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Some higher norm inequalities for composition of power operators

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Abstract

In this paper, we first prove the local L^s norm estimate of composite operators $\Delta^k G^m(u)$ by use of the L^s norm of u . Then we establish the local and global higher norm inequalities of $\Delta^k G^m(u)$. Simultaneously, we also give a global higher norm estimate with Radon measure. Finally, as applications of these results, we give two examples to estimate the higher norm of $\Delta^k G^m(u)$.

Keywords: Power operators; Radon measure; Whitney cover; Higher norm inequalities

1 Introduction

In this paper, our purpose is investigating some higher norm inequalities for composition of power operators $\Delta^k G^m$ on a bounded domain M , where k, m are positive integers, Δ is Laplace–Beltrami operator, and G is Green's operator. The norm estimate of operators applied to differential forms is an important and interesting research topic in some areas of mathematic analysis and has achieved fruitful results; see [1–11] for more detail. Some these results improved the development of some other branches of mathematics and mathematical physics; see [12–18] for details. In previous related research about norm estimates of operators the study mostly concentrated on estimates of the L^t norm of operators and applying them to differential forms in terms of the L^t norm of differential forms. Therefore, if $s > t$, then we could not estimate the L^s norm of operators by the L^t norm of differential forms from the literature. This motivated us to research the higher norm of operators than of differential forms. Since the norm estimate of a composite operator is more complicated than that of a single operator, in this paper, we choose the composition of power operators $\Delta^k G^m$ to be the research object. In this paper, we first give the local L^s norm estimate of $\Delta^k G^m(u)$ by the L^s norm of u in Theorem 2.5. Then based on Theorem 2.5, we prove the local and global higher norm inequalities of $\Delta^k G^m(u)$ separately presented in Theorems 2.6–2.8 and 3.2. Simultaneously, we also establish the global higher norm estimate with Radon measure in Theorem 3.3. Finally, we give two examples as applications of Theorem 3.2.

We start this paper by introducing some notations and definitions in [19]. Let $M \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain, B be a ball, and σB be the ball with the same center as

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B satisfying $\text{diam}(\sigma B) = \sigma \text{diam}(B)$. We do not distinguish balls from cubes in this paper. By $\Lambda^k = \Lambda^k(R^n)$ ($k = 1, 2, \dots, n$) we denote the linear space of all k -forms $u(x) = \sum_I u_I(x) dx_I = \sum u_{i_1 i_2 \dots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ with summation over all ordered k -tuples $I = (i_1, i_2, \dots, i_k)$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$, $k = 1, 2, \dots, n$. If the coefficient $u_I(x)$ of k -forms is differentiable on M , then we call $u(x)$ a differential k -form on M . As usual, we use $C^\infty(M, \Lambda^k)$ to denote the space of smooth k -forms in a domain M , $D'(M, \Lambda^k)$ to denote the space of all differential k -forms. Let $L^p(M, \Lambda^k)$ be the set of differential k -forms $u(x) = \sum_I u_I(x) dx_I$ on M with $\int_M |u_I|^p < \infty$ for all ordered k -tuples I . The norm of a k -form $u(x)$ on M is defined by

$$\|u(x)\|_{p,M} = \left(\int_M |u(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_M \left(\sum_I |u_I(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}, \quad (1.1)$$

and then $L^p(M, \Lambda^k)$ is a Banach space. As usual, we use \star to denote the Hodge star operator and $\text{dist}(x, M)$ to denote the distance of the point x from the set M . Also, we use $d : D'(M, \Lambda^k) \rightarrow D'(M, \Lambda^{k+1})$ to denote the differential operator and $d^* : D'(M, \Lambda^{k+1}) \rightarrow D'(M, \Lambda^k)$ to denote the Hodge codifferential operator defined by $d^* = (-1)^{n-k+1} \star d \star$ on $D'(M, \Lambda^{k+1})$. The n -dimensional Lebesgue measure of a set $E \subseteq R^n$ is denoted by $|E|$. For any differential form u , the average of u over B is defined as $u_B = \frac{1}{|B|} \int_B u dx$. All integrals involved in this paper are the Lebesgue integrals. The Laplace–Beltrami operator Δ is defined by $\Delta = dd^* + d^*d$. We define Green's operator G on the space of smooth k -forms in M by setting $G(u)$ to be a solution of Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection; see [1, 7, 19–22] for more detail about the Laplace–Beltrami operator Δ , Green's operator G , and projection operator H . We call w a weight if $w \in L^1_{\text{loc}}(R^n)$ and $w > 0$ a.e. For any Radon measure ν defined by $d\nu = w(x) dx$, we define the L^p -norm of a measurable function f with Radon measure over M by

$$\|f\|_{p,M,\nu} = \left(\int_M |f|^p d\nu \right)^{\frac{1}{p}} = \left(\int_M |f|^p w(x) dx \right)^{\frac{1}{p}}, \quad (1.2)$$

and the Radon measure of E by $\nu(E) = \int_E d\nu = \int_E w(x) dx$.

