

RESEARCH

Open Access



On stable entire solutions of sub-elliptic system involving advection terms with negative exponents and weights

Belgacem Rahal^{1*}

*Correspondence:
rahhalbelgacem@gmail.com

¹Institut Supérieur des Sciences
Appliquées et de Technologie de
Kairouan, Kairouan, Tunisia

Abstract

We examine the weighted Grushin system involving advection terms given by

$$\begin{cases} \Delta_G u - a \cdot \nabla_G u = (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-p} & \text{in } \mathbb{R}^n, \\ \Delta_G v - a \cdot \nabla_G v = (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} & \text{in } \mathbb{R}^n, \end{cases}$$

where $\Delta_G u = \Delta_x u + |x|^{2\alpha} \Delta_y u$, $z = (x, y) \in \mathbb{R}^n := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is the Grushin operator, $\alpha \geq 0$, $p \geq q > 1$, $\|z\|^{2(\alpha+1)} = |x|^{2(\alpha+1)} + |y|^2$, $\gamma \geq 0$ and a is a smooth divergence-free vector that we will specify later. Inspired by recent progress in the study of the Lane–Emden system, we establish some Liouville-type results for bounded stable positive solutions of the system. In particular, we prove the comparison principle to establish our result. As consequences, we obtain a Liouville-type theorem for the weighted Grushin equation involving advection terms

$$\Delta_G u - a \cdot \nabla_G u = (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-p} \quad \text{in } \mathbb{R}^n.$$

The main tools in the proof of the main result are the comparison principle, nonlinear integral estimates via the stability assumption and the bootstrap argument. Our results generalize and improve the previous work in (Duong et al. in *Complex Var. Elliptic Equ.* 64(12):2117–2129, 2019).

MSC: Primary 35J55; 35J65; secondary 35B65

Keywords: Liouville-type theorem; Grushin operator; Stable solutions; Elliptic system; Weighted Grushin system; Weighted Grushin equation

1 Introduction

In this paper, we study the Liouville-type theorem for bounded stable positive classical solutions of the weighted nonlinear degenerate elliptic system involving advection terms

$$\begin{cases} \Delta_G u - a \cdot \nabla_G u = (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-p} & \text{in } \mathbb{R}^n, \\ \Delta_G v - a \cdot \nabla_G v = (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} & \text{in } \mathbb{R}^n, \end{cases} \quad (1.1)$$

© The Author(s) 2020. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

and of the scalar equation

$$\Delta_G u - a \cdot \nabla_G u = \left(1 + \|z\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} u^{-p} \quad \text{in } \mathbb{R}^n, \quad (1.2)$$

where $\Delta_G u = \Delta_x u + |x|^{2\alpha} \Delta_y u$, $z = (x, y) \in \mathbb{R}^n := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is the Grushin operator, Δ_x and Δ_y are Laplace operators with respect to $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, and $\|z\|^{2(\alpha+1)} := |x|^{2(\alpha+1)} + |y|^2$. Here we always assume that $\alpha \geq 0$, $p \geq q > 1$, $\gamma \geq 0$ and a is a smooth divergence-free vector field:

$$\operatorname{div}_G a = 0 \quad \text{and} \quad |a(z)| \leq \frac{\epsilon}{1 + \|z\|} \quad \text{for all } z = (x, y) \in \mathbb{R}^n, \quad \epsilon \text{ small enough}, \quad (1.3)$$

where $\operatorname{div}_G = \operatorname{div}_x + |x|^\alpha \operatorname{div}_y$.

For simplicity of the presentation, we define the following parameters which play an important role in the sequel:

$Q := n_1 + (1 + \alpha)n_2$ is the homogeneous dimension of \mathbb{R}^n ;

$$\tau_0^+ := \sqrt{\frac{pq(p-1)}{q-1}} + \sqrt{\frac{pq(p-1)}{q-1}} + \sqrt{\frac{pq(p-1)}{q-1}},$$

$$\tau_0^- := \sqrt{\frac{pq(p-1)}{q-1}} - \sqrt{\frac{pq(p-1)}{q-1}} + \sqrt{\frac{pq(p-1)}{q-1}},$$

$$\sigma^+ := \sqrt{\frac{pq(p+1)}{q+1}} + \sqrt{\frac{pq(p+1)}{q+1}} - \sqrt{\frac{pq(p+1)}{q+1}},$$

$$\sigma^- := \sqrt{\frac{pq(p+1)}{q+1}} - \sqrt{\frac{pq(p+1)}{q+1}} - \sqrt{\frac{pq(p+1)}{q+1}}.$$

We start by noting that, in the case $a \equiv 0$ and $\gamma = 0$, the system (1.1) and Eq. (1.2) reduce to

$$\begin{cases} \Delta_G u = v^{-p} & \text{in } \mathbb{R}^n, \\ \Delta_G v = u^{-q} & \text{in } \mathbb{R}^n, \end{cases} \quad (1.4)$$

and

$$\Delta_G u = u^{-p} \quad \text{in } \mathbb{R}^n. \quad (1.5)$$

In the case $\alpha = 0$, Eq. (1.5) arises in many branches of applied sciences and has been studied in a number of recent works; see [6, 20] and the references therein. The nonexistence of positive stable classical solutions of (1.5) was examined in [20]. This result was then generalized in [6] to positive stable weak solutions of a weighted equation. More precisely, the authors of [6] figured out the critical exponent and established an optimal Liouville-type theorem for this class of solutions. When $\alpha > 0$, the Liouville-type theorem for a special class of solutions of (1.4) and (1.5). “the so-called stable solutions” has been studied by Duong, Lan, Le and Nguyen [10]. We summarize here some results in [10].

Theorem 1.1 ([10]) *Assume that $p \geq q > 1$ and*

$$Q < 2 + \frac{8}{p+q+2} \tau_0^+,$$

then (1.4) has no bounded stable positive solution.

Corollary 1.1 ([10]) *Let $p > 1$ and*

$$Q < 2 + \frac{4}{p+1} (p + \sqrt{p(p+1)}), \quad (1.6)$$

then Eq. (1.5) has no bounded stable positive solution.

We remark that for Eq. (1.5) (with $\alpha = 0$), the critical exponent on the right-hand side of (1.6) was first found in [20]. This exponent has been shown to be optimal in the class of positive stable weak solutions; see [6].

