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# Some inequalities obtained by fractional integrals of positive real orders

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## Abstract

The primary objective of this study is to handle new generalized Hermite–Hadamard type inequalities with the help of the Katugampola fractional integral operator, which generalizes the Hadamard and Riemann–Liouville fractional integral operators into one system. In order to do this, a new fractional integral identity is obtained. Then, by using this identity, some inequalities for the class of functions whose derivatives in absolute values at certain powers are  $\rho$ -convex are derived. It is observed that the obtained inequalities are generalizations of some results in the literature.

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**Keywords:** Katugampola fractional integrals; Hadamard type inequalities;  $\rho$ -convex functions

## 1 Introduction

It is a well known fact that inequalities have important roles to play in the studies of linear programming, extremum problems, optimization, error estimates and game theory (see for example [2]). Over the years, only integer real order integrals were taken into account while handling new results about integral inequalities. However, in recent years, fractional calculus has been considered by many scientists (see [3–5, 7–10, 12, 13]). There are some inequalities in the literature that accelerated studies on integral inequalities. One of the most famous and practical inequalities in the literature was the Hermite–Hadamard inequality given in the following theorem.

**Theorem 1.1** *Let  $f$  be defined from interval  $I$  (a nonempty subset of  $\mathbb{R}$ ) to  $\mathbb{R}$  to be a convex function on  $I$  and  $a, b \in I$  with  $a < b$ . Then the double inequality given in the following holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Zhang and Wan introduced  $p$ -convex functions in [13] and İşcan gave a different version of this definition in [3] as follows.

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**Definition 1.1** Let  $I$  be an interval composed of positive real numbers and  $p \in \mathbb{R} \setminus \{0\}$ .  $f : I \rightarrow \mathbb{R}$  is called a  $p$ -convex function if it satisfies

$$f\left(\left[tx^p + (1-t)y^p\right]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y) \quad (2)$$

for all  $t \in [0, 1]$  and  $x, y \in I$ .

It is easy to see that ordinary convexity is retrieved from  $p$ -convexity for  $p = 1$  and harmonically convexity is retrieved from  $p$ -convexity for  $p = -1$ .

Now we will mention some kinds of fractional integral operators and the definition in the space  $X_c^p(a, b)$ .

The first of them is the Riemann–Liouville fractional integral, which makes the integration of fractional order possible (see [9]).

**Definition 1.2** Let  $f \in L_1[a, b]$ .  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$ , which are called left-sided and right-sided Riemann–Liouville integrals of order  $\alpha > 0$  with  $a \geq 0$ , are defined by

$$J_{a+}^\alpha f = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \quad (3)$$

and

$$J_{b-}^\alpha f = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b, \quad (4)$$

respectively, where  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ . Here  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

**Definition 1.3** ([9]) The left-sided and right-sided Hadamard fractional integrals of order  $\alpha \in \mathbb{R}^+$  are defined as

$$H_{a+}^\alpha \varphi = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(\ln \frac{x}{t})^{1-\alpha}} \frac{dt}{t}, \quad x > a > 0, \quad (5)$$

$$H_{b-}^\alpha \varphi = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{(\ln \frac{t}{x})^{1-\alpha}} \frac{dt}{t}, \quad 0 < x < b, \quad (6)$$

where  $\Gamma$  is the gamma function.

**Definition 1.4** ([8]) Let us consider the space  $X_c^p(a, b)$  ( $1 \leq p \leq \infty$ ,  $c \in \mathbb{R}$ ) of the Lebesgue measurable complex-valued mappings  $f$  on  $[a, b]$  which satisfy  $\|f\|_{X_c^p} < \infty$  where the norm is defined for the case  $1 \leq p \leq \infty$ ,  $c \in \mathbb{R}$  as follows:

$$\|f\|_{X_c^p} = \left( \int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \quad (7)$$

and, for the case  $p = \infty$ ,

$$\|f\|_{X_c^p} = \operatorname{ess\,sup}_{a \leq t \leq b} [t^c |f(t)|] \quad (c \in \mathbb{R}). \quad (8)$$

Katugampola revealed a new fractional integration operator which generalizes both the Riemann–Liouville and the Hadamard fractional integration operators. This integration operator possesses the semigroup properties (see [4, 5]) and is defined as follows.

