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New two types of semi-implicit viscosity iterations for approximating the fixed points of nonexpansive operators associated with contraction operators and applications

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Abstract

Motivated and inspired by the growing contribution with respect to iterative approximations from some researchers in the literature, we design and investigate two types of brand-new semi-implicit viscosity iterative approximation methods for finding the fixed points of nonexpansive operators associated with contraction operators in complete CAT(0) spaces and for solving related variational inequality problems. Under some suitable assumptions, strong convergence theorems of the sequences generated by the approximation iterative methods are devised, and a numerical example and some applications to related variational inequality problems are included to verify the effectiveness and practical utility of the convergence theorems. Our main results presented in this paper do not only improve, extend and refine some corresponding consequences in the literature, but also show that the additional variational inequalities, general variational inequality systems and equilibrium problems can be solved via approximation of the iterative sequences. Finally, we provide an open question for future research.

Keywords: Semi-implicit viscosity approximation method; Nonexpansive operator associated with contraction operator; Application to related variational inequality problem; Complete CAT(0) space; Strong convergence

1 Introduction

In this paper, we consider the following two kinds of new semi-implicit viscosity approximation methods of iterative forms (in short, (TVIM-I) and (TVIM-II), respectively) for nonexpansive operator T associated with contraction operator in CAT(0) space X:

$$\begin{cases} v_n = a_n f(u_n) \oplus (1-a_n)T(\frac{u_n \oplus v_n}{2}), \\ u_{n+1} = b_n u_n \oplus (1-b_n)v_n, \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} v_n = a_n f(u_n) \oplus (1 - a_n) T(\frac{u_n \oplus u_{n+1}}{2}), \\ u_{n+1} = b_n u_n \oplus (1 - b_n) v_n, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $u_1 \in E \subseteq X$ is an arbitrary given element, $f : E \to E$ is a contraction operator and number sequences $\{a_n\}, \{b_n\} \subseteq (0, 1)$ satisfy the following conditions:

$$\begin{cases} \lim_{n\to\infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \\ 0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1. \end{cases} \tag{1.3}$$

Remark 1.1 (i) The iterative procedures (TVIM-I) (1.1) and (TVIM-II) (1.2) with the implicit midpoint rule are well-defined. Indeed, defining an operator $G_1 : E \to E$ by $G_1(v) = a_1 f(u_1) \oplus (1 - a_1) T(\frac{u_1 \oplus v}{2})$ for all $v \in E$, then one has for each $x, y \in E$,

$$\begin{aligned} & d\bigl(G_1(x), G_1(y)\bigr) \\ & \quad = d\left(a_1 f(u_1) \oplus (1 - a_1) T\left(\frac{u_1 \oplus x}{2}\right), a_1 f(u_1) \oplus (1 - a_1) T\left(\frac{u_1 \oplus y}{2}\right)\right) \\ & \quad \leq (1 - a_1) d\left(T\left(\frac{u_1 \oplus x}{2}\right), T\left(\frac{u_1 \oplus y}{2}\right)\right) \\ & \quad \leq (1 - a_1) d\left(\frac{u_1 \oplus x}{2}, \frac{u_1 \oplus y}{2}\right) \\ & \quad \leq \frac{1 - a_1}{2} d(x, y). \end{aligned}$$

This together with $0 < \frac{1-a_1}{2} < 1$ shows that G_1 is a contraction. Thus, by the Banach contraction principle, we know that G_1 has a unique fixed point v_1, i.e., $v_1 = a_1 f(u_1) \oplus (1 - a_1) T(\frac{u_1 \oplus v_1}{2})$, and so $u_2 = b_1 u_1 \oplus (1 - b_1) v_1$. Continuing in the same way, the existence of u_n ($n \geq 3$) is established. Hence, the iterative process (TVIM-I) (1.1) is well-defined.

Similarly, for given $u_1 \in E$, let us address an operator $G_2 : E \to E$ as follows:

$$G_2(v) = b_1 u_1 \oplus (1 - b_1)\left(a_1 f(u_1) \oplus (1 - a_1) T\left(\frac{u_1 \oplus v}{2}\right)\right), \quad \forall v \in E.$$

Then we have

$$d\bigl(G_2(x), G_2(y)\bigr) \leq \frac{(1 - a_1)(1 - b_1)}{2} d(x, y), \quad \forall x, y \in E,$$

which implies that G_2 is a contraction by $0 < \frac{(1-a_1)(1-b_1)}{2} < 1$ and so we obtain a unique fixed point $u_2 \in E$ of G_2. As well, we acquire the iteration variable u_n from (TVIM-II) (1.2) for $n \geq 3$. That is to say that the definition of (TVIM-II) is well-defined.

(ii) One can easily to see that the iterative process (TVIM-I) (1.1) is different from the iteration (TVIM-II) (1.2). Further, we note that (TVIM-I) and (TVIM-II) are brand new and not studied in the literature.

In the last several decades, in order to solve ordinary differential equations, differential algebraic equations, minimization problems, fixed point problems and other related problems, this type of viscosity iterative approximations (TVIM-I) and (TVIM-II) and many special cases has been widely employed to find fixed points of nonexpansive mappings in the setting of Hilbert spaces, Banach spaces and geodesic spaces. And we also notice that a complete CAT(0) space (i.e., also Hadamard space [1]) has something to do with the simply connected Riemannian manifold and includes pre-Hilbert space, $\mathbb{R}$-tree, Euclidean building, and many others as special cases. See, for example, [2–9] and the references therein. Actually the equilibrium problem is extension of the fixed point problem and so the implicit midpoint rule has been extensively used to nonexpansive operators (see [10]). Thus, it was suggested many much of iteration methods of the explicit and implicit methods, such as Halpern iteration, Mann iteration, Ishikawa iteration, and Noor iteration. However, by using the theory of cosine families, Xiao et al. [11] revealed that the explicit, implicit and viscosity iteration processes, respectively, are applicable to the nonexpansive cosine families, and the results show that implicit and viscosity iterations are superior the explicit iteration in convergence. Further, as Xu et al. [10] pointed out "The implicit midpoint rule is one of the powerful methods for solving ordinary differential equations". Reviewing the past work, it is particularly worth mentioning that Xu et al. [10] studied the following viscosity implicit midpoint rule for nonexpansive operator T in Hilbert space $\mathbb{H}$:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \tag{1.4}$$