Similar to the celebrated Lane–Emden system in the case of positive exponents, the system (1.4) is also a natural extension of Eq. (1.5). It is worth to remark that there are many papers developing various useful tools to study the nonexistence of positive stable solutions (see for example [2, 12, 13, 17, 22, 26] and the references therein. For other results on Grushin operators, Wei et al. [25] established a Liouville-type theorem for weak stable solutions of weighted p -Laplace-type Grushin equation in the case of negative exponent nonlinearity. Some important and interesting results can be found in [21].

Recently, elliptic problems involving advection terms, i.e. $a \neq 0$, have received considerable attention [3, 4, 8, 11]. In particular, Duong and Nguyen [11] studied the equation

$$-\Delta_G u + \nabla_G w \cdot \nabla_G u = \|z\|^s |u|^{p-1} u \quad \text{in } \mathbb{R}^n, s \geq 0. \quad (1.7)$$

Taking advantage of the variational structure, and using the approach of Farina [13], he established some Liouville-type theorems for the class of stable sign-changing weak solutions. Now, we state this result as follows.

Theorem 1.2 ([11]) *Suppose that there is a nonnegative constant θ such that*

$$|\nabla_G w| \leq \frac{C}{\|z\|^\theta + 1}.$$

Assume in addition that

$$\lim_{R \rightarrow +\infty} R^{-\frac{(1+\min(\theta;1))(p+\beta)+s(\beta+1)}{p-1}} \int_{R < \|z\| < 2R} e^{-w} = 0,$$

for $\beta \in (1, 2p + 2\sqrt{p(p-1)} - 1)$. Then any stable weak solution u to (1.7) must be the trivial one.

In the general case where $a \neq 0$, elliptic problems with advectons have no variational structure and this requires another approach to obtaining a classification of stable solutions. Recall that, in this case, see e.g. [3], a positive classical solution u of

$$-\Delta u + a \cdot \nabla u = u^p \quad \text{in } \mathbb{R}^n \quad (1.8)$$

is called stable if there is a smooth positive function F such that

$$-\Delta F + a \cdot \nabla F \geq pu^{p-1}F.$$

Recently, relying on Farina's approach [13] and the generalized Hardy inequality, Cowan [3] established a Liouville-type theorem for stable positive solution of (1.8) under the smallness condition imposed on the divergence-free a .

On the other direction, the Liouville-type theorem for the class of stable solutions for system

$$\begin{cases} -\Delta u + a \cdot \nabla u = v^p & \text{in } \mathbb{R}^n, \\ -\Delta v + a \cdot \nabla v = u^q & \text{in } \mathbb{R}^n, \end{cases} \quad (1.9)$$

was examined by Duong [8]. He established a Liouville-type result for stable positive solutions of the system in the case $p \geq q \geq 1$ and $pq > 1$. In particular, when $p = q$, his result is a natural extension of Cowan [3] to the equation with advection. Furthermore, we would also like to mention that when $pq \leq 1$, the system (1.9) has no positive supersolutions (see Theorem 1.3 [9]).

For the general equation or system with $\gamma \neq 0$, the Liouville property is less understood and is more delicate to deal with than $\gamma = 0$. There exist many excellent papers using Farina's approach to the Hardy–Hénon equation and the weighted nonlinear elliptic equations. We refer to [7, 22, 24] and the references therein. Inspired by the ideas in [2, 16], Hu [18] adopt the new approach of a combination of second order stability, Souplet's inequality [23] and a bootstrap iteration to establish Liouville-type theorems for the semi-stable solutions of

$$\begin{cases} -\Delta u = (1 + |x|^2)^{\frac{\gamma}{2}} v^p & \text{in } \mathbb{R}^n, \\ -\Delta v = (1 + |x|^2)^{\frac{\gamma}{2}} u^q & \text{in } \mathbb{R}^n, \end{cases} \quad (1.10)$$

and of the scalar equation

$$-\Delta u = (1 + |x|^2)^{\frac{\gamma}{2}} u^p \quad \text{in } \mathbb{R}^n. \quad (1.11)$$

In particular, Hu [18] has obtained the following result.

Theorem 1.3 ([18])

1. Suppose $\gamma > 0$, $2\sigma^- < p \leq q$ and

$$n < 2 + \frac{(4 + 2\gamma)(q + 1)}{pq - 1} \sigma^+.$$

Then there is no classical positive semi-stable solution of (1.10). In particular, there is no classical positive semi-stable solution of (1.10) for any $2 \leq p \leq q$ if $n \leq 10 + 4\gamma$.

2. Let $p > \frac{4}{3}$, $\gamma > 0$ and

$$n < 2 + \frac{2(2 + \gamma)}{p - 1} (p + \sqrt{p(p - 1)}).$$

Then there does not exist a classical positive semi-stable solution of (1.11).

In this paper, we propose to study the system (1.1) which can be regarded as a natural generalization of the scalar equation (1.2). Motivated by [8, 10, 18], we give the classification of bounded stable positive solutions of (1.1) under the assumption (1.3). Before stating our main results, let us recall the definition of such solutions motivated by [8, 10].

Definition 1.1 A positive solution $(u, v) \in C^2(\mathbb{R}^n) \times C^2(\mathbb{R}^n)$ of (1.1) is called stable if there are positive smooth functions ξ, η such that

$$\begin{cases} -\Delta_G \xi + a \cdot \nabla_G \xi = p(1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-p-1} \eta & \text{in } \mathbb{R}^n, \\ -\Delta_G \eta + a \cdot \nabla_G \eta = q(1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q-1} \xi & \text{in } \mathbb{R}^n. \end{cases} \quad (1.12)$$

The main result in this paper is the following.

Theorem 1.4 Assume that $p \geq q > 1$ and

$$Q < 2 + \frac{4(\gamma + 2)}{p + q + 2} \tau_0^+, \quad (1.13)$$

then (1.1) has no bounded stable positive solution.

The key in our proof is the comparison principle and nonlinear integral estimates. However, the techniques used to prove the comparison principle in [14, 18] for the Laplace operator do not seem applicable to the system (1.1) because the operator Δ_G no longer has symmetry and it degenerates on the manifold $\{0\} \times \mathbb{R}^{n_2}$. Then, in this paper, we establish the comparison principle for Grushin operators by developing the idea in [1, 10, 12, 15]. In addition, the L^1 -estimate to the bootstrap iteration in [2] does not work in the case of Grushin operator, we instead switch to the L^2 -estimate in the bootstrap argument. We also employ the idea in [1, 10, 12, 15] to prove the “inverse” comparison principle which is crucial to proving our result.