**Definition 1.5** Let  $[a, b] \subset \mathbb{R}$  be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order  $(\alpha > 0)$  of  $f \in X_c^p(a, b)$  are defined by

$${}^\rho I_{a+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) dt \quad (9)$$

and

$${}^\rho I_{b-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) dt \quad (10)$$

with  $a < x < b$  and  $\rho > 0$  if the integral exists. Equations (9) and (10) look quite the same as the Erdelyi–Kober operator. But besides the Hadamard fractional integrals not being a direct consequence of the Erdelyi–Kober operator, they are a direct consequence of the Katugampola fractional integral operators.

**Theorem 1.2** ([5]) Let  $\alpha > 0$  and  $\rho > 0$ . Then, for  $x > a$ ,

$$1. \quad \lim_{\rho \rightarrow 1} {}^\rho I_{a+}^\alpha f(x) = J_{a+}^\alpha f(x), \quad (11)$$

$$2. \quad \lim_{\rho \rightarrow 0^+} {}^\rho I_{a+}^\alpha f(x) = H_{a+}^\alpha f(x). \quad (12)$$

For right-sided operators, a similar conclusion can be drawn.

For more studies of fractional integral inequalities, see [10, 12] and the references therein.

Erdelyi *et al.* were deeply involved in hypergeometric functions given in the following (see [1]):

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, b-c)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b > 0, |z| < 1 \quad (13)$$

and the regularized hypergeometric function is

$${}_p\tilde{F}_q[a_1, \dots, a_p; b_1, \dots, b_q; z] \equiv \frac{{}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z]}{\Gamma(b_1) \cdots \Gamma(b_q)} \quad (14)$$

given in [11]. We will define  $T_f(\alpha, \rho; a, x, b)$  by

$$\begin{aligned} T_f(\alpha, \rho; a, x, b) &= \frac{\rho}{b-a} \left[ (x^\rho - a^\rho)^\alpha f(a) + (b^\rho - x^\rho)^\alpha f(b) \right] \\ &\quad - \frac{\rho^{\alpha+1} \Gamma(\alpha+1)}{b-a} \left[ {}^\rho I_{x-}^\alpha f(a) + {}^\rho I_{x+}^\alpha f(b) \right] \end{aligned} \quad (15)$$

and  $\Gamma$  is the Euler Gamma function, i.e.,  $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$ .

Kavurmacı *et al.* obtained new Ostrowski type results after proving the next lemma in 2011 in [6].

**Lemma 1.1** ([6]) *Let  $f$  be defined from an interval  $I$  to  $\mathbb{R}$  as a differentiable mapping on the interior of  $I$ , where  $a, b \in I$ ,  $a < b$  and  $f' \in L[a, b]$ . Then the equality given here is valid:*

$$\begin{aligned} & \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{(x-a)^2}{b-a} \int_0^1 (t-1)f'(tx + (1-t)a) dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t)f'(tx + (1-t)b) dt. \end{aligned} \quad (16)$$

Kavurmacı *et al.* presented the next lemma to handle Ostrowski type inequalities for Riemann–Liouville fractional integrals in 2012 in [7].

**Lemma 1.2** ([7]) *Let  $f$  be defined from interval  $I$  to  $\mathbb{R}$  as a differentiable function on  $I^\circ$ , where  $a$  and  $b$  belong to  $I$  with  $a < b$  and  $f' \in L[a, b]$ . Then we get*

$$\begin{aligned} & \frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a} [J_{x^-}^\alpha f(a) + J_{x^+}^\alpha f(b)] \\ &= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 (t^\alpha - 1)f'(tx + (1-t)a) dt \\ & \quad + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 (1-t^\alpha)f'(tx + (1-t)b) dt \end{aligned} \quad (17)$$

for all  $x \in [a, b]$  and  $\alpha > 0$ .

In this paper, a new kernel and Ostrowski type new theorems including the Katugampola fractional integral operator have been retrieved inspired by Lemma 1.2.