The nonlinear partial differential equation

$$d^* A(x, du) = 0 \quad (1.3)$$

is called the A -harmonic equation, where $A : M \times \Lambda^k(R^n) \rightarrow \Lambda^k(R^n)$ satisfies the following conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1} \quad \text{and} \quad A(x, \xi)\xi \geq |\xi|^p \quad (1.4)$$

for almost every $x \in M$ and all $\xi \in \Lambda^k(R^n)$, where $a > 0$ is a constant, and $1 < p < \infty$ is a fixed exponent associated with (1.3). For more details about A -harmonic equation, see [3, 4, 8, 19, 23, 24].

2 Local higher norm inequalities for $\Delta^k G^m(u)$

In this section, we first give the L^s norm estimate of $\Delta^k G^m(u)$ in terms of the L^s norm of u . Then based on this result, we establish local higher norm inequalities for $\Delta^k G^m(u)$ in two cases.

Let ψ be a strictly increasing convex function on $[0, +\infty)$ with $\psi(0) = 0$, and let u be a differential form on a bounded domain M . Then $\psi(\kappa|u| + |u_M|) \in L^1(M, \nu)$ for any real number $\kappa > 0$ and $\nu(\{x \in M : |u - u_M| > 0\}) > 0$, where ν is the Radon measure defined by $d\nu = w(x) dx$ for a weight $w(x)$. For any positive constant a , we have

$$\int_M \psi\left(\frac{1}{2}|u - u_M|\right) d\nu \leq C_1 \int_M \psi(a|u|) d\nu \leq C_2 \int_M \psi(2a|u - u_M|) d\nu$$

for some constants $C_1 > 0$ and $C_2 > 0$. Let $\psi(u) = u^s$, $s > 1$, $w(x) = 1$, and let M be a ball B in this inequality. Then there exist two positive constants C_3 and C_4 , independent of u , such that

$$\|u - u_B\|_{s,B} \leq C_3 \|u\|_{s,B} \leq C_4 \|u - u_B\|_{s,B} \quad (2.1)$$

for all balls B with $|\{x \in B : |u - u_B| > 0\}| > 0$. Inequality (2.1) indicates the norm $\|u - u_B\|_{s,B}$ comparable to the norm $\|u\|_{s,B}$.

Lemma 2.1 ([5]) *Let $u \in C^\infty(M, \Lambda)$ be a smooth differential form defined on M . Then there exists a constant $C = C(s)$, independent of u , such that*

$$\|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} + \|dG(u)\|_{s,B} + \|d^*G(u)\|_{s,B} + \|G(u)\|_{s,B} \leq C(s)\|u\|_{s,\sigma B}$$

for any ball B with $\sigma B \subset M$, where $\sigma > 1$ and $1 < s < \infty$ are constants.

Lemma 2.2 ([25]) *Let u be a solution of A -harmonic equation (1.3) in a domain M . Then there exists a constant C , independent of u , such that*

$$\|u\|_{s,B} \leq C|B|^{(t-s)/st} \|u\|_{t,\sigma B}$$

for all balls B with $\sigma B \subset M$, where $\sigma > 1$ and $0 < s, t < \infty$ are constants.

Lemma 2.3 ([24]) *Let $u \in D'(Q, \Lambda^l)$ and $du \in L^p(Q, \Lambda^{l+1})$. Then $u - u_Q$ is in $L^{np/(n-p)}(Q, \Lambda^l)$, and*

$$\left(\int_Q |u - u_Q|^{np/(n-p)} dx \right)^{(n-p)/np} \leq C_p(n) \left(\int_Q |du|^p dx \right)^{1/p}$$

for a cube or a ball Q in R^n , where $l = 0, 1, 2, \dots, n-1$ and $1 < p < n$.

Lemma 2.4 *Let u be a smooth differential form defined on M , G be Green's operator, and Δ be the Laplace–Beltrami operator defined by $\Delta = dd^* + d^*d$. Then the Laplace–Beltrami operator Δ and Green's operator G are commutable, that is,*

$$\Delta G(u) = G\Delta(u). \quad (2.2)$$

Proof From [19, p. 88] we find that Green's operator G commutes with d and d^* . Therefore, for any differential form u , we have

$$Gd(u) = dG(u), Gd^*(u) = d^*G(u). \quad (2.3)$$

From (2.3) and the definition of the Laplace–Beltrami operator Δ we obtain

$$\begin{aligned}\Delta G(u) &= (dd^* + d^*d)G(u) \\ &= dd^*G(u) + d^*dG(u) \\ &= Gdd^*(u) + Gd^*d(u) \\ &= G(dd^* + d^*d)(u) \\ &= G\Delta(u).\end{aligned}$$

Thus we complete the proof of Lemma 2.4. \square

Now we will give the following local L^s -norm estimate of composite operator $\Delta^k G^m$, which will be used in the proof of higher norm theorems.

Theorem 2.5 *Let $u \in L^s_{\text{loc}}(M, \Lambda^l)$ ($1 < s < \infty$, $l = 1, 2, \dots, n$) be a smooth differential form defined on M , G be Green's operator, and Δ be the Laplace–Beltrami operator. Then there exists a constant C , independent of u , such that*

$$\|\Delta^k G^m(u)\|_{s,B} \leq C\|u\|_{s,\sigma B} \quad (2.4)$$

for all balls B with $\sigma B \subset M$ and any positive integer $k \leq m$, where $\sigma > 1$.