Recall that a classical solution of (1.2) is called stable if

$$p \int_{\mathbb{R}^n} (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-p-1} \phi^2 dx dy \leq \frac{1}{4} \int_{\mathbb{R}^n} |a\phi + 2\nabla_G \phi|^2 dx dy, \quad (1.14)$$

for all $\phi \in C_c^1(\mathbb{R}^n)$.

When $p = q$, by using the comparison principle below, we obtain a direct consequence of Theorem 1.4 for the scalar equation (1.2).

Corollary 1.2 Let $p > 1$ and

$$Q < 2 + \frac{2(\gamma + 2)}{p + 1} (p + \sqrt{p(p+1)}). \quad (1.15)$$

Then Eq. (1.2) has no bounded stable positive solution.

We remark also that the method used in the present paper can be applied to study the weighted systems, and to more general class of degenerate operator, such as the Δ_λ oper-

ator (see [19, 22]) of the form

$$\Delta_{\lambda} := \sum_{i=1}^n \partial_{x_i} (\lambda_i^2 \partial_{x_i}), \quad \lambda = (\lambda_1, \dots, \lambda_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

Here $\lambda_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, n$ are nonnegative continuous functions satisfying some properties such that Δ_{λ} is homogeneous of degree two with respect to a group dilation in \mathbb{R}^n .

The organization of this paper is as follows. In Sect. 2, we establish the stability inequality and the comparison principle for the system (1.1) and then prove an a priori estimate of the solutions. In Sect. 3, we give the proof of the main result.

2 Stability inequality and comparison principle

2.1 Stability inequality

Lemma 2.1 Assume that (u, v) is a positive stable solution of the system (1.1) with (1.3) is satisfied. Then, for $\phi \in C_c^1(\mathbb{R}^n)$, we have

$$\sqrt{pq} \int_{\mathbb{R}^n} (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-\frac{p-1}{2}} u^{-\frac{q-1}{2}} \phi^2 dx dy \leq \frac{1}{4} \int_{\mathbb{R}^n} |a\phi + 2\nabla_G \phi|^2 dx dy. \quad (2.1)$$

Proof We follow the idea in [2, 8]. Let $\phi \in C_c^1(\mathbb{R}^n)$. Multiplying the first equation in (1.12) by $\frac{\phi^2}{\xi}$, we obtain

$$\int_{\mathbb{R}^n} \left(-\Delta_G \xi \frac{\phi^2}{\xi} + a \cdot \nabla_G \xi \frac{\phi^2}{\xi} \right) dx dy = p \int_{\mathbb{R}^n} (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-p-1} \eta \frac{\phi^2}{\xi} dx dy. \quad (2.2)$$

Using integration by parts and Young's inequality: $2zz' - z'^2 \leq z^2$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(-\Delta_G \xi \frac{\phi^2}{\xi} + a \cdot \nabla_G \xi \frac{\phi^2}{\xi} \right) dx dy \\ &= \int_{\mathbb{R}^n} \left(2 \frac{\phi}{\xi} \nabla_G \phi \nabla_G \xi - |\nabla_G \xi|^2 \frac{\phi^2}{\xi^2} + a \cdot \nabla_G \xi \frac{\phi^2}{\xi} \right) dx dy \\ &= \int_{\mathbb{R}^n} \left((2\nabla_G \phi + a\phi) \frac{\phi}{\xi} \nabla_G \xi - |\nabla_G \xi|^2 \frac{\phi^2}{\xi^2} \right) dx dy \\ &\leq \frac{1}{4} \int_{\mathbb{R}^n} |a\phi + 2\nabla_G \phi|^2 dx dy. \end{aligned} \quad (2.3)$$

Consequently, combining (2.2) and (2.3), it follows that

$$p \int_{\mathbb{R}^n} (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-p-1} \eta \frac{\phi^2}{\xi} dx dy \leq \frac{1}{4} \int_{\mathbb{R}^n} |a\phi + 2\nabla_G \phi|^2 dx dy. \quad (2.4)$$

By the same argument, we also have

$$q \int_{\mathbb{R}^n} (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q-1} \xi \frac{\phi^2}{\eta} dx dy \leq \frac{1}{4} \int_{\mathbb{R}^n} |a\phi + 2\nabla_G \phi|^2 dx dy. \quad (2.5)$$

We now add the inequalities (2.4) and (2.5) to obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(1 + \|\mathbf{z}\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} \left(pv^{-p-1}\eta \frac{\phi^2}{\xi} + qu^{-q-1}\xi \frac{\phi^2}{\eta}\right) dx dy \\ & \leq \frac{1}{2} \int_{\mathbb{R}^n} |a\phi + 2\nabla_G \phi|^2 dx dy. \end{aligned} \quad (2.6)$$

Now note that

$$2\sqrt{pq}v^{-\frac{p-1}{2}}u^{-\frac{q-1}{2}}\phi^2 \leq pv^{-p-1}\eta \frac{\phi^2}{\xi} + qu^{-q-1}\xi \frac{\phi^2}{\eta}.$$

Putting this back into (2.6) gives the desired result. \square

2.2 Comparison principle

In this subsection, we shall prove the comparison principle for the system (1.1) without stability assumption.

Lemma 2.2 *Suppose that (u, v) is a bounded positive solution of (1.1). Assume that $1 < q \leq p$ and (1.3) hold. Then*

$$(p-1)v^{p-1} \leq (q-1)u^{q-1}. \quad (2.7)$$

Proof Let $d = \frac{q-1}{p-1} \leq 1$ and $l = d^{\frac{1}{p-1}}$. The inequality (2.7) is equivalent to

$$v \leq lu^d. \quad (2.8)$$

Put $w = v - lu^d$. A direct calculation leads to

$$\begin{aligned} \Delta_G w &= \Delta_G v - ldu^{d-1}\Delta_G u - ld(d-1)u^{d-2}|\nabla_G u|^2 \\ &\geq \Delta_G v - ldu^{d-1}\Delta_G u \\ &= a \cdot \nabla_G v + \left(1 + \|\mathbf{z}\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} u^{-q} - ldu^{d-1} \left(a \cdot \nabla_G u + \left(1 + \|\mathbf{z}\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} v^{-p}\right) \\ &= a \cdot \nabla_G w + u^{d-1} \left(1 + \|\mathbf{z}\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} (u^{-dp} - l^p v^{-p}). \end{aligned}$$

Therefore

$$\begin{aligned} \Delta_G w - a \cdot \nabla_G w &\geq u^{d-1} \left(1 + \|\mathbf{z}\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} (u^{-dp} - l^p v^{-p}) \\ &\geq u^{d-1} (u^{-dp} - l^p v^{-p}) \\ &= u^{d-1} \left(\frac{1}{u^{dp}} - \frac{l^p}{v^p}\right) \\ &= u^{d-1} \frac{v^p - l^p u^{dp}}{u^{dp} v^p}. \end{aligned} \quad (2.9)$$

We now prove (2.8) by contradiction. Suppose that

$$M = \sup_{\mathbb{R}^n} w > 0 \quad (M \leq +\infty). \quad (2.10)$$

Next, we divide the proof into two cases.