where $\alpha_n \in (0,1)$ for $n \geq 1$ and $f : \mathbb{H} \to \mathbb{H}$ is a contraction operator. It follows that (1.4) is a special case of (1.1) or (1.2) with $b_n \equiv 0$. Under certain assumptions to the sequence of parameters, the authors proved that the sequence $\{x_n\}$ decided by (1.4) converges strongly to a point $q \in F(T) := \{x \in \mathbb{H} : x = T(x)\}$ denote the set of fixed point of operator T, which is also the unique solution of the following variational inequality:

$$\langle (I-f)q, x-q \rangle \geq 0, \quad x \in F(T), \tag{1.5}$$

where I is the identity operator of $\mathbb{H}$. In connection with of (modified) viscosity implicit rules have been studied by many authors. See, for example, [5, 8, 12, 13] and the references therein. On the other hand, Kaewkhao et al. [6] thought out the following two-step explicit viscosity iteration method (in short, (TSVIM)) for the nonexpansive operator T in complete CAT(0) space X:

$$\begin{cases} y_n = \alpha_n f(x_n) \oplus (1-\alpha_n) T(x_n), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $x_1 \in E$ is an arbitrary fixed element and $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$ under some appropriate conditions, and the contraction coefficient of f is $k \in [0, 1/2)$.

Remark 1.2 We note that aiming at the open question 2 presented in [6], Chang et al. [14] also investigated (TSVIM) (1.6) changing the contraction coefficient $k \in [0, 1/2)$ by $k \in [0,1)$ and satisfying some suitable conditions such as (1.3). Further, it is easy to see that (1.6) is explicit viscosity iteration and can not be reduced from our iterative processes

(TVIM-I) (1.1) and (TVIM-II) (1.2). Furthermore, the one-step explicit viscosity iteration process is achieved when $\beta_n \equiv 0$ in (1.6), but that is not possible for (1.1) and (1.2) under normal circumstances.

One also note that under some certain conditions imposed on parameters $\{\alpha_n\}$ and $\{\beta_n\}$, Kaewkhao et al. [6] analyzed the convergence of the sequence $\{x_n\}$ generated by (TSVIM) for a fixed point $q \in F(T)$, which meets the variational inequality as follows:

$$\langle \overrightarrow{qf(q)}, \overrightarrow{xq} \rangle \geq 0, \quad x \in F(T), \tag{1.7}$$

and the result presented in [6] gives a positive answer to the open question (that is, can the nice projection property $\mathcal{N}$ be omitted) due to Piatek [15]. Very recently, we [3, 4] further extended and improved some corresponding results of Kaewkhao et al. [6], Piatek [15], Chang et al. [14] and so on.

At the end of this section, the arrangement for rest of this article is emerged as follows: We shall give some required concepts and lemmas as preliminaries in Sect. 2. In Sect. 3, we show that the sequences $\{u_n\}$ generated by (TVIM-I) or (TVIM-II) converges strongly to a fixed point $q \in F(T)$ in complete CAT(0) spaces, where q satisfies the variational inequality (1.7). To reflect the validity and significance of (TVIM-I) and (TVIM-II) in regard to (TSVIM) and other relevant viscosity iterative approximation methods for nonexpansive operators associated with contraction operators in the literature, we also display a numerical example and some applications to related variational inequality problems, i.e., a general system of variational inequalities and equilibrium problems in Hilbert spaces in Sect. 4. Finally, concluding remarks are made and an open question for future research is proposed.

2 Preliminaries

To prove our main results, the task of this section is to present some very important and highly necessary concepts and lemmas. For more detailed property of CAT(0) and Δ-convergence, one can refer to [16] and our recent work in [3, 4, 8].

Throughout this paper, letting (X, d) be a CAT(0) space and $E \subseteq X$ a nonempty closed convex subset, then [17, Lemma 2.1] implies that there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = (1-t)d(x, y) \quad \text{and} \quad d(y, z) = td(x, y), \quad \forall x, y \in X, t \in [0, 1], \tag{2.1}$$

and the unique point z in (2.1) is denoted by $tx \oplus (1-t)y$.

Lemma 2.1 ([17]) *For all $x, y, z \in X$ and any $t \in [0, 1]$, the following two statements hold*:

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$;

(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$.

Lemma 2.2 ([18]) *For each $x, y \in X$ and every $t, s \in [0, 1]$, we have*

$$d\big(tx \oplus (1-t)y, sx \oplus (1-s)y\big) \leq |t-s| d(x, y).$$

Lemma 2.3 ([19]) *Assume that* $\{x_n\}, \{y_n\} \subset X$ *are two bounded sequences and* $\{\beta_n\} \subset [0,1]$ *is a sequence with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *If*

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n, \quad \forall n \in \mathbb{N},$$
$$\limsup_{n\to\infty} \bigl(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)\bigr) \le 0,$$

then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

By [16, Proposition 2.4], one knows that, for any $x \in X$, there is a unique point $x_0 \in E$, i.e., *unique nearest point* of x in E, such that

$$d(x, x_0) = \inf\bigl\{d(x,y) : y \in E\bigr\}.$$

We recall that the metric projection of X onto E is an operator $P_E : X \to E$ defined by

$$P_E(x) = x_0: \text{ the unique nearest point of } x \text{ in E.}$$

Since it is impossible to formulate the concept of demi-closedness in a CAT(0) space, as represented in linear spaces, let us officially say that "$I - T$ is demi-closed at zero" when the conditions satisfy $E \supset \{x_n\}$ Δ-converges to $q \in X$ and $d(x_n, Tx_n) \to 0$ yield $q \in F(T)$.

Lemma 2.4 ([2]) *Every bounded sequence containing in a complete* CAT(0) *space always has a* Δ-*convergent subsequence*.

Lemma 2.5 ([17]) *Suppose that* E *is a closed convex subset of a complete* CAT(0) *space* (X,d) *and* $\{x_n\} \subset E$ *is a bounded sequence. Then the asymptotic center* $A(\{x_n\}) \in E$, *where* $A(\{x_n\}) := \arg\inf_{x\in X} \limsup_{n\to\infty} d(x, x_n) = \{z \in X : d(z, x_n) = \inf_{x\in X} \limsup_{n\to\infty} d(x, x_n)\}$.