Proof We prove this theorem in two steps: (1) First, let $k = m$. For any differential form u , from Lemma 2.4 we have $G\Delta(u) = \Delta G(u)$, and thus

$$\Delta^k G^k(u) = (\Delta G)^k(u). \quad (2.5)$$

Hence (2.4) is equivalent to

$$\|(\Delta G)^k(u)\|_{s,B} \leq C\|u\|_{s,\sigma B}. \quad (2.6)$$

Now we use mathematical induction to prove (2.6). Let $k = 1$. By Lemma 2.1 we have

$$\begin{aligned}\|\Delta G(u)\|_{s,B} &= \|(dd^* + d^*d)G(u)\|_{s,B} \\ &\leq \|dd^*G(u)\|_{s,B} + \|d^*dG(u)\|_{s,B} = C_1\|u\|_{s,\sigma_1 B},\end{aligned} \quad (2.7)$$

where $\sigma_1 B \subset M$ with $\sigma_1 > 1$. Assume that (2.6) holds for $k = k'$, $k' = 1, 2, \dots$, that is,

$$\|(\Delta G)^{k'}(u)\|_{s,B} \leq C_2\|u\|_{s,\sigma_2 B}, \quad (2.8)$$

where $\sigma_2 > 1$ is a constant such that $\sigma_2 B \subset M$. Now we prove that (2.6) holds for $k = k' + 1$. From (2.7) and (2.8) we have

$$\begin{aligned}\|(\Delta G)^{k'+1}(u)\|_{s,B} &= \|\Delta G((\Delta G)^{k'}(u))\|_{s,B} \\ &\leq C_3\|(\Delta G)^{k'}(u)\|_{s,\sigma_3 B} \leq C_4\|u\|_{s,\sigma_4 B},\end{aligned} \quad (2.9)$$

where $\sigma_4 > \sigma_3 > 1$ and $\sigma_4 B \subset M$. From (2.9) it follows that if $k = m$, then (2.4) holds.

(2) Next, let $k < m$. From $\Delta G(u) = G\Delta(u)$ we obtain

$$\Delta^k G^m(u) = G^{m-k}((\Delta G)^k(u)). \quad (2.10)$$

For any differential form u , from Lemma 2.1 we have

$$\|G(u)\|_{s,B} \leq C_5 \|u\|_{s,\sigma_5 B} \quad (2.11)$$

for some constant $\sigma_5 > 1$ with $\sigma_5 B \subset M$. Using mathematical induction and (2.11), we can easily prove that

$$\|G^{k'}(u)\|_{s,B} \leq C_6 \|u\|_{s,\sigma_6 B} \quad (2.12)$$

for any positive integer k' and some constant $\sigma_6 > 1$ such that $\sigma_6 B \subset M$. Combining (2.10), (2.12), and (2.6), we have

$$\begin{aligned} \|\Delta^k G^m(u)\|_{s,B} &= \|G^{m-k}((\Delta G)^k(u))\|_{s,B} \\ &\leq C_7 \|(\Delta G)^k(u)\|_{s,\sigma_6 B} \\ &\leq C_8 \|u\|_{s,\sigma_7 B} \end{aligned} \quad (2.13)$$

for some constants $\sigma_7 > \sigma_6 > 1$ such that $\sigma_7 B \subset M$. Estimate (2.13) shows that (2.4) holds for $k < m$. This completes the proof of Theorem 2.5. \square

Next, based on Theorem 2.5, we will prove local higher norm inequalities for the composite operator $\Delta^k G^m$ in two cases.

Theorem 2.6 *Let $u \in L^t_{\text{loc}}(M, \Lambda^l)$ be a smooth differential form on M , G be Green's operator, and Δ be the Laplace–Beltrami operator, $l = 1, 2, \dots, n$, $1 < t < n$. Then $\Delta^k G^m(u) \in L^s_{\text{loc}}(M, \Lambda^l)$ for any s such that $0 < s < nt/(n-t)$. Moreover, there exists a constant C , independent of u , such that*

$$\|\Delta^k G^m(u)\|_{s,B} \leq C|B|^{1/s+1/n-1/t} \|u\|_{t,\sigma B} \quad (2.14)$$

for all balls B with $\sigma B \subset M$ and $|B| > d_0$ and any positive integer $k < m$, where $\sigma > 1$ and $d_0 > 0$ are constants.

Proof We prove this theorem in the following two cases: (1) First, assume that the measure $|\{x \in B : |\Delta^k G^m(u) - (\Delta^k G^m(u))_B| > 0\}| = 0$. Then $\Delta^k G^m(u) = (\Delta^k G^m(u))_B$ almost everywhere in B . Thus $\Delta^k G^m(u)$ is a closed form and is a solution of A -harmonic equation (1.3). Hence Lemma 2.2 holds for $\Delta^k G^m(u)$. Replacing u by $\Delta^k G^m(u)$ in Lemma 2.2, we have

$$\|\Delta^k G^m(u)\|_{s,B} \leq C_1 |B|^{(t-s)/st} \|\Delta^k G^m(u)\|_{t,\sigma_1 B}, \quad (2.15)$$

where $\sigma_1 > 1$ is a constant such that $\sigma_1 B \subset M$. Since $|B| > d_0 > 0$, there exists a constant $C_2 > 0$ such that $\frac{1}{|B|^{1/n}} \leq C_2$. Combining (2.15) and Theorem 2.5, we obtain

