac{1}{nH(1,1)}\big(H_{t_1}(1,1) + H_{t_2}(1,1) + H_{t_1t_1}(1,1) + H_{t_2t_2}(1,1) + 2H_{t_1t_2}(1,1)\big)}\right) \\ &\qquad\times \omega\left(g, \frac{\sqrt{n} + b_n - c_n}{n}\right).\end{aligned} \tag{5.3}$$

Furthermore, if g *is differentiable on* $[0,\infty)$ *and* $g' \in \wp_b([0,\infty))$, *then*

$$\begin{aligned}&\big|P_n(g)(x) - g(x)\big| \\ &\quad\leq \left(\sqrt{\frac{x}{n} + \frac{1}{n^2H(1,1)}\big(H_{t_1}(1,1) + H_{t_2}(1,1) + H_{t_1t_1}(1,1) + H_{t_2t_2}(1,1) + 2H_{t_1t_2}(1,1)\big)}\right)\end{aligned}$$

$$\times\left(1+\sqrt{x+\frac{1}{nH(1,1)}\bigl(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\bigr)}\right)$$
$$\times\,\omega\left(g',\frac{\sqrt{n}+b_n-c_n}{n}\right)+\|g'\|_\infty\frac{b_n-c_n}{n}. \tag{5.4}$$

Proof From [[4], Theorem 5.2.4], it follows that, for every $\gamma>0$,

$$\begin{aligned}&\bigl|S_n(g)(x)-g(x)\bigr|\\&\quad\le\left(1+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\bigl(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\bigr)}\right)\\&\qquad\times\omega(g,\gamma).\end{aligned}\tag{5.5}$$

From this and from (5.1) and (5.2), we have

$$\begin{aligned}&\bigl|P_n(g)(x)-g(x)\bigr|\\&\quad\le\frac{n}{b_n-c_n}\bigl|S_n\bigl(\tau_n(G)\bigr)(x)-\tau_n(G)(x)\bigr|+\left|\frac{n}{b_n-c_n}\tau_n(G)(x)-g(x)\right|\\&\quad\le\left(1+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\bigl(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\bigr)}\right)\\&\qquad\times\frac{n}{b_n-c_n}\omega\bigl(\tau_n(G),\gamma\bigr)+\omega\left(g,\frac{b_n-c_n}{n}\right)\\&\quad\le\left(1+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\bigl(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\bigr)}\right)\\&\qquad\times\omega\left(g,\gamma+\frac{b_n-c_n}{n}\right)+\omega\left(g,\frac{b_n-c_n}{n}\right)\\&\quad\le\left(2+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\bigl(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\bigr)}\right)\\&\qquad\times\omega\left(g,\gamma+\frac{b_n-c_n}{n}\right).\end{aligned}$$

Taking $\gamma=\frac{1}{\sqrt{n}}$, we have (5.3).

And for (5.4), assume that g is differentiable on $[0,\infty)$ and $g'\in\wp_b([0,\infty))$; then

$$\omega\left(g,\frac{b_n-c_n}{n}\right)\le\|g'\|_\infty\frac{b_n-c_n}{n}. \tag{5.6}$$

Moreover, from [[4], Theorem 5.2.4], we have, for $\gamma>0$,

$$\begin{aligned}&\bigl|S_n(g)(x)-g(x)\bigr|\\&\quad\le\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\bigl(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\bigr)}\end{aligned}$$

$$\times\left(1+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\big(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\big)}\right)$$
$$\times\,\omega\big(g',\gamma\big). \tag{5.7}$$

Since g is differentiable, $\tau_n(G)$ is also differentiable with bounded and continuous derivative, and for every $x \geq 0$ and $n \geq 1$,

$$\tau_n(G)'(x) = g\left(x+\frac{b_n}{n}\right) - g\left(x+\frac{c_n}{n}\right).$$

Now, if $x, y \geq 0$ are two elements such that $|x-y| \leq \gamma$, then by Lagrange's theorem, there exist $\eta_{n,x} \in [x+\frac{c_n}{n}, x+\frac{b_n}{n}]$ and $\xi_{n,y} \in [y+\frac{c_n}{n}, y+\frac{b_n}{n}]$ such that

$$\begin{aligned}\big|\tau_n(G)'(x)-\tau_n(G)'(y)\big| &= \left|g\left(x+\frac{b_n}{n}\right)-g\left(x+\frac{c_n}{n}\right)-g\left(y+\frac{b_n}{n}\right)+g\left(y+\frac{c_n}{n}\right)\right| \\ &= \frac{b_n-c_n}{n}\big|g'(\eta_{n,x})-g'(\xi_{n,y})\big| \leq \frac{b_n-c_n}{n}\omega\left(g',\gamma+\frac{b_n-c_n}{n}\right).\end{aligned}$$

Hence,

$$\omega\big(\tau_n(G)',\gamma\big) \leq \frac{b_n-c_n}{n}\omega\left(g',\gamma+\frac{b_n-c_n}{n}\right), \tag{5.8}$$

and by (5.6), (5.7), and (5.8),

$$\begin{aligned}&\big|P_n(g)(x)-g(x)\big| \\ &\leq \frac{n}{b_n-c_n}\big|S_n\big(\tau_n(G)\big)(x)-\tau_n(G)(x)\big| + \left|\frac{n}{b_n-c_n}\tau_n(G)(x)-g(x)\right| \\ &\leq \frac{n}{b_n-c_n} \\ &\quad\times\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\big(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\big)} \\ &\quad\times\left(1+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\big(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\big)}\right) \\ &\quad\times\,\omega\big(\tau_n(G)',\gamma\big)+\omega\left(g,\frac{n}{b_n-c_n}\right) \\ &\leq \sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\big(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\big)} \\ &\quad\times\left(1+\frac{1}{\gamma}\sqrt{\frac{x}{n}+\frac{1}{n^2H(1,1)}\big(H_{t_1}(1,1)+H_{t_2}(1,1)+H_{t_1t_1}(1,1)+H_{t_2t_2}(1,1)+2H_{t_1t_2}(1,1)\big)}\right) \\ &\quad\times\,\omega\left(g',\gamma+\frac{b_n-c_n}{n}\right)+\big\|g'\big\|_\infty\frac{b_n-c_n}{n}.\end{aligned}$$

In particular, for $\gamma = \frac{1}{n^{\frac{1}{2}}}$, we have (5.4). □

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