Denote a pair (i.e., a vector) $(a,b) \subset X \times X$ by $\overrightarrow{ab}$. In 2008, Berg and Nikolaev [20] introduced a map $\langle \cdot, \cdot \rangle$: $(X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\bigl[d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\bigr], \quad \forall a,b,c,d \in X,$$

which is the concept of quasi-linearization. By [20, Corollary 3], one can easily to see that a geodesic space X is a CAT(0) space if and only if X satisfies the Cauchy–Schwarz inequality,

$$\bigl|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle\bigr| \le d(a,b)d(c,d), \quad \forall a,b,c,d \in X.$$

Further, Wangkeeree and Preechasilp [21] proved the following results on quasi-linearization.

Lemma 2.6 ([21, Lemma 2.10]) *If for all* $t \in [0,1]$, $u_t := tu \oplus (1-t)v$, *where* $u, v \in X$, *then, for all* $x, y \in X$, *one has the following presentations*:

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \le t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

3 Main results

In this section, we will prove our main theorems for solving variational inequality systems, equilibrium problems, fixed point problems and other correlative differential equations. Firstly, we investigate the strong convergence of the iteration (TVIM-I). The following lemma is required.

Lemma 3.1 ([22, Lemma 2.1]) *Let $\{s_n\}$ be a non-negative real number sequence with*

$$s_{n+1} \le (1-\gamma_n)s_n + \delta_n, \quad \forall n \ge 1,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $s_n \to 0$ as $n \to \infty$.

Theorem 3.1 *Let E be a nonempty closed convex subset of a complete* CAT(0) *space (X,d). If $T : E \to E$ is a nonexpansive operator with $F(T) \neq \emptyset$, $f : E \to E$ is a contraction with coefficient $k \in [0,1)$, and* (1.3) *holds, then, for any given $u_1 \in E$, the sequence $\{u_n\}$ generated by* (TVIM-I) *converges strongly to $q \in F(T)$ such that $q = P_{F(T)}f(q)$, which is also a unique solution of the variational inequality* (1.7).

Proof The proof will be presented by the following five steps:

Step (I). We show that $\{u_n\}$ is bounded. Indeed, take $p \in F(T)$ arbitrarily. Then from Lemma 2.1, it follows that

$$\begin{aligned}
d(v_n,p) &\le a_n d\big(f(u_n),p\big) + (1-a_n)d\left(T\left(\frac{u_n \oplus v_n}{2}\right), T(p)\right) \\
&\le a_n\big[d\big(f(u_n),f(p)\big) + d\big(f(p),p\big)\big] + (1-a_n)d\left(\frac{u_n \oplus v_n}{2}, p\right) \\
&\le a_n k d(u_n,p) + a_n d\big(f(p),p\big) + \frac{1-a_n}{2}\big[d(u_n,p) + d(v_n,p)\big] \\
&\le \frac{1-(1-2k)a_n}{2} d(u_n,p) + \frac{1-a_n}{2} d(v_n,p) + a_n d\big(f(p),p\big).
\end{aligned}$$

That is,

$$d(v_n,p) \le \frac{1-(1-2k)a_n}{1+a_n} d(u_n,p) + \frac{2a_n}{1+a_n} d\big(f(p),p\big). \tag{3.1}$$

Further, from (TVIM-I) and (3.1), we know that

$$\begin{aligned}
&d(u_{n+1},p) \\
&\quad\le b_n d(u_n,p) + (1-b_n)d(v_n,p) \\
&\quad\le \left[1 - \frac{2(1-k)(1-b_n)a_n}{1+a_n}\right] d(u_n,p) + \frac{2(1-k)(1-b_n)a_n}{1+a_n}\frac{d(f(p),p)}{1-k} \\
&\quad\le \max\left\{d(u_n,p), \frac{d(f(p),p)}{1-k}\right\}.
\end{aligned}$$

By induction, one also has

$$d(u_n,p) \le \max\left\{d(u_1,p), \frac{d(f(p),p)}{1-k}\right\}.$$

Hence, $\{u_n\}$ is bounded and so are $\{v_n\}$, $\{f(u_n)\}$ and $\{T(\frac{u_n \oplus v_n}{2})\}$.

Step (II). $\lim_{n\to\infty} d(u_n, T(u_n)) = 0$ is proposed. Combining [23, Lemma 3] and Lemma 2.2, we know that

$$\begin{aligned}
d(v_n,v_{n+1}) \le{}& d\left(a_n f(u_n) \oplus (1-a_n)T\left(\frac{u_n \oplus v_n}{2}\right),\right.\\
&\left. a_n f(u_n) \oplus (1-a_n)T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right)\\
&+ d\left(a_n f(u_n) \oplus (1-a_n)T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right),\right.\\
&\left. a_n f(u_{n+1}) \oplus (1-a_n)T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right)\\
&+ d\left(a_n f(u_{n+1}) \oplus (1-a_n)T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right),\right.\\
&\left. a_{n+1} f(u_{n+1}) \oplus (1-a_{n+1})T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right)\\
\le{}& a_n d\bigl(f(u_n), f(u_{n+1})\bigr) + (1-a_n) d\left(\frac{u_n \oplus v_n}{2}, \frac{u_{n+1} \oplus v_{n+1}}{2}\right)\\
&+ |a_n - a_{n+1}| d\left(f(u_{n+1}), T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right)\\
\le{}& a_n k d(u_n,u_{n+1}) + \frac{1-a_n}{2}\bigl[d(u_n,u_{n+1}) + d(v_n,v_{n+1})\bigr]\\
&+ |a_n - a_{n+1}| d\left(f(u_{n+1}), T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right)\\
\le{}& \frac{1-a_n(1-2k)}{1+a_n} d(u_n,u_{n+1})\\
&+ \frac{2|a_n - a_{n+1}|}{1+a_n} d\left(f(u_{n+1}), T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right),
\end{aligned}$$

that is,

$$\begin{aligned}
&d(v_n,v_{n+1}) - d(u_n,u_{n+1})\\
&\quad\le \frac{2|a_n - a_{n+1}|}{1+a_n} d\left(f(u_{n+1}), T\left(\frac{u_{n+1} \oplus v_{n+1}}{2}\right)\right) - \frac{2(1-k)a_n}{1+a_n} d(u_n,u_{n+1}).
\end{aligned}$$