$$\begin{aligned} \|\Delta^k G^m(u)\|_{s,B} &\leq C_1 |B|^{(t-s)/st} \|\Delta^k G^m(u)\|_{t,\sigma_1 B} \\ &\leq C_3 |B|^{(t-s)/st} \|u\|_{t,\sigma_2 B} \\ &= C_3 |B|^{\frac{1}{s} + \frac{1}{n} - \frac{1}{t}} \frac{1}{|B|^{1/n}} \|u\|_{t,\sigma_2 B} \\ &\leq C_4 |B|^{\frac{1}{s} + \frac{1}{n} - \frac{1}{t}} \|u\|_{t,\sigma_2 B}, \end{aligned} \quad (2.16)$$

where $\sigma_2 > \sigma_1 > 1$ with $\sigma_2 B \subset M$.

(2) Second, if the the measure $|\{x \in B : |\Delta^k G^m(u) - (\Delta^k G^m(u))_B| > 0\}| > 0$, then (2.1) holds for $\Delta^k G^m(u)$, and thus we have

$$\|\Delta^k G^m(u)\|_{nt/(n-t),B} \leq C_5 \|\Delta^k G^m(u) - (\Delta^k G^m(u))_B\|_{nt/(n-t),B}. \quad (2.17)$$

Note that $G\Delta(u) = \Delta G(u)$ and $k < m$. Then $d\Delta^k G^m(u) = dG(\Delta^k G^{m-1}(u))$ and $k \leq m-1$. Thus, combining Lemma 2.1 and Theorem 2.5, we have

$$\|d\Delta^k G^m(u)\|_{t,B} = \|dG(\Delta^k G^{m-1}(u))\|_{t,B} \leq C_6 \|\Delta^k G^{m-1}(u)\|_{t,\sigma_3 B} \leq C_7 \|u\|_{t,\sigma_4 B}, \quad (2.18)$$

where $\sigma_4 > \sigma_3 > 1$ are constants such that $\sigma_4 B \subset M$. Since $1 < t < n$, from Lemma 2.3 and (2.18) we have

$$\begin{aligned} &\|\Delta^k G^m(u) - (\Delta^k G^m(u))_B\|_{nt/(n-t),B} \\ &= \left(\int_B |\Delta^k G^m(u) - (\Delta^k G^m(u))_B|^{nt/(n-t)} dx \right)^{(n-t)/nt} \\ &\leq C_8 \left(\int_B |d\Delta^k G^m(u)|^t dx \right)^{1/t} \\ &= C_8 \|d\Delta^k G^m(u)\|_{t,B} \\ &\leq C_9 \|u\|_{t,\sigma_4 B}. \end{aligned} \quad (2.19)$$

By the monotonicity property of the L^t -space, we obtain the inequality

$$\left(\frac{1}{|B|} \int_B |\Delta^k G^m(u)|^s dx \right)^{1/s} \leq \left(\frac{1}{|B|} \int_B |\Delta^k G^m(u)|^{nt/(n-t)} dx \right)^{(n-t)/nt} \quad (2.20)$$

for any s such that $0 < s < nt/(n-t)$. Inequality (2.20) shows that

$$\|\Delta^k G^m(u)\|_{s,B} \leq |B|^{1/s+1/n-1/t} \|\Delta^k G^m(u)\|_{nt/(n-t),B}. \quad (2.21)$$

Combining (2.21), (2.17), and (2.19), we have

$$\|\Delta^k G^m(u)\|_{s,B} \leq C_{10} |B|^{1/s+1/n-1/t} \|u\|_{t,\sigma_4 B}. \quad (2.22)$$

From (2.16) and (2.22) we get (2.14) for $1 < t < n$. Thus we complete the proof of Theorem 2.6. \square

In Theorem 2.6, since $1 < t < n$, then $\frac{nt}{n-t} \rightarrow +\infty$ as $t \rightarrow n^-$. Since $0 < s < nt/(n-t)$, s can be greater than t , and thus the composite operator $\Delta^k G^m(u)$ has higher norm than the differential form u . Next, we prove (2.14) for $t \geq n$.

Theorem 2.7 *Let $u \in L^t_{\text{loc}}(M, \Lambda^l)$ be a smooth differential form on M , G be Green's operator, and Δ be the Laplace–Beltrami operator on M , $l = 1, 2, \dots, n$, $t \geq n$. Then $\Delta^k G^m(u) \in L^s_{\text{loc}}(M, \Lambda^l)$ for any constant $s > 0$. Moreover, there exists a constant C , independent of u , such that*

$$\|\Delta^k G^m(u)\|_{s,B} \leq C|B|^{1/s+1/n-1/t} \|u\|_{t,\sigma B} \quad (2.23)$$

for all balls B with $\sigma B \subset M$ and $|B| > d_0$ and any positive integer $k < m$, where $\sigma > 1$ and $d_0 > 0$ are constants.

Proof We prove this theorem in the following two cases: (1) First, if the measure $|\{x \in B : |\Delta^k G^m(u) - (\Delta^k G^m(u))_B| > 0\}| = 0$, then applying the same method as in the proof of Theorem 2.6, we can prove (2.23) for any ball B with $\sigma B \subset M$, where $\sigma > 1$ is a constant.