## 2 Main results

**Lemma 2.1** *Let  $f$  be defined from interval  $I$  which consists of positive real numbers to  $\mathbb{R}$  as a differentiable function on  $I^\circ$ , where  $a, b \in I$  with  $a < b$  and  $f' \in L[a, b]$ . Then we have*

$$\begin{aligned} T_f(\alpha, \rho; a, x, b) &= \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(t^\alpha - 1)f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha)f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt \end{aligned} \quad (18)$$

for all  $x \in [a, b]$ ,  $\rho > 0$  and  $\alpha > 0$ .

*Proof* With the help of partial integration we have

$$\begin{aligned} I_1 &= \int_0^1 \frac{(t^\alpha - 1)f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \\ &= \frac{\rho f(a)}{x^\rho - a^\rho} - \frac{\alpha \rho}{x^\rho - a^\rho} \int_0^1 t^{\alpha-1} f([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}}) dt. \end{aligned} \quad (19)$$

By changing the variable  $[tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}} = u$  we get

$$\begin{aligned} I_1 &= \frac{\rho f(a)}{x^\rho - a^\rho} - \frac{\alpha\rho}{x^\rho - a^\rho} \int_a^x \left( \frac{u^\rho - a^\rho}{x^\rho - a^\rho} \right)^{\alpha-1} \frac{\rho u^{\rho-1}}{x^\rho - a^\rho} f(u) du \\ &= \frac{\rho f(a)}{x^\rho - a^\rho} - \frac{\alpha\rho^2}{(x^\rho - a^\rho)^{\alpha+1}} \int_a^x \frac{u^{\rho-1}}{(u^\rho - a^\rho)^{1-\alpha}} f(u) du \\ &= \frac{\rho f(a)}{x^\rho - a^\rho} - \frac{\alpha\rho^2 \Gamma(\alpha)}{(x^\rho - a^\rho)^{\alpha+1} \rho^{1-\alpha}} {}^\rho I_{x^-}^\alpha f(a) \\ &= \frac{\rho f(a)}{x^\rho - a^\rho} - \frac{\rho^{\alpha+1} \Gamma(\alpha+1)}{(x^\rho - a^\rho)^{\alpha+1}} {}^\rho I_{x^-}^\alpha f(a). \end{aligned} \quad (20)$$

Similarly we have

$$I_2 = \int_0^1 \frac{(1-t^\alpha) f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt, \quad (21)$$

$$\begin{aligned} I_2 &= \frac{\rho(1-t^\alpha)}{x^\rho - b^\rho} f([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}}) \Big|_0^1 \\ &\quad + \frac{\alpha\rho}{x^\rho - b^\rho} \int_0^1 t^{\alpha-1} f([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}}) dt \\ &= \frac{\rho f(b)}{b^\rho - x^\rho} - \frac{\alpha\rho}{b^\rho - x^\rho} \int_0^1 t^{\alpha-1} f([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}}) dt. \end{aligned} \quad (22)$$

By changing the variable  $[tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}} = u$  we get

$$\begin{aligned} I_2 &= \frac{\rho f(b)}{b^\rho - x^\rho} - \frac{\alpha\rho}{b^\rho - x^\rho} \int_b^x \left( \frac{u^\rho - b^\rho}{x^\rho - b^\rho} \right)^{\alpha-1} \frac{\rho u^{\rho-1}}{x^\rho - b^\rho} f(u) du \\ &= \frac{\rho f(b)}{b^\rho - x^\rho} - \frac{\alpha\rho^2}{(b^\rho - x^\rho)^{\alpha+1}} \int_x^b \frac{u^{\rho-1}}{(b^\rho - u^\rho)^{1-\alpha}} f(u) du \\ &= \frac{\rho f(b)}{b^\rho - x^\rho} - \frac{\alpha\rho^2 \Gamma(\alpha)}{(b^\rho - x^\rho)^{\alpha+1} \rho^{1-\alpha}} {}^\rho I_{x^+}^\alpha f(b) \\ &= \frac{\rho f(b)}{b^\rho - x^\rho} - \frac{\rho^{\alpha+1} \Gamma(\alpha+1)}{(b^\rho - x^\rho)^{\alpha+1}} {}^\rho I_{x^+}^\alpha f(b). \end{aligned} \quad (23)$$