This together with $\lim_{n\to\infty} a_n = 0$ implies that

$$\limsup_{n\to\infty}\bigl[d(v_{n+1},v_n) - d(u_{n+1},u_n)\bigr] \le 0.$$

By Lemma 2.3, one has $\lim_{n\to\infty} d(u_n, v_n) = 0$. Thus,

$$\begin{aligned} d\bigl(u_n, T(u_n)\bigr) &\le d(u_n, v_n) + d\left(v_n, T\left(\frac{u_n \oplus v_n}{2}\right)\right) + d\left(T\left(\frac{u_n \oplus v_n}{2}\right), T(u_n)\right) \\ &\le d(u_n, v_n) + a_n d\left(f(u_n), T\left(\frac{u_n \oplus v_n}{2}\right)\right) + d\left(\frac{u_n \oplus v_n}{2}, u_n\right) \\ &\le \frac{3}{2} d(u_n, v_n) + a_n d\left(f(u_n), T\left(\frac{u_n \oplus v_n}{2}\right)\right) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.2}$$

Step (III). The following result should be proved:

$$\omega_\Delta\{u_n\} := \bigcup_{\{\zeta_n\}\subseteq\{u_n\}} \bigl\{A\bigl(\{\zeta_n\}\bigr)\bigr\} \subseteq F(T), \tag{3.3}$$

where $A(\{\zeta_n\})$ is the asymptotic center of $\{\zeta_n\}$. In fact, if $\zeta \in \omega_\Delta\{u_n\}$, then there is a subsequence $\{\zeta_n\}$ of $\{u_n\}$ such that $A(\{\zeta_n\}) = \{\zeta\}$. Further, by Lemma 2.4, we know that there exists a subsequence $\{v_n\}$ of $\{\zeta_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v$. It follows from (3.2) that $\lim_{n\to\infty} d(v_n, T(v_n)) = 0$. From Lemma 2.5 and the demi-closedness of $I - T$ at zero, it follows that $v \in E$ and $v \in F(T)$. Afterwards, $\zeta = v$ will be given. If not, by the uniqueness of the asymptotic centers ζ, v of $\{\zeta_n\}$, $\{v_n\}$, respectively, we get

$$\begin{aligned} \limsup_{n\to\infty} d(\zeta_n, \zeta) &< \limsup_{n\to\infty} d(\zeta_n, v) \le \limsup_{n\to\infty} d(u_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v) < \limsup_{n\to\infty} d(v_n, \zeta) \\ &\le \limsup_{n\to\infty} d(\zeta_n, \zeta), \end{aligned}$$

a contradiction. Hence, we get $\zeta = v \in F(T)$.

Step (IV). Now, we obtain $\limsup_{n\to\infty} \langle \overrightarrow{f(q)q}, \overrightarrow{v_n q} \rangle \le 0$, where $q \in F(T)$ is a unique solution of the variational inequality (1.7). It follows from [24, Theorem 2.4] that $P_{F(T)} f$ is a contraction operator, and there is unique fixed point $q \in F(T)$ such that $q = P_{F(T)} f(q)$ satisfying $\langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle \ge 0$ for any $p \in F(T)$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\} \subseteq \{u_n\}$ such that $\{u_{n_i}\}$ Δ-converges to a point $p \in F(T)$. By [25, Theorem 2.6] and (3.3), one knows that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(q)q}, \overrightarrow{u_n q} \rangle = \lim_{i\to\infty} \langle \overrightarrow{f(q)q}, \overrightarrow{u_{n_i} q} \rangle = \langle \overrightarrow{f(q)q}, \overrightarrow{pq} \rangle \le 0.$$

Thus, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle \overrightarrow{f(q)q}, \overrightarrow{v_n q} \rangle &= \limsup_{n\to\infty} \bigl(\langle \overrightarrow{f(q)q}, \overrightarrow{v_n u_n} \rangle + \langle \overrightarrow{f(q)q}, \overrightarrow{u_n q} \rangle \bigr) \\ &\le \limsup_{n\to\infty} \bigl(d\bigl(f(q), q\bigr) d(v_n, u_n) + \langle \overrightarrow{f(q)q}, \overrightarrow{u_n q} \rangle \bigr) \\ &\le 0. \end{aligned} \tag{3.4}$$

Step (V). We prove that $\{u_n\}$ converges strongly to q which satisfies $q = P_{F(T)}f(q)$ and

$$\langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle \geq 0, \quad \forall p \in F(T).$$

For any $n \in Z^+$, take $z_n = a_n q \oplus (1-a_n)T(\frac{u_n \oplus v_n}{2})$. Then it follows from [21, Lemma 2.9], Lemmas 2.6 and 2.1 that

$$\begin{aligned}
d^2(v_n, q) &\leq d^2(z_n, q) + 2\langle \overrightarrow{v_n z_n}, \overrightarrow{v_n q} \rangle \\
&\leq (1-a_n)^2 d^2\left(T\left(\frac{u_n \oplus v_n}{2}\right), q\right) \\
&\quad + 2\left[a_n \langle \overrightarrow{f(u_n) z_n}, \overrightarrow{v_n q} \rangle + (1-a_n)\left\langle \overrightarrow{T\left(\frac{u_n \oplus v_n}{2}\right) z_n}, \overrightarrow{v_n q} \right\rangle\right] \\
&\leq (1-a_n)^2 d^2\left(T\left(\frac{u_n \oplus v_n}{2}\right), T(q)\right) \\
&\quad + 2\left[a_n^2 \langle \overrightarrow{f(u_n) q}, \overrightarrow{v_n q} \rangle + a_n(1-a_n)\left\langle \overrightarrow{f(u_n) T\left(\frac{u_n \oplus v_n}{2}\right)}, \overrightarrow{v_n q} \right\rangle\right. \\
&\quad + (1-a_n)a_n \left\langle \overrightarrow{T\left(\frac{u_n \oplus v_n}{2}\right) q}, \overrightarrow{v_n q} \right\rangle \\
&\quad \left. + (1-a_n)^2 \left\langle \overrightarrow{T\left(\frac{u_n \oplus v_n}{2}\right) T\left(\frac{u_n \oplus v_n}{2}\right)}, \overrightarrow{v_n q} \right\rangle\right] \\
&\leq (1-a_n)^2 d^2\left(\frac{u_n \oplus v_n}{2}, q\right) \\
&\quad + 2\left[a_n^2 \langle \overrightarrow{f(u_n) q}, \overrightarrow{v_n q} \rangle + a_n(1-a_n)\left\langle \overrightarrow{f(u_n) T\left(\frac{u_n \oplus v_n}{2}\right)}, \overrightarrow{v_n q} \right\rangle\right. \\
&\quad \left. + a_n(1-a_n)\left\langle \overrightarrow{T\left(\frac{u_n \oplus v_n}{2}\right) q}, \overrightarrow{v_n q} \right\rangle\right] \\
&\leq (1-a_n)^2 d^2\left(\frac{u_n \oplus v_n}{2}, q\right) + 2a_n \langle \overrightarrow{f(u_n) q}, \overrightarrow{v_n q} \rangle \\
&\leq (1-a_n)^2 \left[\frac{1}{2} d^2(u_n, q) + \frac{1}{2} d^2(v_n, q) - \frac{1}{4} d^2(u_n, v_n)\right] \\
&\quad + 2a_n \langle \overrightarrow{f(u_n) f(q)}, \overrightarrow{v_n q} \rangle + 2a_n \langle \overrightarrow{f(q) q}, \overrightarrow{v_n q} \rangle \\
&\leq \frac{(1-a_n)^2}{2}\left[d^2(u_n, q) + d^2(v_n, q)\right] \\
&\quad + 2a_n d\big(f(u_n), f(q)\big) d(v_n, q) + 2a_n \langle \overrightarrow{f(q) q}, \overrightarrow{v_n q} \rangle \\
&\leq \frac{1-2(1-k)a_n}{2}\left[d^2(u_n, q) + d^2(v_n, q)\right] + M a_n^2 + 2a_n \langle \overrightarrow{f(q) q}, \overrightarrow{v_n q} \rangle,
\end{aligned}$$