Case 1. If there exists z^* such that $\sup_{\mathbb{R}^n} w = w(z^*) = v(z^*) - lu^d(z^*) > 0$, then

$$\frac{\partial w}{\partial z_i}(z^*) = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial z_i^2}(z^*) \leq 0 \quad \text{for } i = 1, \dots, n.$$

This implies that

$$\nabla_G w(z^*) = 0 \quad \text{and} \quad \Delta_G w(z^*) \leq 0.$$

However, the right-hand side of (2.9) at z^* is positive. This is a contradiction.

Case 2. If the supremum of w is attained at infinity.

Take a cut-off function $\chi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ verifying $\chi = 1$ on $B_1 \times B_1$ and $\chi = 0$ outside $B_2 \times B_{2^{1+\alpha}}$. Put $\phi_R(x) = \chi^m(\frac{x}{R}, \frac{y}{R^{1+\alpha}})$. Here $m > 0$ will be chosen later. A simple calculation yields

$$|\Delta_G \phi_R| \leq CR^{-2} \phi_R^{\frac{m-2}{m}}, \quad \phi_R^{-1} |\nabla_G \phi|^2 \leq CR^{-2} \phi_R^{\frac{m-2}{m}}. \quad (2.11)$$

Let $w_R = \phi_R w$ which is a compactly supported function. Then there exists $z_R = (x_R, y_R) \in B_{2R} \times B_{(2R)^{1+\alpha}}$ such that

$$w_R(z_R) = \sup_{\mathbb{R}^n} w_R(z) = \max_{\mathbb{R}^n} w_R(z) \rightarrow M \quad \text{as } R \rightarrow +\infty. \quad (2.12)$$

This implies that

$$\nabla_G w_R(z_R) = 0 \quad \text{and} \quad \Delta_G w_R(z_R) \leq 0. \quad (2.13)$$

In what follows, all the estimates are taken at the point z_R . First, using $\nabla_G w_R(z_R) = 0$, we have

$$0 = \nabla_G w_R = \nabla_G \phi_R w + \phi_R \nabla_G w.$$

Hence,

$$\nabla_G w = -\phi_R^{-1} \nabla_G \phi_R w. \quad (2.14)$$

Since $\Delta_G w_R(z_R) \leq 0$, we get

$$0 \geq \Delta_G w_R = \Delta_G \phi_R w + 2 \nabla_G \phi_R \cdot \nabla_G w + \phi_R \Delta_G w.$$

Thus,

$$\phi_R \Delta_G w \leq (2 \phi_R^{-1} |\nabla_G \phi_R|^2 - \Delta_G \phi_R) w. \quad (2.15)$$

Combining (2.11) and (2.15), one has

$$\phi_R \Delta_G w \leq CR^{-2} \phi_R^{\frac{m-2}{m}} w. \quad (2.16)$$

Using (2.11), (2.14) and the fact that $|a(\mathbf{z})| \leq \frac{\epsilon}{\|\mathbf{z}\|+1}$, one has

$$|a \cdot \nabla_G w \phi_R| \leq CR^{-2} \phi_R^{\frac{m-1}{m}} w. \quad (2.17)$$

Recall that $v - lu^d = w$ and at z_R , we get

$$\frac{v^p}{w^p} - \frac{(lu^d)^p}{w^p} \geq 1. \quad (2.18)$$

Multiplying (2.9) by ϕ_R and using (2.16), (2.17) and (2.18), we obtain

$$\phi_R u^{d-1} \frac{w^p}{u^{dp} v^p} \leq CR^{-2} \phi_R^{\frac{m-2}{m}} w.$$

Recall that the constant C is independent of R . Consequently,

$$\phi_R^{\frac{m+2}{m}} u^{d-1} \frac{w^p}{u^{dp} v^p} \leq CR^{-2} \phi_R w.$$

Choosing $m = \frac{2}{p-1}$ (or $p = \frac{m+2}{m}$), we get

$$u^{d-1} \frac{w_R^p}{u^{dp} v^p} \leq CR^{-2} w_R \quad \text{or} \quad u^{d-1} \frac{w_R^{p-1}}{u^{dp} v^p} \leq CR^{-2}. \quad (2.19)$$

It follows from (2.19), the boundedness of (u, v) and $d \leq 1$ that

$$w_R^{p-1}(z_R) \leq CR^{-2}.$$

Finally, letting $R \rightarrow +\infty$, we get $M = 0$, which contradicts (2.10). The proof is complete. \square

Combining the proof of Lemma 2.2 with the idea in [1, 10, 12, 15], we have the inverse comparison principle as follows.

Lemma 2.3 *Suppose that (u, v) is a bounded positive solution of (1.1). Assume that $1 < q \leq p$ and (1.3) hold. Then we have*

$$u \leq \|v\|_{\infty}^{\frac{p-q}{q-1}} v, \quad (2.20)$$

where $\|v\|_{\infty} = \sup_{\mathbb{R}^n} v$.