By multiplying (20) and (23) with  $\frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a}$  and  $\frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a}$ , respectively, and then summing them side by side, we have

$$\begin{aligned} &\frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(t^\alpha - 1) f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \\ &\quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha) f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt \\ &= \frac{\rho f(a)(x^\rho - a^\rho)^\alpha}{b-a} - \frac{\rho^{\alpha+1} \Gamma(\alpha+1) {}^\rho I_{x^-}^\alpha f(a)}{b-a} \\ &\quad + \frac{\rho f(b)(b^\rho - x^\rho)^\alpha}{b-a} - \frac{\rho^{\alpha+1} \Gamma(\alpha+1) {}^\rho I_{x^+}^\alpha f(b)}{b-a}. \end{aligned} \quad (24)$$

By rearranging the last equality we get the desired equality.  $\square$

**Remark 2.1** If we choose  $\rho \rightarrow 1$  in Lemma 2.1, we get Lemma 1.2 proved in [7].

**Remark 2.2** By choosing  $\rho \rightarrow 1$  and  $\alpha = 1$  in Lemma 2.1, we get Lemma 1.1 proved in [6].

**Theorem 2.1** Let  $f$  be defined from interval  $I$  which consists of positive real numbers to  $\mathbb{R}$  as a differentiable mapping on  $I^\circ$  and  $a, b \in I$  with  $a < b$  such that  $f' \in L[a, b]$ . If  $|f'|$  is  $\rho$ -convex on  $I$  we have

$$\begin{aligned} |T_f(\alpha, \rho; a, x, b)| &\leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \{ |f'(x)|K(a) + |f'(a)|L(a) \} \\ &\quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \{ |f'(x)|K(b) + |f'(b)|L(b) \} \end{aligned} \quad (25)$$

where

$$\begin{aligned} K(a) &= \frac{\rho(x^{1+\rho} - a^\rho(-a\rho + x + \rho x))}{(1+\rho)(x^\rho - a^\rho)^2} \\ &\quad - a^{1-\rho} \Gamma(2+\alpha)_1 \tilde{F}_2 \left( 2+\alpha, \frac{\rho-1}{\rho}; 3+\alpha; 1 - \left( \frac{x}{a} \right)^\rho \right), \end{aligned} \quad (26)$$

$$\begin{aligned} K(b) &= \frac{\rho(x^{1+\rho} - b^\rho(-b\rho + x + \rho x))}{(1+\rho)(x^\rho - b^\rho)^2} \\ &\quad - b^{1-\rho} \Gamma(2+\alpha)_1 \tilde{F}_2 \left( 2+\alpha, \frac{\rho-1}{\rho}; 3+\alpha; 1 - \left( \frac{x}{b} \right)^\rho \right), \end{aligned} \quad (27)$$

$$L(a) = \frac{(tx^\rho + a^\rho)^{\frac{1-\rho}{\rho}} [{}_2F_1(1, p-1; p+2; \frac{a^\rho}{tx^\rho + a^\rho})]^{\frac{1}{\rho}}}{(\rho+1)^{\frac{1}{\rho}}} \left( \frac{\Gamma(q+1)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+q+\frac{1}{\alpha})} \right)^{\frac{1}{q}}, \quad (28)$$

$$L(b) = \frac{(tx^\rho + b^\rho)^{\frac{1-\rho}{\rho}} [{}_2F_1(1, p-1; p+2; \frac{b^\rho}{tx^\rho + b^\rho})]^{\frac{1}{\rho}}}{(\rho+1)^{\frac{1}{\rho}}} \left( \frac{\Gamma(q+1)\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+q+\frac{1}{\alpha})} \right)^{\frac{1}{q}}, \quad (29)$$

and for all  $x \in (a, 2^{\frac{1}{\rho}}a)$  (if  $2^{\frac{1}{\rho}}a < b$ , otherwise  $x \in (a, b)$ ),  $\alpha > 0$ ,  $\rho > 1$ ,  $q > 1$  and  $\frac{1}{\rho} + \frac{1}{q} = 1$ .