where $M > 0$ is a constant such that

$$\max\left\{\sup_{n \in Z^+}\left\{d^2(u_n, q)\right\}, \sup_{n \in Z^+}\left\{d^2(v_n, q)\right\}\right\} \leq M.$$

It follows that

$$d^2(v_n,q) \le \frac{1-2(1-k)a_n}{1+2(1-k)a_n} d^2(u_n,q) + \frac{2a_n^2}{1+2(1-k)a_n} M + \frac{4a_n}{1+2(1-k)a_n} \langle \overrightarrow{f(q)q}, \overrightarrow{v_n q} \rangle. \tag{3.5}$$

By (TVIM-I) and Lemma 2.1, we have

$$\begin{aligned} &d^2(u_{n+1},q) \\ &\quad \le b_n d^2(u_n,q) + (1-b_n)d^2(v_n,q) - b_n(1-b_n)d^2(u_n,v_n) \\ &\quad \le b_n d^2(u_n,q) + (1-b_n)d^2(v_n,q). \end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.6), we get

$$d^2(u_{n+1},q) \le (1-\gamma_n)d^2(u_n,q) + \delta_n, \quad \forall n \ge 1, \tag{3.7}$$

where $\gamma_n = \frac{4(1-k)a_n(1-b_n)}{1+2(1-k)a_n}$ and

$$\delta_n = \frac{2a_n^2(1-b_n)}{1+2(1-k)a_n} M + \frac{4a_n(1-b_n)}{1+2(1-k)a_n} \langle \overrightarrow{f(q)q}, \overrightarrow{v_n q} \rangle.$$

By (1.3) and (3.4), now we know that $\gamma_n \in (0,1)$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and

$$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty} \left[\frac{a_n}{2(1-k)} M + \frac{1}{1-k} \langle \overrightarrow{f(q)q}, \overrightarrow{v_n q} \rangle \right] \le 0.$$

The conclusion follows by Lemma 3.1 and (3.7). This completes the proof. □

In the next moment, we investigate the strong convergence of the iterative approximation method (TVIM-II).

Theorem 3.2 *Suppose that T, f, E and X are the same as in Theorem 3.1. If (1.3) and the following condition (C^*) holds: $\lim_{n\to\infty} |b_n - b_{n+1}| = 0$, then, for any chosen $u_1 \in E$, the sequence $\{u_n\}$ defined by (TVIM-II) converges strongly to a unique solution $q = P_{F(T)} f(q) \in F(T)$ of the variational inequality (1.7).*

Proof Above all, we show that $\{u_n\}$ is bounded. Indeed, for any given $p \in F(T)$, by Lemma 2.1, one has

$$\begin{aligned} &d(u_{n+1},p) \\ &\quad \le b_n d(u_n,p) + (1-b_n) d\left(a_n f(u_n) \oplus (1-a_n) T\left(\frac{u_n \oplus u_{n+1}}{2} \right), p \right) \\ &\quad \le b_n d(u_n,p) + (1-b_n)a_n \bigl[d\bigl(f(u_n), f(p)\bigr) + d\bigl(f(p),p\bigr) \bigr] \\ &\qquad + (1-b_n)(1-a_n) d\left(T\left(\frac{u_n \oplus u_{n+1}}{2} \right), T(p) \right) \end{aligned}$$

$$
\begin{aligned}
&\leq b_n d(u_n,p) + (1-b_n)a_n\big[kd(u_n,p) + d\big(f(p),p\big)\big] \\
&\quad + (1-b_n)(1-a_n)d\left(\frac{u_n \oplus u_{n+1}}{2},p\right) \\
&\leq b_n d(u_n,p) + k(1-b_n)a_n d(u_n,p) + (1-b_n)a_n d\big(f(p),p\big) \\
&\quad + \frac{(1-b_n)(1-a_n)}{2}\big[d(u_n,p) + d(u_{n+1},p)\big] \\
&\leq \frac{1+b_n-(1-2k)a_n(1-b_n)}{2}d(u_n,p) \\
&\quad + \frac{(1-a_n)(1-b_n)}{2}d(u_{n+1},p) + a_n(1-b_n)d\big(f(p),p\big)
\end{aligned}
$$

and so

$$
\begin{aligned}
&d(u_{n+1},p) \\
&\leq \left(1-\frac{2a_n(1-b_n)(1-k)}{1+b_n+a_n(1-b_n)}\right)d(u_n,p) + \frac{2a_n(1-b_n)(1-k)}{1+b_n+a_n(1-b_n)}\cdot\frac{d(f(p),p)}{1-k} \\
&\leq \max\left\{d(u_n,p),\frac{d(f(p),p)}{1-k}\right\}.
\end{aligned}
$$

By induction, we obtain $d(u_n,p) \leq \max\{d(u_1,p),\frac{d(f(p),p)}{1-k}\}$. Hence, $\{u_n\}$ is bounded.