Proof Let $l = \|v\|_{\infty}^{\frac{p-q}{q-1}}$ and $w = u - lv$. We need to show that $w \leq 0$. Notice that

$$\begin{aligned} \Delta_G w - a \cdot \nabla_G w &= \Delta_G u - a \cdot \nabla_G u - l(\Delta_G v - a \cdot \nabla_G v) \\ &= (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-p} - l(1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} \\ &= \frac{v^{-p}}{\|v\|_{\infty}^{-p}} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} \|v\|_{\infty}^{-p} - l(1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} \\ &\geq (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} \left(\frac{v^{-q}}{\|v\|_{\infty}^{-q}} \|v\|_{\infty}^{-p} - lu^{-q} \right) \end{aligned}$$

$$\begin{aligned} &= \|v\|_{\infty}^{q-p} \left(1 + \|z\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} \left(v^{-q} - l^q v^{-q}\right) \\ &\geq \|v\|_{\infty}^{q-p} \left(v^{-q} - l^q v^{-q}\right). \end{aligned} \quad (2.21)$$

In order to obtain the proof, it suffices to use the arguments as in Lemma 2.2 by noting that (2.9) is replaced by (2.21). The details are then omitted. \square

In what follows, the constant C does not depend on a positive parameter R and may change from line to line.

Lemma 2.4 *Suppose that (u, v) be a bounded stable positive solution of (1.1). Assume that $1 < q \leq p$ and (1.3) hold. Then for $R > 0$ there exists $C > 0$ independent of R such that*

$$\int_{B_R \times B_{R^{1+\alpha}}} u^{-q} dx dy \leq CR^{Q - \frac{2(pq-q)}{pq-1} - \frac{\gamma(pq-1)}{pq-q}} \quad (2.22)$$

and

$$\int_{B_R \times B_{R^{1+\alpha}}} v^{-\frac{p+q+2}{2}} dx dy \leq CR^{Q-2-\gamma}. \quad (2.23)$$

Proof Using Lemma 2.2, we get

$$v^{-\frac{p+1}{2}} \geq \left(\frac{p-1}{q-1}\right)^{\frac{p+1}{2(p-1)}} u^{-\frac{(q-1)(p+1)}{2(p-1)}}. \quad (2.24)$$

Take a cut-off function $\phi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ verifying $\phi = 1$ on $B_1 \times B_1$ and $\phi = 0$ outside $B_2 \times B_{2^{\alpha+1}}$. For $R > 0$, put $\phi_R(x) = \phi^m(\frac{x}{R}, \frac{\gamma}{R^{1+\alpha}})$, where $m \geq 2$ which is fixed. Then there exists $C > 0$ independent of R such that

$$|\nabla_G \phi_R| \leq CR^{-1} \phi_R^{\frac{m-1}{m}}. \quad (2.25)$$

By virtue of (2.1) and (2.24), we derive

$$\begin{aligned} &\left(\frac{p-1}{q-1}\right)^{\frac{p+1}{2(p-1)}} \sqrt{pq} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} \left(1 + \|z\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} u^{-\frac{pq+1}{p-1}} \phi_R^2 dx dy \\ &\leq \sqrt{pq} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} \left(1 + \|z\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} v^{-\frac{p+1}{2}} u^{-\frac{q+1}{2}} \phi_R^2 dx dy \\ &\leq \frac{1}{4} \int_{\mathbb{R}^n} |a\phi_R + 2\nabla_G \phi_R|^2 dx dy. \end{aligned} \quad (2.26)$$

Recall that $\frac{pq-1}{pq-q} > 1$. Then, by combining the Hölder inequality, (2.25) and (2.26), we get

$$\begin{aligned} &\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} u^{-q} \phi_R^2 dx dy \\ &\leq \left(\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} \left(1 + \|z\|^{2(\alpha+1)}\right)^{\frac{\gamma}{2(\alpha+1)}} u^{-\frac{pq+1}{p-1}} \phi_R^2 dx dy \right)^{\frac{pq-q}{pq-1}} \end{aligned}$$

$$\begin{aligned}
 & \times \left(\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{-\frac{\gamma(pq-1)^2}{2(\alpha+1)(q-1)(pq-q)}} \phi_R^2 \right)^{\frac{q-1}{pq-1}} \\
 & \leq C \left(\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} |a\phi_R + 2\nabla_G \phi_R|^2 dx dy \right)^{\frac{pq-q}{pq-1}} \\
 & \quad \times \left(\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{-\frac{\gamma(pq-1)^2}{2(\alpha+1)(q-1)(pq-q)}} \phi_R^2 dx dy \right)^{\frac{q-1}{pq-1}} \\
 & \leq CR^{-2\frac{pq-q}{pq-1}} \left(\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} \phi_R^{\frac{2(m-1)}{m}} dx dy \right)^{\frac{pq-q}{pq-1}} \\
 & \quad \times \left(\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{-\frac{\gamma(pq-1)^2}{2(\alpha+1)(q-1)(pq-q)}} \phi_R^2 dx dy \right)^{\frac{q-1}{pq-1}} \\
 & \leq CR^{-2\frac{pq-q}{pq-1}} R^{Q\frac{pq-q}{pq-1}} R^{-\frac{\gamma(pq-1)}{pq-q}} R^{Q\frac{q-1}{pq-1}} = CR^{Q-\frac{2(pq-q)}{pq-1}-\frac{\gamma(pq-1)}{pq-q}}. \tag{2.27}
 \end{aligned}$$

Hence, the desired integral estimate (2.22) follows. Finally, (2.23) follows from using the same argument as above where we use (2.20) instead of (2.7). \square

3 Proof of the main result

3.1 Beginning of the proof

In this subsection, we give a preparation for the bootstrap iteration. Using Lemmas 2.2 and 2.1, we get the following.

Lemma 3.1 *Under the same assumptions of Lemma 2.4, suppose that*

$$\tau_0^- < t < \tau_0^+.$$

Then we have

$$\begin{aligned}
 & \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} v^{-2t-1} \phi^2 dx dy \\
 & \leq C \int_{\mathbb{R}^n} v^{-2t} (|\nabla_G \phi|^2 + \phi |\Delta_G \phi| + |a| |\nabla_G(\phi^2)|) dx dy,
 \end{aligned}$$

for all $\phi \in C_c^2(\mathbb{R}^n)$ satisfying $0 \leq \phi \leq 1$. Here C does not depend on (u, v) .