**Proof** Using Lemma 2.1 and the properties of the absolute value we get

$$\begin{aligned} |T_f(\alpha, \rho; a, x, b)| &\leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha)|f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})|}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \\ &\quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha)|f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})|}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt. \end{aligned}$$

Then by taking into account the  $\rho$ -convexity of  $|f'|$  and the Hölder inequality we get

$$\begin{aligned} &T_f(\alpha, \rho; a, x, b) \\ &\leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{(t-t^{\alpha+1})|f'(x)|}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt + \int_0^1 \frac{(1-t)(1-t^\alpha)|f'(a)|}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right\} \\ &\quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{(t-t^{\alpha+1})|f'(x)|}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt + \int_0^1 \frac{(1-t)(1-t^\alpha)|f'(b)|}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{(t-t^{\alpha+1})|f'(x)|}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right. \\
&\quad + \left( \int_0^1 \frac{(1-t)^\rho}{(tx^\rho + (1-t)a^\rho)^{\rho-1}} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}} |f'(a)| \Big\} \\
&\quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left\{ \int_0^1 \frac{(t-t^{\alpha+1})|f'(x)|}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt \right. \\
&\quad + \left( \int_0^1 \frac{(1-t)^\rho}{(tx^\rho + (1-t)b^\rho)^{\rho-1}} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}} |f'(b)| \Big\}. \quad (30)
\end{aligned}$$

By the necessary computations we have

$$K(a) = \int_0^1 \frac{(t-t^{\alpha+1})}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt, \quad (31)$$

$$K(b) = \int_0^1 \frac{(t-t^{\alpha+1})}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt, \quad (32)$$

$$L(a) = \left( \int_0^1 \frac{(1-t)^\rho}{(tx^\rho + (1-t)a^\rho)^{\rho-1}} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}}, \quad (33)$$

$$L(b) = \left( \int_0^1 \frac{(1-t)^\rho}{(tx^\rho + (1-t)b^\rho)^{\rho-1}} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}}, \quad (34)$$

where  $K(a)$ ,  $K(b)$ ,  $L(a)$  and  $L(b)$  are defined as in (26), (27), (28) and (29), respectively. So the proof is completed.  $\square$

**Theorem 2.2** Let  $f$  be defined from an interval  $I$  which consists of positive real numbers to  $\mathbb{R}$  as a differentiable mapping on  $I^\circ$  and  $a, b \in I$  with  $a < b$  such that  $f' \in L[a, b]$ . If  $|f'|^q$  is  $\rho$ -convex on  $I$  we have

$$\begin{aligned}
&|T_f(\alpha, \rho; a, x, b)| \\
&\leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} M^{\frac{1}{r}}(a) (|f'(x)|^q N_1 + |f'(a)|^q N_2)^{\frac{1}{q}} \\
&\quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} M^{\frac{1}{r}}(b) (|f'(x)|^q N_1 + |f'(b)|^q N_2)^{\frac{1}{q}}, \quad (35)
\end{aligned}$$

where

$$M(a) = \frac{\rho(ax)^{-\rho r} (-a^{\rho+r} x^{\rho r} + a^{\rho r} x^{\rho+r})}{(\rho(1-r) + r)(x^\rho - a^\rho)}, \quad (36)$$

$$M(b) = \frac{\rho(bx)^{-\rho r} (-b^{\rho+r} x^{\rho r} + b^{\rho r} x^{\rho+r})}{(\rho(1-r) + r)(x^\rho - a^\rho)}, \quad (37)$$

$$N_1 = \frac{\Gamma(1+q)\Gamma(\frac{2+\alpha}{\alpha})}{2\Gamma(1+q+\frac{2}{\alpha})}, \quad (38)$$

$$N_2 = \frac{\Gamma(1+q)}{2} \left( \frac{2\Gamma(1+\frac{1}{\alpha})}{\Gamma(1+q+\frac{1}{\alpha})} - \frac{\Gamma(1+\frac{2}{\alpha})}{\Gamma(1+q+\frac{2}{\alpha})} \right), \quad (39)$$

and for all  $x \in (a, b]$ ,  $\alpha > 0$ ,  $\rho > 0$ ,  $r > 1$ ,  $q > 1$ ,  $\frac{1}{r} + \frac{1}{q} = 1$ ,  $r \neq \frac{\rho}{\rho-1}$ .