After calculation, we get

$$
\begin{aligned}
&d(v_n,v_{n+1}) - d(u_n,u_{n+1}) \\
&\leq \frac{a_n(2k-1)}{1+b_n}d(u_n,u_{n+1}) + \frac{|b_n-b_{n+1}|}{1+b_n}d(u_{n+1},v_{n+1}) \\
&\quad + \frac{2|a_n-a_{n+1}|}{1+b_n}d\left(f(u_{n+1}),T\left(\frac{u_{n+1}\oplus u_{n+2}}{2}\right)\right).
\end{aligned}
$$

Similarly, by the condition (C^*), we have

$$
\limsup_{n\to\infty}\big[d(v_{n+1},v_n) - d(u_{n+1},u_n)\big] \leq 0
$$

and

$$
\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty}\left[\frac{\alpha_n}{2(1-k)}M' + \frac{1}{1-k}\langle\overrightarrow{f(q)q},\overrightarrow{v_nq}\rangle\right] \leq 0,
$$

where

$$
\delta_n = \frac{2\alpha_n^2(1-\beta_n)M'}{1+2(1-k)\alpha_n+\beta_n-2\alpha_n\beta_n} + \frac{4\alpha_n(1-\beta_n)\langle\overrightarrow{f(q)q},\overrightarrow{y_nq}\rangle}{1+2(1-k)\alpha_n+\beta_n-2\alpha_n\beta_n},
$$

$M' > 0$ is a constant with $\sup_{n\in Z^+}\{d^2(x_n,q)\} \leq M'$, and

$$
\gamma_n = \frac{4(1-k)\alpha_n(1-\beta_n)}{1+2(1-k)\alpha_n+\beta_n-2\alpha_n\beta_n}.
$$

Thus, in a similar way to *Steps* (II)–(V) of the proof in Theorem 3.1, the rest of the proof can be completed and it is omitted. □

Remark 3.1 In order to show $\lim_{n\to\infty} d(u_n, v_n) = 0$ via Lemma 2.3, one can easily see that the condition (C^*) in Theorem 3.2 is very important and prerequisite for the new semi-implicit viscosity iterative approximation (TVIM-II) (1.2). It is worth noting that the condition (C^*) is not needed for the explicit viscosity iteration (TSVIM) (1.6), and one can refer to the last section in this paper for more discussion on the existence value of the condition (C^*).

4 Numerical simulation and applications

To verify the effectiveness of our main results, we shall propose a numerical example and some applications to more general variational inequality systems and equilibrium problems in this section.

4.1 Numerical example

In the sequel, a numerical example is given to show the effectiveness of Theorems 3.1 and 3.2.

Let two iteration processes $\{x_n\}$ and $\{y_n\}$ both converge to a certain fixed point p of an operator T. If

$$\lim_{n\to\infty} \frac{|x_n - p|}{|y_n - p|} = 0, \tag{4.1}$$

then it can be recalled that the convergence rate of $\{x_n\}$ is faster than that of $\{y_n\}$ (see [26]).

Example 4.1 Let $f, T : \mathbb{R} \to \mathbb{R}$ be defined by $f(u) = \frac{u}{6}$ and $T(u) = \frac{u}{2}$ for any $u \in \mathbb{R}$, respectively. It is easy to see that $F(T) = \{0\}$. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{3}$ for $n \in \mathbb{Z}^+$. Let $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$ and $\{u_n^{(3)}\}$ be three sequences generated by (TSVIM), (TVIM-I) and (TVIM-II), respectively. One can clearly see that $\{u_n^{(k)}\}$ converges to 0 for $k = 1, 2, 3$, and can easily rewrite (TSVIM), (TVIM-I), and (TVIM-II), respectively, as follows:

$$u_{n+1}^{(1)} = \frac{6n-2}{9n} u_n^{(1)}, \tag{TSVIM}$$

$$u_{n+1}^{(2)} = \frac{13n+1}{9(3n+1)} u_n^{(2)}, \tag{TVIM-I}$$

$$u_{n+1}^{(3)} = \frac{9n-1}{3(5n+1)} u_n^{(3)}. \tag{TVIM-II}$$

Taking $u_1^{(k)} = 1$ for $k = 1, 2, 3$, then one can easily check that

$$\lim_{n\to\infty} \frac{|u_n^{(2)} - 0|}{|u_n^{(3)} - 0|} = \lim_{n\to\infty} \frac{3[13(n-1)+1][5(n-1)+1]}{9[3(n-1)+1][9(n-1)-1]} \cdot \frac{u_{n-1}^{(2)}}{u_{n-1}^{(3)}},$$

that is,

$$\lim_{n\to\infty} \frac{|u_n^{(2)} - 0|}{|u_n^{(3)} - 0|} \Bigg/ \frac{|u_{n-1}^{(2)} - 0|}{|u_{n-1}^{(3)} - 0|} = \lim_{n\to\infty} \frac{65 + \frac{112}{n} + \frac{48}{n^2}}{81 + \frac{144}{n} - \frac{60}{n^2}} = \frac{65}{81} \in (0, 1),$$

Table 1 Numerical results for (TSVIM), (TVIM-I) and (TVIM-II)

No.	(TSVIM)	(TVIM-I)	(TVIM-II)	No.	(TSVIM)	(TVIM-I)	(TVIM-II)
1	1.00000	1.00000	1.00000	12	0.00380	0.00016	0.00144
2	0.44444	0.38889	0.44444	13	0.00246	0.00008	0.00084
3	0.24691	0.16667	0.22900	14	0.00160	0.00004	0.00049
4	0.14632	0.07407	0.12402	15	0.00104	0.00002	0.00029
5	0.08942	0.03355	0.06890	16	0.00068	0.00001	0.00017
6	0.05564	0.01538	0.03887	17	0.00044	0.00000	0.00010
7	0.03503	0.00711	0.02215	18	0.00029	0.00000	0.00006
8	0.02224	0.00330	0.01272	19	0.00019	0.00000	0.00003
9	0.01421	0.00154	0.00734	20	0.00012	0.00000	0.00002
10	0.00912	0.00072	0.00425	21	0.00008	0.00000	0.00001
11	0.00588	0.00034	0.00248	22	0.00005	0.00000	0.00000

and the series of positive term $\sum_{n=2}^{\infty} \frac{|u_n^{(2)}-0|}{|u_n^{(3)}-0|}$, is convergent. Thus, one has

$$\lim_{n\to\infty} \frac{|u_n^{(2)}-0|}{|u_n^{(3)}-0|} = \lim_{n\to\infty} \frac{7}{8} \cdot \frac{99}{119} \cdot \frac{32}{39} \cdots \frac{(13n+1)(5n+1)}{3(3n+1)(9n-1)} = 0.$$

Similarly,

$$\lim_{n\to\infty} \frac{|u_n^{(3)}-0|}{|u_n^{(1)}-0|} \Big/ \frac{|u_{n-1}^{(3)}-0|}{|u_{n-1}^{(1)}-0|} = \frac{9}{10} \in (0,1) \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{|u_n^{(3)}-0|}{|u_n^{(1)}-0|} = 0.$$

Thus, it follows from the notion of convergence rate with respect to (4.1) that (TVIM-I) converges faster than (TVIM-II), and the iterative number (in short, No.) of arriving at the convergence point for (TVIM-II) is smaller than that of (TSVIM), which are also listed in Table 1.