(2) Next, let the measure $|\{x \in B : |\Delta^k G^m(u) - (\Delta^k G^m(u))_B| > 0\}| > 0$. Select $p = \max\{1, s/t\}$ and $r = \frac{npt}{n+pt}$. Then $1 < r < n$. Since $t \geq n$, that is, $n - t \leq 0$, we have $r - t = \frac{t(p(n-t)-n)}{n+pt} < 0$, that is, $r < t$. Since $1 < r < n$, Lemma 2.3 holds for $\Delta^k G^m(u)$ and r . Thus replacing u and p by $\Delta^k G^m(u)$ and r , respectively, in Lemma 2.3, we have

$$\left(\int_B |\Delta^k G^m(u) - (\Delta^k G^m(u))_B|^{nr/(n-r)} dx \right)^{(n-r)/nr} \leq C_1 \left(\int_B |d\Delta^k G^m(u)|^r dx \right)^{1/r}. \quad (2.24)$$

From Lemmas 2.4 and 2.1 and Theorem 2.5 we obtain

$$\begin{aligned} \left(\int_B |d\Delta^k G^m(u)|^r dx \right)^{1/r} &= \left(\int_B |dG(\Delta^k G^{m-1}(u))|^r dx \right)^{1/r} \\ &\leq C_2 \left(\int_{\sigma_1 B} |\Delta^k G^{m-1}(u)|^r dx \right)^{1/r} \\ &\leq C_3 \left(\int_{\sigma_2 B} |u|^r dx \right)^{1/r} \end{aligned} \quad (2.25)$$

for all balls B with $\sigma_2 B \subset M$, where $\sigma_2 > \sigma_1 > 1$. Since $r < t$, applying the monotonicity property of the L^t space, we have

$$\left(\frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} |u|^r dx \right)^{1/r} \leq \left(\frac{1}{|\sigma_2 B|} \int_{\sigma_2 B} |u|^t dx \right)^{1/t}. \quad (2.26)$$

Inequality (2.26) is equivalent to

$$\left(\int_{\sigma_2 B} |u|^r dx \right)^{1/r} \leq C_4 |B|^{1/r-1/t} \left(\int_{\sigma_2 B} |u|^t dx \right)^{1/t}. \quad (2.27)$$

Since the measure $|\{x \in B : |\Delta^k G^m(u) - (\Delta^k G^m(u))_B| > 0\}| > 0$, estimates (2.1) hold for $\Delta^k G^m(u)$. In the second half of (2.1), we replace u and s by $\Delta^k G^m(u)$ and $nr/(n-r)$:

$$\begin{aligned} & \left(\int_B |\Delta^k G^m(u)|^{nr/(n-r)} dx \right)^{(n-r)/nr} \\ & \leq C_5 \left(\int_B |\Delta^k G^m(u) - (\Delta^k G^m(u))_B|^{nr/(n-r)} dx \right)^{(n-r)/nr}. \end{aligned} \quad (2.28)$$

Since $p = \max\{1, s/t\}$, we have $pt \geq s$. Since $nr/(n-r) = pt$, we have $nr/(n-r) \geq s$, and thus applying the monotonicity property of the L^t space, we have

$$\left(\frac{1}{|B|} \int_B |\Delta^k G^m(u)|^s dx \right)^{1/s} \leq \left(\frac{1}{|B|} \int_B |\Delta^k G^m(u)|^{nr/(n-r)} dx \right)^{(n-r)/nr}. \quad (2.29)$$

Combining (2.29), (2.28), (2.24), (2.25), and (2.27), we obtain

$$\begin{aligned} & \left(\int_B |\Delta^k G^m(u)|^s dx \right)^{1/s} \\ & \leq |B|^{1/s-(n-r)/nr} \left(\int_B |\Delta^k G^m(u)|^{nr/(n-r)} dx \right)^{(n-r)/nr} \\ & \leq C_5 |B|^{1/s-(n-r)/nr} \left(\int_B |\Delta^k G^m(u) - (\Delta^k G^m(u))_B|^{nr/(n-r)} dx \right)^{(n-r)/nr} \\ & \leq C_6 |B|^{1/s-(n-r)/nr+1/r-1/t} \left(\int_{\sigma_2 B} |u|^t dx \right)^{1/t} \\ & = C_6 |B|^{1/s+1/n-1/t} \left(\int_{\sigma_2 B} |u|^t dx \right)^{1/t}. \end{aligned} \quad (2.30)$$

Inequality (2.30) implies (2.23). Therefore we complete the proof of this theorem. \square

In Theorem 2.7, from the condition $t \geq n$ we have $\frac{1}{s} + \frac{1}{n} - \frac{1}{t} > 0$, which is also presented in Theorem 2.6, and thus combining Theorems 2.6 and 2.7, we easily obtain the following theorem for any $t > 1$.