*Proof* Using Lemma 2.1 and the properties of the absolute value we get

$$\begin{aligned} & |T_f(\alpha, \rho; a, x, b)| \\ & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha) |f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})|}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha) |f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})|}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt. \end{aligned} \quad (40)$$

By using the Hölder inequality we have

$$\begin{aligned} |T_f(\alpha, \rho; a, x, b)| & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 ((tx^\rho + (1-t)a^\rho)^{\frac{1}{\rho}-1})^r dt \right)^{\frac{1}{r}} \\ & \quad \times \left( \int_0^1 (1-t^\alpha)^q |f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 ((tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1})^r dt \right)^{\frac{1}{r}} \\ & \quad \times \left( \int_0^1 (1-t^\alpha)^q |f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})|^q dt \right)^{\frac{1}{q}}. \end{aligned} \quad (41)$$

Since  $|f'|^q$  is  $\rho$ -convex on  $I$  we get

$$\begin{aligned} |T_f(\alpha, \rho; a, x, b)| & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 ((tx^\rho + (1-t)a^\rho)^{\frac{1}{\rho}-1})^r dt \right)^{\frac{1}{r}} \\ & \quad \times \left( \int_0^1 (1-t^\alpha)^q t |f'(x)|^q dt + \int_0^1 (1-t^\alpha)^q (1-t) |f'(a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 ((tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1})^r dt \right)^{\frac{1}{r}} \\ & \quad \times \left( \int_0^1 (1-t^\alpha)^q t |f'(x)|^q dt + \int_0^1 (1-t^\alpha)^q (1-t) |f'(b)|^q dt \right)^{\frac{1}{q}}. \end{aligned} \quad (42)$$

With simple calculation we get

$$M(a) = \int_0^1 ((tx^\rho + (1-t)a^\rho)^{\frac{1}{\rho}-1})^r dt, \quad (43)$$

$$M(b) = \int_0^1 ((tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1})^r dt, \quad (44)$$

$$N_1 = \int_0^1 (1-t^\alpha)^q t dt, \quad \text{and} \quad N_2 = \int_0^1 (1-t^\alpha)^q (1-t) dt, \quad (45)$$

where  $M(a)$ ,  $M(b)$ ,  $N_1$  and  $N_2$  are defined as in (36), (37), (38), and (39), respectively. So we get the desired result.  $\square$

**Theorem 2.3** Let  $f$  be defined from an interval  $I$  which consists of positive real numbers to  $\mathbb{R}$  as a differentiable mapping on  $I^\circ$  and  $a, b \in I$  with  $a < b$  such that  $f' \in L[a, b]$ . If  $|f'|^q$



is  $\rho$ -convex on  $I$  we have

$$\begin{aligned} & |T_f(\alpha, \rho; a, x, b)| \\ & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left( \frac{\rho(x-a)}{x^\rho - a^\rho} \right)^{1-\frac{1}{q}} (|f'(x)|^q K(a) + |f'(a)|^q L(a))^{\frac{1}{q}} \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left( \frac{\rho(b-x)}{b^\rho - x^\rho} \right)^{1-\frac{1}{q}} (|f'(x)|^q K(b) + |f'(a)|^q L(b))^{\frac{1}{q}}, \end{aligned} \quad (46)$$

where  $K(a)$ ,  $K(b)$ ,  $L(a)$  and  $L(b)$  are defined as in Theorem 2.1 and for all  $x \in (a, 2^{\frac{1}{\rho}}a)$  (if  $2^{\frac{1}{\rho}}a < b$ , otherwise  $x \in (a, b)$ ),  $\alpha > 0$ ,  $\rho > 1$ ,  $q > 1$ ,  $\frac{1}{\rho} + \frac{1}{q} = 1$ .