Remark 4.1 The numerical results in Table 1 show that Theorems 3.1 and 3.2, respectively, corresponding to the iterative forms (TVIM-I) and (TVIM-II) extend and improve corresponding work of Kaewkhao et al. [6] and Chang et al. [14] and many others in the literature, which are associated with the iteration (TSVIM) (1.6).

4.2 More general variational inequality systems

Let C be a nonempty closed convex subset of the real Hilbert space $\mathbb{H}$ and $\{A_i\}_{i=1}^N : C \to \mathbb{H}$ be a family of operators. In [27], Cai and Bu considered the problem of finding $(u_1^*, u_2^*, \ldots, u_N^*) \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N u_N^* + u_1^* - x_N^*, x - x_1^* \rangle \geq 0, & \forall u \in C, \\ \langle \lambda_{N-1} A_{N-1} u_{N-1}^* + u_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, & \forall u \in C, \\ \ldots, \\ \langle \lambda_2 A_2 u_2^* + u_3^* - x_2^*, x - x_3^* \rangle \geq 0, & \forall u \in C, \\ \langle \lambda_1 A_1 u_1^* + u_2^* - x_1^*, x - x_2^* \rangle \geq 0, & \forall u \in C. \end{cases} \tag{4.2}$$

Equation (4.2) is a more general variational inequality system in Hilbert spaces, where $\lambda_i > 0$ for any $i \in \{1, 2, \ldots, N\}$.

Lemma 4.1 ([27]) *Let C be a nonempty closed convex subset of the real Hilbert space $\mathbb{H}$, For $i = 1, 2, \dots, N$, let $A_i : C \to \mathbb{H}$ be δ_i-inverse strongly monotone for some positive real number δ_i, i.e.,*

$$\langle A_i x - A_i y, x - y\rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C.$$

If $0 < \lambda_i < 2\delta_i$ for any $i \in \{1, 2, \dots, N\}$, then, for all $x \in C$, the operator $G : C \to C$ defined as

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots P_C(I - \lambda_1 A_1)x \tag{4.3}$$

is nonexpansive.

Lemma 4.2 ([28]) *Let C be a nonempty closed convex subset of the real Hilbert space $\mathbb{H}$, Let $A_i : C \to \mathbb{H}$ be a nonlinear operator, where $i = 1, 2, \dots, N$. For given $u_k^* \in C$, $i = 1, 2, \dots, N$, $(u_1^*, u_2^*, \dots, u_N^*)$ is a solution of the problem (4.2) if and only if*

$$\begin{aligned} &u_1^* = P_C(I - \lambda_N A_N)u_N^*, \\ &u_k^* = P_C(I - \lambda_{k-1}A_{k-1})u_{k-1}^*, \quad k = 2, 3, \dots, N, \end{aligned} \tag{4.4}$$

that is,

$$u_1^* = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)u_1^*.$$

By Lemmas 4.1 and 4.2, one knows that $u_1^* = G(u_1^*)$, that is, u_1^* is a fixed point of the operator G defined by (4.3). Further, if we find the fixed point u_1^* of G, it is easy to get the other points by (4.4), and one can solve (4.2). From Lemmas 4.1 and 4.2, and Theorems 3.1 and 3.2, now we have the following two results.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. For $i = 1, 2, \dots, N$, let $A_i : C \to \mathbb{H}$ be a δ_i-inverse strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by (4.3), and $f : C \to C$ be a contraction with coefficient $k \in [0, 1)$. Then the sequence $\{u_n\}$, defined as follows:*

$$\begin{aligned} &v_n = a_n f(u_n) + (1 - a_n)G\left(\frac{u_n + v_n}{2}\right), \\ &u_{n+1} = b_n u_n + (1 - b_n)v_n, \quad \forall n \geq 1, \end{aligned}$$

converges strongly to a fixed point u_1^* of the nonexpansive operator G, where $u_1 \in C$ is any given element, $0 < \lambda_i < 2\delta_i$ for any $i = 1, 2, \dots, N$, and $\{a_n\}$, $\{b_n\} \subseteq (0, 1)$ satisfy (1.3). That is, it follows from (4.4) that $(u_1^*, u_2^*, \dots, u_N^*)$ is a solution of the variational inequality system (4.2), and u_1^* satisfies $u_1^* = P_{F(G)}f(u_1^*)$, which is also a unique solution of the variational inequality

$$\langle u_1 - f(u_1), u - u_1\rangle \geq 0, \quad \forall u \in F(G). \tag{4.5}$$

Theorem 4.2 *Assume that A_i $(i = 1,2,\dots,N)$, G, f, C and $\mathbb{H}$ are the same as in Theorem 4.1. Choosing any $u_1 \in C$, define a sequence $\{u_n\}$ by*

$$\begin{aligned} v_n &= a_n f(u_n) + (1-a_n)G\left(\frac{u_n + u_{n+1}}{2}\right), \\ u_{n+1} &= b_n u_n + (1-b_n)v_n, \quad \forall n \geq 1, \end{aligned}$$

where $0 < \lambda_i < 2\delta_i$ for any $i \in \{1,2,\dots,N\}$, and $\{a_n\}$, $\{b_n\} \subseteq (0,1)$ satisfy (1.3) and the condition (C^*) in Theorem 3.2. Then the sequence $\{u_n\}$ converges strongly to a fixed point u_1^* of the nonexpansive operator G. Further, a solution $(u_1^*, u_2^*, \dots, u_N^*)$ of more general system of variational inequalities problem (4.2) is obtained by (4.4), and $u_1^* = P_{F(G)}f(u_1^*)$ is also a unique solution of (4.5).