Theorem 2.8 *Let $u \in L^t_{\text{loc}}(M, \Lambda^l)$ be a smooth differential form defined on M , G be Green's operator, and Δ be the Laplace–Beltrami operator defined on M , $l = 1, 2, \dots, n$, $t > 1$. Then $\Delta^k G^m(u) \in L^s_{\text{loc}}(M, \Lambda^l)$ for any constant $s > 0$ such that $\frac{1}{s} + \frac{1}{n} - \frac{1}{t} > 0$. Moreover, there exists a constant C , independent of u , such that*

$$\|\Delta^k G^m(u)\|_{s,B} \leq C |B|^{1/s+1/n-1/t} \|u\|_{t,\sigma B} \quad (2.31)$$

for all balls B with $\sigma B \subset M$ and $|B| > d_0$ and any positive integer $k < m$, where $\sigma > 1$ and $d_0 > 0$ are constants.

3 Global higher norm inequalities for composite operator $\Delta^k G^m$

In this section, based on Theorem 2.8, we will prove the global higher norm estimate for the composite operator $\Delta^k G^m(u)$ in any bounded domain $M \subset R^n$. Then we will establish the corresponding global higher norm estimate with Radon measure. In the following

proof of related theorems, we need the following modified Whitney cover in [25]; see [23] for more detail about Whitney covers.

Lemma 3.1 *Each $\Omega \subset \mathbb{R}^n$ has a modified Whitney cover of cubes $W = \{Q_i\}$ that satisfy*

$$\bigcup_i Q_i = \Omega,$$

$$\sum_{Q \in W} \chi_{\sqrt{5/4}Q} \leq N \cdot \chi_\Omega$$

for all $x \in \mathbb{R}^n$ and some $N > 1$, and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if Ω is a δ -John, then there is a distinguished cube $Q_0 \in W$ that can be connected with every cube $Q \in W$ by a chain of cubes Q_0, Q_1, \dots, Q_k from W and such that $Q \subset \rho Q_i$, $i = 1, 2, \dots, k$, for some $\rho = \rho(n, \delta)$.

Now we will give the global higher norm inequality for the composite operator $\Delta^k G^m(u)$ based on Theorem 2.8.

Theorem 3.2 *Let $u \in L^t_{\text{loc}}(M, \Lambda^l)$ be a smooth differential form defined on a bounded domain M , G be Green's operator, and Δ be the Laplace–Beltrami operator, $l = 1, 2, \dots, n$, $t > 1$. Then $\Delta^k G^m(u) \in L^s_{\text{loc}}(M, \Lambda^l)$ for any constant $s > 0$ such that $\frac{1}{s} + \frac{1}{n} - \frac{1}{t} > 0$. Moreover, there exists a constant C , independent of u , such that*

$$\|\Delta^k G^m(u)\|_{s,M} \leq C|M|^{1/s+1/n-1/t} \|u\|_{t,M} \quad (3.1)$$

for any positive integer $k < m$.

Proof From Lemma 3.1, we know that there exists a sequence of cubes $W = \{B_i\}$ such that $\bigcup_i B_i = M$ and $\sum_{B_i \in W} \chi_{\sqrt{5/4}B_i} \leq N \cdot \chi_M(x)$ for all $x \in M$, where $N > 1$ is some constant. Hence, for $u \in L^t_{\text{loc}}(M, \Lambda^l)$, using Theorem 2.8, we have

$$\begin{aligned} \|\Delta^k G^m(u)\|_{s,M} &\leq \sum_{B_i \in W} \|\Delta^k G^m(u)\|_{s,B_i} \\ &\leq \sum_{B_i \in W} C_1 |B_i|^{1/s+1/n-1/t} \|u\|_{t,\sigma B_i} \\ &\leq C_1 |M|^{1/s+1/n-1/t} \sum_{B_i \in W} \left(\int_{\sigma B_i} |u|^t dx \right)^{1/t} \\ &= C_1 |M|^{1/s+1/n-1/t} \sum_{B_i \in W} \left(\int_M |u|^t dx \cdot \chi_{\sigma B_i} \right)^{1/t} \\ &= C_1 |M|^{1/s+1/n-1/t} N \cdot \left(\int_M |u|^t dx \right)^{1/t} \\ &= C_1 N |M|^{1/s+1/n-1/t} \|u\|_{t,M} \\ &= C_2 |M|^{1/s+1/n-1/t} \|u\|_{t,M}, \end{aligned} \quad (3.2)$$

where $C_2 = C_1 N$ is independent of u and all B_i . Thus we complete the proof of Theorem 3.2. \square

In Theorem 3.2, if we assume that $s > t$, then Theorem 3.2 reduces to the global higher norm estimate for composite operator $\Delta^k G^m$. Next, we consider the following global norm comparison equipped with Radon measure based on Theorem 3.2.

Theorem 3.3 *Let $u \in L^t_{\text{loc}}(M, \Lambda^l)$ be a smooth differential form defined on a bounded domain M , G be Green's operator, Δ be the Laplace–Beltrami operator, and $\bar{h}_1(y)$ and $\bar{h}_2(y)$ be two continuous nonnegative functions defined on $(0, +\infty)$ with conditions: (1) $\lim_{y \rightarrow 0} \bar{h}_1(y) = 0$; (2) $\lim_{y \rightarrow 0} \bar{h}_2(y) = \infty$, $l = 1, 2, \dots, n$, $t > 1$. Then $\Delta^k G^m(u) \in L^s_{\text{loc}}(M, \Lambda^l)$ for any constant $s > 0$ such that $\frac{1}{s} + \frac{1}{n} - \frac{1}{t} > 0$. Moreover, there exists a constant C , independent of u , such that*

$$\|\Delta^k G^m(u)\|_{s, M, \nu_1} \leq C|M|^{1/s+1/n-1/t} \|u\|_{t, M, \nu_2} \quad (3.3)$$

for any positive integer $k < m$ and the Radon measure ν_1, ν_2 defined by

$$d\nu_1 = \bar{h}_1(\text{dist}(x, \partial M)) dx, \quad d\nu_2 = \bar{h}_2(\text{dist}(x, \partial M)) dx.$$