Proof Using Lemma 2.1 with the test function $v^{-t}\phi$ we have

$$\begin{aligned}
 & \sqrt{pq} \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{\frac{-p-1}{2}} u^{\frac{-q-1}{2}} v^{-2t} \phi^2 dx dy \\
 & \leq \frac{1}{4} \int_{\mathbb{R}^n} |av^{-t}\phi + 2\nabla_G(v^{-t}\phi)|^2 dx dy \\
 & = \frac{1}{4} \int_{\mathbb{R}^n} (|av^{-t}\phi|^2 + 4a \cdot \nabla_G(v^{-t}\phi)(v^{-t}\phi) + 4|\nabla_G(v^{-t}\phi)|^2) dx dy \\
 & = \frac{1}{4} \int_{\mathbb{R}^n} (|av^{-t}\phi|^2 + 4|\nabla_G(v^{-t}\phi)|^2) dx dy,
 \end{aligned}$$

where in the last equality, we have used $\operatorname{div}_G a = 0$.

Applying the Hardy inequality related to Grushin type operators, see e.g. [5], one obtains

$$\int_{\mathbb{R}^n} |a(\mathbf{z})v^{-t}\phi|^2 dx dy \leq \epsilon^2 \int_{\mathbb{R}^n} \frac{|v^{-t}\phi|^2}{\|\mathbf{z}\|^2} dx dy \leq \frac{4\epsilon^2}{(Q-2)^2} \int_{\mathbb{R}^n} |\nabla_G(v^{-t}\phi)|^2 dx dy.$$

Combining the above two estimates, we derive that

$$\begin{aligned} & \sqrt{pq} \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-\frac{p-1}{2}} u^{-\frac{q-1}{2}} v^{-2t} \phi^2 dx dy \\ & \leq \left(1 + \frac{\epsilon^2}{(Q-2)^2}\right) \int_{\mathbb{R}^n} |\nabla_G(v^{-t}\phi)|^2 dx dy \\ & = \left(1 + \frac{\epsilon^2}{(Q-2)^2}\right) \left(t^2 \int_{\mathbb{R}^n} |\nabla_G v|^2 v^{-2t-2} \phi^2 dx dy \right. \\ & \quad \left. + \int_{\mathbb{R}^n} v^{-2t} |\nabla_G \phi|^2 dx dy - \frac{1}{2} \int_{\mathbb{R}^n} v^{-2t} \Delta_G(\phi^2) dx dy\right). \end{aligned} \quad (3.1)$$

Multiplying the second equation in (1.1) by $v^{-2t-1}\phi^2$ and integrating by parts we arrive at

$$\begin{aligned} & (2t+1) \int_{\mathbb{R}^n} |\nabla_G v|^2 v^{-2t-2} \phi^2 dx dy - \frac{1}{2t} \int_{\mathbb{R}^n} v^{-2t} \Delta_G(\phi^2) dx dy \\ & - \frac{1}{2t} \int_{\mathbb{R}^n} v^{-2t} a \cdot \nabla_G(\phi^2) dx dy \\ & = \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} v^{-2t-1} \phi^2 dx dy. \end{aligned} \quad (3.2)$$

Combining (3.1) and (3.2), we obtain

$$\begin{aligned} & \sqrt{pq} \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} v^{-\frac{p-1}{2}} u^{-\frac{q-1}{2}} v^{-2t} \phi^2 dx dy \\ & \leq \frac{t^2}{2t+1} \left(1 + \frac{\epsilon^2}{(Q-2)^2}\right) \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} v^{-2t-1} \phi^2 dx dy \\ & + C \int_{\mathbb{R}^n} v^{-2t} (|\nabla_G \phi|^2 + \phi |\Delta_G \phi| + |a| |\nabla_G(\phi^2)|) dx dy. \end{aligned} \quad (3.3)$$

Using Lemma 2.2 and (3.3), we get

$$\begin{aligned} & \left(\sqrt{\frac{pq(p-1)}{q-1}} - \frac{t^2}{2t+1} \left(1 + \frac{\epsilon^2}{(Q-2)^2}\right)\right) \int_{\mathbb{R}^n} (1 + \|\mathbf{z}\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} v^{-2t-1} \phi^2 dx dy \\ & \leq C \int_{\mathbb{R}^n} v^{-2t} (|\nabla_G \phi|^2 + \phi |\Delta_G \phi| + |a| |\nabla_G(\phi^2)|) dx dy. \end{aligned}$$

Since $\tau_0^- < t < \tau_0^+$, we have $\sqrt{\frac{pq(p-1)}{q-1}} - \frac{t^2}{2t+1} \left(1 + \frac{\epsilon^2}{(Q-2)^2}\right) > 0$ provided ϵ is sufficiently small, which completes the proof. \square

3.2 End of the proof

The bootstrap argument in this subsection is quite similar to that in [10, 12]. For completeness, we present the details.

Take a cut-off function $\phi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ verifying

$$\phi = 1 \quad \text{on } B_1 \times B_1 \quad \text{and} \quad \phi = 0 \quad \text{outside } B_2 \times B_{2^{\alpha+1}}. \quad (3.4)$$

Let w be a smooth function and let $\kappa = \frac{Q}{Q-2}$ if $Q > 2$. For $Q > 2$, put $w_R(x) = w(Rx, R^{1+\alpha}y)$, then, by using the Sobolev inequality (see [26]) and integration by parts, we have

$$\begin{aligned} & \left(\int_{B_1 \times B_1} w_R^{2\kappa} dx dy \right)^{\frac{1}{2\kappa}} \\ & \leq \left(\int_{B_2 \times B_{2^{\alpha+1}}} (w_R \phi)^{2\kappa} dx dy \right)^{\frac{1}{2\kappa}} \\ & \leq C \left(\int_{B_2 \times B_{2^{\alpha+1}}} |\nabla_G(w_R \phi)|^2 dx dy \right)^{\frac{1}{2}} \\ & = C \left(\int_{B_2 \times B_{2^{\alpha+1}}} |\nabla_G w_R|^2 \phi^2 + w_R^2 |\nabla_G \phi|^2 + \frac{1}{2} \nabla_G(w_R^2) \nabla_G(\phi^2) dx dy \right)^{\frac{1}{2}} \\ & = C \left(\int_{B_2 \times B_{2^{\alpha+1}}} |\nabla_G w_R|^2 \phi^2 + w_R^2 |\nabla_G \phi|^2 + \frac{1}{2} w_R^2 (-\Delta_G(\phi^2)) dx dy \right)^{\frac{1}{2}} \\ & \leq C \left(\int_{B_2 \times B_{2^{\alpha+1}}} (R^2 |\nabla_G w|^2 + w^2)(Rx, \dots, R^{1+\alpha}y) dx dy \right)^{\frac{1}{2}}. \end{aligned}$$