4.3 Equilibrium problems

Let X be a real topological vector space with the topological dual space X^*, $\langle\cdot,\cdot\rangle$ be a pair of X and X^*, and $C \subset X$ be a closed convex subset. The equilibrium problem for the function $\phi : C \times C \to \mathbb{R}$ is to find a point $u^* \in C$ such that

$$\phi\big(u^*, v\big) \geq 0, \quad \forall y \in C, \tag{4.6}$$

where $\mathbb{R}$ is the set of real numbers. The set of solutions of (4.6) is denoted by $\mathrm{EP}(\phi)$.

If $\phi(u,v) = \langle T(u), v-u\rangle$ for a given operator $T : C \to X^*$, then the problem (4.6) is equivalent to finding $u^* \in X$ such that

$$u^* \in C, \quad \big\langle T\big(u^*\big), v - u^*\big\rangle \geq 0, \quad \forall v \in C,$$

which is called a variational inequality of the topological vector space X.

To find solutions of the equilibrium problem (4.6), we assume that the bifunction ϕ satisfies the following conditions (see [29]):

(H_1) $\phi(x,x) = 0$, for any $x \in C$;

(H_2) ϕ is monotone, i.e., $\phi(x,y) + \phi(y,x) \leq 0$, for any $x, y \in C$;

(H_3) ϕ is upper-hemicontinuous, i.e., for any $x, y, z \in C$,

$$\limsup_{t \to 0^+} \phi\big(tz + (1-t)x, y\big) \leq \phi(x,y);$$

(H_4) $\phi(x,\cdot)$ is convex and lower semicontinuous for any $x \in C$.

Based on [29, Corollary 1] and [30, Lemma 2.12], and Theorems 3.1 and 3.2, the following results can be established.

Theorem 4.3 *Let $\mathbb{H}$ be a real Hilbert space, $C \subset \mathbb{H}$ be a nonempty. If (1.3) meets, $\phi : C \times C \to \mathbb{R}$ is a bifunction satisfying (H_1)–(H_4), $f : C \to C$ is a contraction with coefficient $k \in [0,1)$ and for any $x \in \mathbb{H}$, $T_r : \mathbb{H} \to C$ is defined as*

$$T_r(x) = \left\{ z \in C : \phi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

with $F(T_r) \neq \emptyset$, then, for any given $u_1 \in C$, the sequence $\{u_n\}$ generated by

$$v_n = a_n f(u_n) + (1 - a_n) T_r \left(\frac{u_n + v_n}{2} \right),$$
$$u_{n+1} = b_n u_n + (1 - b_n) v_n,$$

converges strongly to a fixed point u^* of the nonexpansive operator T_r such that $u^* = P_{F(T_r)} f(u^*)$, which is a solution of the equilibrium problem (4.6), and is also a unique solution of the variational inequality

$$\bigl\langle u^* - f\bigl(u^*\bigr), u - u^* \bigr\rangle \geq 0, \quad \forall u \in F(T_r). \tag{4.7}$$

Theorem 4.4 *Suppose that ϕ, T_r, f, C and $\mathbb{H}$ are the same as in Theorem 4.3. If (1.3) and (C^*) in Theorem 3.2 hold, then, for any chosen $u_1 \in C$, the sequence $\{u_n\}$ defined by*

$$u_{n+1} = b_n u_n + (1 - b_n) \left[a_n f(u_n) + (1 - a_n) T_r \left(\frac{u_n + u_{n+1}}{2} \right) \right]$$

converges strongly to a fixed point u^* of T_r, where $u^* = P_{F(T_r)} f(u^*)$ is a solution of the equilibrium problem (4.6), and is also a unique solution of the variational inequality (4.7).

5 Concluding remarks

Motivated and inspired by the recent work of Kaewkhao et al. [6], Xu et al. [10], and Chang et al. [14] on (viscosity) iterative approximation methods for the implicit midpoint rule of nonexpansive operators, in this paper, we introduced and studied the following brand-new semi-implicit viscosity iteration approximation methods involving nonexpansive operator T and contraction operator f in complete CAT(0) spaces X:

$$\begin{cases} v_n = a_n f(u_n) \oplus (1 - a_n) T(\frac{u_n \oplus v_n}{2}), \\ u_{n+1} = b_n u_n \oplus (1 - b_n) v_n, \quad \forall n \geq 1, \end{cases}$$

and

$$\begin{cases} v_n = a_n f(u_n) \oplus (1 - a_n) T(\frac{u_n \oplus u_{n+1}}{2}), \\ u_{n+1} = b_n u_n \oplus (1 - b_n) v_n, \quad \forall n \geq 1, \end{cases} \tag{5.1}$$

where $u_1 \in E \subseteq X$ is an arbitrary fixed element and $\{a_n\}, \{b_n\} \subseteq (0, 1)$. Under some certain assumptions to the sequences, we proved strong convergence theorems of the two kinds of two-step viscosity approximation with the implicit midpoint rule, which show that the limit solves an additional variational inequality, variational inequality systems, equilibrium problems, differential equations and other related fixed point problems.

Further, on behalf of verifying effectiveness for our main convergence results presented in this paper, we gave a numerical example and some applications to related variational inequality problems, such as more general variational inequality systems and equilibrium problems. Our results presented in this paper extend and improve corresponding work due to Kaewkhao et al. [6] and Chang et al. [14] and many other researchers.

However, choosing the sequence $\{b_n\}$ as follows:

$$b_n = \frac{2 + (-1)^n}{4},$$

then one easily see that $\{b_n\} \subseteq (0, 1)$, and the inequality

$$0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$$

holds, but the condition (C^*) in Theorem 3.2 is not met. Indeed, (C^*) in Theorem 3.2 is an extra condition compared to Theorem 3.1. If the condition (C^*) is not added, does Theorem 3.2 hold? This remains an open question for future work of research: Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d). Assume that $T : E \to E$ is a nonexpansive operator with $F(T) \neq \emptyset$, $f : E \to E$ is a contraction with coefficient $k \in [0, 1)$, the sequences $\{a_n\}, \{b_n\} \subseteq (0, 1)$ satisfy (1.3), and for an arbitrary initial point $u_1 \in E$, $\{u_n\}$ is a sequence generated by (5.1). Will the conclusion of Theorem 3.2 be correct? This is usually presented as an important and big problem in future research.

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Abbreviations
TVIM-I, The first type of semi-implicit viscosity approximation method; TVIM-II, The second type of semi-implicit viscosity approximation method; TSVIM, Two-step explicit viscosity iteration method.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
T-JX carried out the proof of the corollaries and gave some examples to show the main results. H-YL conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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