Proof From Theorem 3.2, we know that there exists a constant C_1 , independent of u , such that

$$\|\Delta^k G^m(u)\|_{s, M} \leq C_1|M|^{1/s+1/n-1/t} \|u\|_{t, M}. \quad (3.4)$$

Since $\lim_{y \rightarrow 0} \bar{h}_1(y) = 0$, for any small positive number ϵ , there exists $\delta(\epsilon) > 0$ such that $\bar{h}_1(\text{dist}(x, \partial M)) < \epsilon$ for all $x \in M$ with $\text{dist}(x, \partial M) < \delta$. Let $M' = \{x \in M, \text{dist}(x, \partial M) < \delta\}$ and $M'' = M - M'$. Then for all $x \in M''$, we have

$$\delta \leq \text{dist}(x, \partial M) < \text{diam}(M).$$

Therefore by the continuity and nonnegativity of \bar{h}_1 we have that there exists $C_2 > 0$ such that

$$0 < \bar{h}_1(\text{dist}(x, \partial M)) < C_2$$

for all $x \in M''$. Thus we have

$$\begin{aligned} & \|\Delta^k G^m(u)\|_{s, M, \nu_1} \\ &= \left(\int_M |\Delta^k G^m(u)|^s \cdot \bar{h}_1(\text{dist}(x, \partial M)) dx \right)^{\frac{1}{s}} \\ &\leq \left(\epsilon \int_{M'} |\Delta^k G^m(u)|^s dx + C_2 \int_{M''} |\Delta^k G^m(u)|^s dx \right)^{\frac{1}{s}} \\ &\leq C_3 \left(\int_M |\Delta^k G^m(u)|^s dx \right)^{\frac{1}{s}}, \end{aligned} \quad (3.5)$$

where $C_3 = \max\{\epsilon^{\frac{1}{s}}, C_2^{\frac{1}{s}}\}$. Combining (3.5) and (3.4), we have

$$\|\Delta^k G^m(u)\|_{s,M,v_1} \leq C_3 \|\Delta^k G^m(u)\|_{s,M} \leq C_4 |B|^{1/s+1/n-1/t} \|u\|_{t,M}. \quad (3.6)$$

Note that $\lim_{y \rightarrow 0} \frac{1}{h_2(y)} = 0$. Then there exists $\delta_1(\epsilon) > 0$ such that $\frac{1}{h_2(\text{dist}(x, \partial M))} < \epsilon$ for all $x \in M$ with $\text{dist}(x, \partial M) < \delta_1$. Let $M'_1 = \{x \in M, \text{dist}(x, \partial M) < \delta_1\}$ and $M''_1 = M - M'_1$. Then for all $x \in M''_1$, we have

$$\delta_1 \leq \text{dist}(x, \partial M) < \text{diam}(M).$$

Therefore by the continuity and nonnegativity of h_2 we have that there exists a constant $C_5 > 0$ such that

$$0 < \frac{1}{h_2(\text{dist}(x, \partial M))} < C_5$$

for all $x \in M''_1$. Therefore we obtain

$$\begin{aligned} \|u\|_{t,M} &= \left(\int_M |u|^t \frac{1}{h_2(\text{dist}(x, \partial M))} dv_2 \right)^{\frac{1}{t}} \\ &\leq \left(\epsilon \int_{M'_1} |u|^t dv_2 + C_5 \int_{M''_1} |u|^t dv_2 \right)^{\frac{1}{t}} \\ &\leq C_6 \left(\int_M |u|^t dv_2 \right)^{\frac{1}{t}} = C_6 \|u\|_{t,M,v_2}, \end{aligned} \quad (3.7)$$

where $C_6 = \max\{\epsilon^{\frac{1}{t}}, C_5^{\frac{1}{t}}\}$. By (3.6) and (3.7) we have

$$\|\Delta^k G^m(u)\|_{s,M,v_1} \leq C_7 |M|^{1/s+1/n-1/t} \|u\|_{t,M,v_2}, \quad (3.8)$$

where C_7 is independent of u . Thus we complete the proof of Theorem 3.3. \square

In Theorem 3.3, choosing $h_1(y) = y^p$ and $h_2(y) = y^{-q}$, $0 < p, q < \infty$, we easily obtain the following corollary.