So we get

$$\left(\int_{B_1 \times B_1} w_R^{2\kappa} dx dy \right)^{\frac{1}{2\kappa}} \leq C \left(\int_{B_2 \times B_{2^{\alpha+1}}} (R^2 |\nabla_G w|^2 + w^2)(Rx, R^{1+\alpha}y) dx dy \right)^{\frac{1}{2}}.$$

From a scaling argument it follows that

$$\begin{aligned} & \left(\int_{B_R \times B_{R^{\alpha+1}}} w^{2\kappa} dx dy \right)^{\frac{1}{\kappa}} \\ & \leq CR^{2+Q(\frac{1}{\kappa}-1)} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} |\nabla_G w|^2 dx dy + CR^{Q(\frac{1}{\kappa}-1)} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} w^2 dx dy. \end{aligned} \quad (3.5)$$

Suppose that (u, v) is a positive stable solution of (1.1). Set

$$w = v^{-t} \quad \text{for } \tau_0^- < t < \tau_0^+.$$

A simple calculation gives

$$|\nabla_G w|^2 = t^2 |\nabla_G v|^2 v^{-2t-2}.$$

Let $\phi_R = \phi(\frac{x}{R}, \frac{y}{R^{1+\alpha}})$ where ϕ is given in (3.4). Then

$$\begin{aligned} \int_{B_R \times B_{R^{\alpha+1}}} |\nabla_G w|^2 dx dy &= C \int_{B_R} |\nabla_G v|^2 v^{-2t-2} dx dy \\ &\leq C \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} |\nabla_G v|^2 v^{-2t-2} \phi_R^2 dx dy. \end{aligned} \quad (3.6)$$

Multiplying the second equation in (1.1) by $v^{-2t-1} \phi_R^2$ and using integration by parts, we obtain

$$\begin{aligned} &\int_{B_{2R} \times B_{(2R)^{\alpha+1}}} |\nabla_G v|^2 v^{-2t-2} \phi_R^2 dx dy \\ &= \frac{1}{2t+1} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} (1 + \|z\|^{2(\alpha+1)})^{\frac{\gamma}{2(\alpha+1)}} u^{-q} v^{-2t-1} \phi_R^2 dx dy \\ &\quad + \frac{1}{2t(2t+1)} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} v^{-2t} \Delta_G(\phi_R^2) dx dy \\ &\quad + \frac{1}{2t(2t+1)} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} v^{-2t} a \cdot \nabla_G(\phi_R^2) dx dy. \end{aligned}$$

Inserting this into (3.6), using Lemma 3.1, we obtain

$$\int_{B_R \times B_{R^{\alpha+1}}} |\nabla_G w|^2 dx dy \leq CR^{-2} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} w^2 dx dy.$$

Substituting the above inequality into (3.5), we have

$$\left(\int_{B_R \times B_{R^{\alpha+1}}} w^{2\kappa} dx dy \right)^{\frac{1}{\kappa}} \leq CR^{Q(\frac{1}{\kappa}-1)} \int_{B_{2R} \times B_{(2R)^{\alpha+1}}} w^2 dx dy. \quad (3.7)$$

We fix a real positive number $\delta = \frac{p+q+2}{4}$ and recall the fact that

$$2\tau_0^- < 0 < 2\delta = \frac{p+q+2}{2}. \quad (3.8)$$

Let m be a nonnegative integer satisfying $\delta\kappa^{m-1} < \tau_0^+ \leq \delta\kappa^m$. We construct an increasing geometric sequence.

$$\tau_0^- < t_1 < t_2 < \cdots < t_m < \tau_0^+,$$

given by

$$2t_1 = 2\delta r, \quad 2t_2 = 2\delta r\kappa, \quad \dots, \quad 2t_m = 2\delta r\kappa^{m-1},$$

where $r \in [1, \kappa]$ is chosen such that t_m is arbitrarily close to τ_0^+ .

To simplify notations below, we use $R_n = 2^n R$. By using (3.7) and an induction argument, we obtain

$$\begin{aligned} \left(\int_{B_R \times B_{R^{\alpha+1}}} v^{-2t_m \kappa} dx dy \right)^{\frac{1}{t_m \kappa}} &\leq C \left(R^{Q(\frac{1}{\kappa}-1)} \right)^{\frac{1}{t_m}} \left(\int_{B_{R_1} \times B_{R_1^{\alpha+1}}} v^{-2t_m} dx \right)^{\frac{1}{t_m}} \\ &= C R^{Q(\frac{1}{t_m \kappa} - \frac{1}{t_m})} \left(\int_{B_{R_1} \times B_{R_1^{\alpha+1}}} v^{-2t_{m-1} \kappa} dx \right)^{\frac{1}{t_{m-1} \kappa}} \\ &\leq C R^{Q(\frac{1}{t_m \kappa} - \frac{1}{t_1})} \left(\int_{B_{R_m} \times B_{R_m^{\alpha+1}}} v^{-2t_1} dx dy \right)^{\frac{1}{t_1}} \\ &= C R^{Q(\frac{1}{t_m \kappa} - \frac{1}{\delta r})} \left(\int_{B_{R_m} \times B_{R_m^{\alpha+1}}} v^{-2\delta r} dx dy \right)^{\frac{1}{\delta r}}. \end{aligned} \quad (3.9)$$

For the last integral, we shall use Hölder's inequality, (3.7) and Lemma 2.4 to obtain

$$\begin{aligned} \int_{B_{R_m} \times B_{R_m^{\alpha+1}}} v^{-2\delta r} dx dy &\leq \left(\int_{B_{R_m} \times B_{R_m^{\alpha+1}}} v^{-2\delta \kappa} dx dy \right)^{\frac{r}{\kappa}} \left(\int_{B_{R_m} \times B_{R_m^{\alpha+1}}} 1 dx dy \right)^{1-\frac{r}{\kappa}} \\ &\leq C \left(R^{Q(\frac{1}{\kappa}-1)} \int_{B_{R_{m+1}} \times B_{R_{m+1}^{\alpha+1}}} v^{-2\delta} dx dy \right)^r R^{Q(1-\frac{r}{\kappa})} \\ &= C R^{Q(1-r)} \left(\int_{B_{R_{m+1}} \times B_{R_{m+1}^{\alpha+1}}} v^{-\frac{p+q+2}{2}} dx dy \right)^r \\ &\leq C R^{Q(1-r)} R^{r(Q-2-\gamma)} = C R^{Q-r(\gamma+2)}. \end{aligned}$$