*Proof* Using Lemma 2.1 and the properties of the absolute value we get

$$\begin{aligned} |T_f(\alpha, \rho; a, x, b)| & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha) |f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})|}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 \frac{(1-t^\alpha) |f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})|}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt. \end{aligned} \quad (47)$$

With the help of the power-mean inequality we have

$$\begin{aligned} |T_f(\alpha, \rho; a, x, b)| & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left( \int_0^1 \frac{(1-t^\alpha) |f'([tx^\rho + (1-t)a^\rho]^{\frac{1}{\rho}})|^q}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \int_0^1 \left( \frac{(1-t^\alpha) |f'([tx^\rho + (1-t)b^\rho]^{\frac{1}{\rho}})|^q}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt \right)^{\frac{1}{q}} \end{aligned} \quad (48)$$

and by using the  $\rho$ -convexity of  $|f'|^q$ , then using the Hölder inequality we have

$$T_f(\alpha, \rho; a, x, b) \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right)^{1-\frac{1}{q}} \quad (49)$$

$$\begin{aligned} & \times \left\{ |f'(x)|^q \int_0^1 (t-t^{\alpha+1})(tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1} dt \right. \\ & \quad \left. |f'(a)|^q \int_0^1 (1-t)(1-t^\alpha)(tx^\rho + (1-t)a^\rho)^{\frac{1}{\rho}-1} dt \right\}^{\frac{1}{q}} \\ & \quad + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left\{ |f'(x)|^q \int_0^1 (t-t^{\alpha+1})(tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1} dt \right. \\ & \quad \left. |f'(b)|^q \int_0^1 (1-t)(1-t^\alpha)(tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1} dt \right\}^{\frac{1}{q}} \\ & \leq \frac{(x^\rho - a^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{(1-t^\alpha)}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right)^{1-\frac{1}{q}} \end{aligned} \quad (50)$$

$$\begin{aligned} & \times \left\{ \frac{|f'(x)|^q \int_0^1 (t - t^{\alpha+1})(tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1} dt}{|f'(a)|^q \left( \int_0^1 (1-t)^\rho (tx^\rho + (1-t)a^\rho)^{1-\rho} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}}} \right\}^{\frac{1}{q}} \\ & + \frac{(b^\rho - x^\rho)^{\alpha+1}}{b-a} \left( \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt \right)^{1-\frac{1}{q}} \\ & \times \left\{ \frac{|f'(x)|^q \int_0^1 (t - t^{\alpha+1})(tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1} dt}{|f'(b)|^q \left( \int_0^1 (1-t)^\rho (tx^\rho + (1-t)b^\rho)^{1-\rho} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}}} \right\}^{\frac{1}{q}}. \end{aligned}$$

By simple computation we get

$$\begin{aligned} \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)a^\rho)^{1-\frac{1}{\rho}}} dt &= \frac{\rho}{x^\rho - a^\rho} \left[ (x-a) - \int_a^x \left( \frac{u^\rho - a^\rho}{x^\rho - a^\rho} \right)^\alpha du \right] \\ &< \frac{\rho(x-a)}{x^\rho - a^\rho}, \end{aligned} \quad (51)$$

$$\begin{aligned} \int_0^1 \frac{1-t^\alpha}{(tx^\rho + (1-t)b^\rho)^{1-\frac{1}{\rho}}} dt &= \frac{\rho}{b^\rho - x^\rho} \left[ (b-x) - \int_x^b \left( \frac{b^\rho - u^\rho}{b^\rho - x^\rho} \right)^\alpha du \right] \\ &< \frac{\rho(b-x)}{b^\rho - x^\rho}, \end{aligned} \quad (52)$$

and

$$K(a) = \int_0^1 (t - t^{\alpha+1})(tx^\rho + (1-t)a^\rho)^{\frac{1}{\rho}-1} dt, \quad (53)$$

$$K(b) = \int_0^1 (t - t^{\alpha+1})(tx^\rho + (1-t)b^\rho)^{\frac{1}{\rho}-1} dt, \quad (54)$$

$$L(a) = \left( \int_0^1 \frac{(1-t)^\rho}{(tx^\rho + (1-t)a^\rho)^{\rho-1}} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}}, \quad (55)$$

$$L(b) = \left( \int_0^1 \frac{(1-t)^\rho}{(tx^\rho + (1-t)b^\rho)^{\rho-1}} dt \right)^{\frac{1}{\rho}} \left( \int_0^1 (1-t^\alpha)^q dt \right)^{\frac{1}{q}}, \quad (56)$$

where  $K(a)$ ,  $K(b)$ ,  $L(a)$  and  $L(b)$  are defined as in (26), (27), (28) and (29), respectively. So the proof is completed.  $\square$

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