Corollary 3.4 *Let $u \in L^t_{\text{loc}}(M, \Lambda^l)$ be a smooth differential form defined on a bounded domain M , G be Green's operator, and Δ be the Laplace–Beltrami operator, $l = 1, 2, \dots, n$, $t > 1$. Then $\Delta^k G^m(u) \in L^s_{\text{loc}}(M, \Lambda^l)$ for any constant $s > 0$. Moreover, there exists a constant C , independent of u , such that*

$$\begin{aligned} &\left(\int_M |\Delta^k G^m(u)|^s \cdot (\text{dist}(x, \partial M))^p dx \right)^{1/s} \\ &\leq C |M|^{1/s+1/n-1/t} \left(\int_M |u|^t \frac{1}{(\text{dist}(x, \partial M))^q} dx \right)^{1/t} \end{aligned} \quad (3.9)$$

for any positive integer $k < m$ and any real numbers $0 < p, q < \infty$.

4 Applications

In many cases, it is very difficult to give the norm estimate for a composite operator. In this section, we give two examples to obtain the upper bounds for the norm of the composite operator $\Delta^k G^m(u)$ as applications of Theorem 3.2.

Example 4.1 Let $M = \{(x_1, x_2) : x_1^2 + x_2^2 \leq a^2\} \subset \mathbb{R}^2$, and let u be the differential 1-form

$$u(x_1, x_2) = x_1 dx_1 - x_2 dx_2 \quad (4.1)$$

defined on M . Then $|M| = \pi a^2$, and u is a differential form satisfying the conditions of Theorem 3.2. Thus we can estimate $\|\Delta^k G^m(u)\|_{s,M}$ in terms of $\|u\|_{t,M}$, where s, t are two independent positive real numbers such that $t > 1$ and $1/s + 1/n - 1/t > 0$. For any $(x_1, x_2) \in M$, we have $|u| = \sqrt{x_1^2 + x_2^2}$. By polar coordinate transformation $x_1 = \rho \cos \theta$, $x_2 = \rho \sin \theta$ we obtain

$$\|u\|_{t,M} = \left(\int_M |u|^t dx \right)^{1/t} = \left(\int_0^{2\pi} d\theta \int_0^a \rho^{t+1} d\rho \right)^{1/t} = \left(\frac{2\pi a^{t+2}}{t+2} \right)^{1/t}. \quad (4.2)$$

Since $n = 2$, for any $s > 0$ such that $1/s + 1/n - 1/t > 0$, by Theorem 3.2 we have

$$\begin{aligned} \|\Delta^k G^m(u)\|_{s,M} &\leq C|M|^{1/s+1/n-1/t} \|u\|_{t,M} \\ &= C(\pi a^2)^{1/s+1/2-1/t} \left(\frac{2\pi a^{t+2}}{t+2} \right)^{1/t} \\ &= C(\pi a^2)^{1/s+1/2} \left(\frac{2a^t}{t+2} \right)^{1/t} \end{aligned} \quad (4.3)$$

for any positive integer $k < m$. Moreover, if we assume that $s = 1/2$ and $t = 1$ in (4.3), then $1/s + 1/n - 1/t > 0$, satisfying the condition of Theorem 3.2, and thus we have

$$\|\Delta^k G^m(u)\|_{1/2,M} \leq C \left(\frac{2}{3} \pi a^3 \right). \quad (4.4)$$

Example 4.2 Let $M = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq a^2\} \subset \mathbb{R}^3$ and $u(x_1, x_2, x_3)$ be a differential 2-form on M defined by

$$\begin{aligned} u &= \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} dx_2 \wedge dx_3 - \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} dx_1 \wedge dx_3 \\ &\quad + \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} dx_1 \wedge dx_2. \end{aligned} \quad (4.5)$$

Then $|M| = \frac{4}{3}\pi a^3$ and $|u| = 1$, and thus we obtain

$$\|u\|_{t,M} = \left(\int_M |u|^t dx \right)^{1/t} = \left(\int_M dx \right)^{1/t} = |M|^{1/t} = \left(\frac{4}{3}\pi a^3 \right)^{1/t} \quad (4.6)$$

for any $t > 1$. Applying Theorem 3.2, we have $\Delta^k G^m(u) \in L_{\text{loc}}^s(M)$ for any constant $s > 0$ such that $1/s + 1/n - 1/t > 0$. Thus we can further obtain the following upper bound for the

norm of the composite operator $\Delta^k G^m(u)$:

$$\begin{aligned}\|\Delta^k G^m(u)\|_{s,M} &\leq C|M|^{1/s+1/n-1/t}\|u\|_{t,M} \\ &= C\left(\frac{4}{3}\pi a^3\right)^{1/s+1/3-1/t}\left(\frac{4}{3}\pi a^3\right)^{1/t} \\ &= C\left(\frac{4}{3}\pi a^3\right)^{1/s+1/3}\end{aligned}\quad (4.7)$$

for any positive integer $k < m$.

Remark Examples 4.1 and 4.2 can be generalized to the n -dimensional space.

5 Conclusion

In this paper, we first give the local L^s norm estimate for the composite operators $\Delta^k G^m(u)$ in terms of the L^s norm of u with commuting Laplace–Beltrami operator Δ and Green’s operator G . At the same time, we also obtain the local and global L^s norms of $\Delta^k G^m(u)$ in terms of the L^t norm of differential forms u for any constant $s > 0$ such that $\frac{1}{s} + \frac{1}{n} - \frac{1}{t} > 0$. Then we establish a global higher norm estimate with Radon measure for composite operators $\Delta^k G^m$. At last, as applications of these results, we give two examples to estimate the higher norm of $\Delta^k G^m(u)$.

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Authors’ contributions

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