Consequently,

$$\left(\int_{B_{R_m} \times B_{R_m^{\alpha+1}}} v^{-2\delta r} dx \right)^{\frac{1}{\delta r}} \leq C R^{\frac{Q}{\delta r} - \frac{\gamma+2}{\delta}}. \quad (3.10)$$

Substituting (3.10) into the last inequality of (3.9), one has

$$\left(\int_{B_R \times B_{R^{\alpha+1}}} v^{-2t_m \kappa} dx \right)^{\frac{1}{t_m \kappa}} \leq C R^{\frac{Q}{\kappa t_m} - \frac{\gamma+2}{\delta}} = C R^{\frac{Q}{\kappa t_m} - \frac{4(\gamma+2)}{p+q+2}}. \quad (3.11)$$

Since $r \in [1, \kappa]$ is chosen such that t_m is close to τ_0^+ , the exponent in the right-hand side of (3.11) is negative. Letting $R \rightarrow +\infty$, we obtain a contradiction.

Acknowledgements

We would like to express our gratitude to the anonymous referee for the careful reading and helpful suggestions.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Competing interests

The author declares to have no competing interests.

Authors' contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

Disclosure statement

No potential conflict of interest was reported by the author.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 December 2019 Accepted: 16 April 2020 Published online: 29 April 2020

References

1. Cheng, Z., Huang, G., Li, C.: On the Hardy–Littlewood–Sobolev type systems. *Commun. Pure Appl. Anal.* **15**(6), 2059–2074 (2016)
2. Cowan, C.: Liouville theorems for stable Lane–Emden systems with biharmonic problems. *Nonlinearity* **26**(8), 2357–2371 (2013)
3. Cowan, C.: Stability of entire solutions to supercritical elliptic problems involving advection. *Nonlinear Anal.* **104**, 1–11 (2014)
4. Cowan, C., Fazly, M.: On stable entire solutions of semi-linear elliptic equations with weights. *Proc. Am. Math. Soc.* **140**(6), 2003–2012 (2012)
5. D'Ambrosio, L.: Hardy inequalities related to Grushin type operators. *Proc. Am. Math. Soc.* **132**(3), 725–734 (2004)
6. Du, Y., Guo, Z.: Positive solutions of an elliptic equation with negative exponent: stability and critical power. *J. Differ. Equ.* **246**, 2387–2414 (2009)
7. Du, Y.H., Guo, Z.M.: Finite Morse-index solutions and asymptotics of weighted nonlinear elliptic equations. *Adv. Differ. Equ.* **18**, 737–768 (2013)
8. Duong, A.T.: A Liouville type theorem for non-linear elliptic systems involving advection terms. *Complex Var. Elliptic Equ.* **63**, 1704–1720 (2018)
9. Duong, A.T.: On the classification of positive supersolutions of elliptic systems involving the advection terms. *J. Math. Anal. Appl.* **478**(2), 1172–1188 (2019)
10. Duong, A.T., Lan, D., Le, P.Q., Nguyen, P.T.: On the nonexistence of stable solutions of sub-elliptic systems with negative exponents. *Complex Var. Elliptic Equ.* **64**(12), 2117–2129 (2019). <https://doi.org/10.1080/17476933.2019.1574772>
11. Duong, A.T., Nguyen, N.T.: Liouville type theorems for elliptic equations involving Grushin operator and advection. *Electron. J. Differ. Equ.* **2017**, 108 (2017)
12. Duong, A.T., Phan, Q.H.: Liouville type theorem for nonlinear elliptic system involving Grushin operator. *J. Math. Anal. Appl.* **454**, 785–801 (2017)
13. Farina, A.: On the classification of solutions of the Lane–Emden equation on unbounded domains of \mathbb{R}^n . *J. Math. Pures Appl.* **87**, 537–561 (2007)
14. Fazly, M., Ghoussoub, N.: On the Hénon–Lane–Emden conjecture. *Discrete Contin. Dyn. Syst.* **34**(6), 2513–2533 (2014)
15. Hajlaoui, H., Harrabi, A., Mtiri, F.: Liouville theorems for stable solutions of the weighted Lane–Emden system. *Discrete Contin. Dyn. Syst.* **37**(1), 265–279 (2017)
16. Hajlaoui, H., Harrabi, A., Ye, D.: On stable solutions of biharmonic problems with polynomial growth. *Pac. J. Math.* **270**, 79–93 (2014)
17. Harrabi, A., Rahal, B.: Liouville type theorems for elliptic equations in half-space with mixed boundary value conditions. *Adv. Nonlinear Anal.* **8**(1), 193–202 (2019). <https://doi.org/10.1515/anona-2016-0168>
18. Hu, L.G.: Liouville type results for semi-stable solutions of the weighted Lane–Emden system. *J. Math. Anal. Appl.* **432**(1), 429–440 (2015). <https://doi.org/10.1016/j.jmaa.2015.06.032>
19. Kogoj, A.E., Lanconelli, E.: On semilinear Δ_λ -Laplace equation. *Nonlinear Anal.* **75**(12), 4637–4649 (2012)
20. Ma, L., Wei, J.C.: Properties of positive solutions to an elliptic equation with negative exponent. *J. Funct. Anal.* **254**(4), 1058–1087 (2008)
21. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: *Nonlinear Analysis—Theory and Methods*. Springer Monographs in Mathematics. Springer, Cham (2019)
22. Rahal, B.: Liouville-type theorems with finite Morse index for semilinear Δ_λ -Laplace operators. *Nonlinear Differ. Equ. Appl.* **25**(3), 21 (2018)
23. Souplet, P.: The proof of the Lane–Emden conjecture in four space dimensions. *Adv. Math.* **221**(5), 1409–1427 (2009)
24. Wang, C., Ye, D.: Some Liouville theorems for Hénon type elliptic equations. *J. Funct. Anal.* **262**, 1705–1727 (2012)
25. Wei, Y., Chen, C., Chen, Q., Yang, H.: Liouville-type theorem for nonlinear elliptic equations involving p -Laplace-type Grushin operators. *Math. Methods Appl. Sci.* **43**(1), 320–333 (2020)
26. Yu, X.: Liouville type theorem for nonlinear elliptic equation involving Grushin operators. *Commun. Contemp. Math.* **17**(2), art. 12 (2015)