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The Crank–Nicolson finite element method for the 2D uniform transmission line equation

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Abstract

We develop the Crank–Nicolson finite element (CNFE) method for the two-dimensional (2D) uniform transmission line equation, study the stability and existence as well as error estimates for the CNFE solutions of the 2D uniform transmission line equation by strict theoretical approaches. We verify the correctness of the obtained theoretical results by means of numerical tests.

MSC: 65N12; 65N30; 65M15

Keywords: Crank–Nicolson finite element method; Uniform transmission line equation; Stability and existence; Error estimate; Numerical test

1 Introduction

Let Θ be an open bounded region in \mathbb{R}^2 . We consider the uniform transmission line equation in the region Θ :

$$\begin{cases} \sigma V_t + V_{tt} - \Delta V + \delta V = F(x, y, t), & (x, y) \in \Theta, t \in (0, T), \\ V(x, y, t) = V_0(x, y, t), & (x, y) \in \partial\Theta, t \in (0, T), \\ V(x, y, 0) = H_0(x, y), \quad V_t(x, y, 0) = H_1(x, y), & (x, y) \in \Theta, \end{cases} \quad (1)$$

where V represents the unknown voltage or current, $V_t = \partial V / \partial t$, $V_{tt} = \partial^2 V / \partial t^2$, $\Delta V = \partial^2 V / \partial x^2 + \partial^2 V / \partial y^2$, $F(x, y, t)$ represents the source term, $V_0(x, y, t)$ is the boundary value, $H_0(x, y)$ and $H_1(x, y)$ stand for initial values, T is the final moment, and $\sigma = RG(\hat{C}L)^{-1} > 0$ and $\delta = (\hat{C}R + LG)(\hat{C}L)^{-1} > 0$ due to G standing for the conductivity in the uniform transmission line, R for the impedance in the uniform transmission line, L for the inductance, and \hat{C} for the electric capacity in the uniform transmission line. For convenience, we will premise that $V_0(x, y, t) = 0$ in theoretical discussion.

The uniform transmission line equation known as the telegraph equation is a crucial physical equation and has largely extensive applications (see [1, 2]). Nevertheless, when its source term, boundary value, or initial values are complex, or its coefficients σ and δ are discontinuous, it has no genuine solution, so that we have to rely on numeric solutions.

The finite element (FE) method is one of most effective numerical methods and is used to solve many partial differential equations (see, e.g., [3–6]), whereas the Crank–Nicolson FE

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(CNFE) method has not been used to solve the uniform transmission line equation. Hence, in this paper, we intend to develop the CNFE method for the 2D uniform transmission line equation, analyze the stability and existence as well as errors for the CNFE solutions of the 2D uniform transmission line equation, and verify the correctness of the obtained theoretical results via numerical tests.

Although the collocation spectral method, space-time finite element method, and finite difference scheme (see [7–14]) have been used to solve the uniform transmission line equation, they are different from the CNFE method, whereas the CNFE method has more merits than the methods mentioned; for example, the CNFE method for the 2D uniform transmission line equation is unconditionally stable, resulting in that it can ensure that the numerical solution is absolutely convergent and can obtain optimal order error estimates for the CNFE solutions unmatched for the above methods.

The paper is organized as follows. In Sect. 2, we firstly construct the CNFE model for the 2D uniform transmission line equation and analyze the existence, uniqueness, stability, and convergence of the CNFE solutions. Afterward, in Sect. 3, we use some numerical tests to verify the correctness of the obtained theoretical results. Lastly, we summarize the main conclusions and give some prospect in Sect. 4.

2 The CNFE method for the 2D uniform transmission line equation

2.1 The weak form of the 2D uniform transmission line equation

We will use the classical Sobolev spaces and norms (see, e.g., [15, 16]). Let $\mathbb{U} = H_0^1(\Theta)$. Using Green’s formula, we can derive the following weak form for the 2D uniform transmission line equation (1).

Problem 1 Find $V \in H^2(0, T; \mathbb{U})$ such that

$$\begin{cases} (V_{tt}, \vartheta) + (\nabla V, \nabla \vartheta) + \sigma(V_t, \vartheta) + \delta(V, \vartheta) = (F, \vartheta), & \vartheta \in \mathbb{U}, \\ V(x, y, 0) = H_0(x, y), \quad V_t(x, y, 0) = H_1(x, y), & (x, y) \in \Theta, \end{cases} \tag{2}$$

where (\cdot, \cdot) stands for the inner product in $L^2(\Theta)$, and $a(\omega, v) = (\nabla \omega, \nabla v)$.

For Problem 1, we have the following conclusion of the existence, uniqueness, and stability of the weak solution.

Theorem 1 *If $F \in L^2(0, T; L^2(\Theta))$, $H_1 \in L^2(\Theta)$, and $H_0 \in H^1(\Theta)$, then the weak formulation (2) has a unique weak solution $V \in H^2(0, T; \mathbb{U})$ meeting the stability*

$$\|V_t\|_0 + \|V\|_1 \leq \tilde{C}(\|H_0\|_1 + \|H_1\|_0 + \|F\|_{L^2(H^{-1})}), \tag{3}$$

where $\tilde{C} = 2\sqrt{\max\{1, \delta, 1/(2\sigma)\} / \min\{1, \delta\}}$.

Proof Since (2) is a linear equation system with respect to unknown function V , it has a unique solution if and only if it has merely a zero solution as $F(x, y, t) = H_0(x, y) = H_1(x, y) = 0$.

Taking $\vartheta = V_t$ in the first equation in Problem 1, we get

$$\frac{d\|V_t\|_0^2}{2dt} + \frac{d\|\nabla V\|_0^2}{2dt} + \sigma\|V_t\|_0^2 + \delta\frac{d\|V\|_0^2}{2dt} = (F, V_t). \tag{4}$$

Integrating (4) from 0 to $t \in [0, T]$ and using the Hölder and Cauchy inequalities, we get

$$\begin{aligned} & \|V_t\|_0^2 + \|\nabla V\|_0^2 + 2\sigma \int_0^t \|V_t\|_0^2 dt + \delta \|V\|_0^2 \\ &= \|G\|_0^2 + \|\nabla H\|_0^2 + \delta \|H\|_0^2 + 2 \int_0^t (F, V_t) dt \\ &\leq \|G\|_0^2 + \|\nabla H\|_0^2 + \delta \|H\|_0^2 + \frac{1}{2\sigma} \int_0^t \|F\|_0^2 dt + 2\sigma \int_0^t \|V_t\|_0^2 dt. \end{aligned} \tag{5}$$

Thereupon, when $F(x, y, t) = H_0(x, y) = H_1(x, y) = 0$, we obtain $\|V\|_0 = \|\nabla V\|_0 = 0$, which implies $u = 0$, that is, the weak formulation (2) has a unique solution $V \in H_0^1(\Theta)$. Further, from (5) we obtain (3). This completes the proof of Theorem 1. \square

2.2 The CNFE method for the 2D uniform transmission line equation

To solve Problem 1 by the CNFE method, let \mathfrak{S}_h be a uniformly regular triangulation on $\bar{\Theta}$ (see, e.g., [16]). The FE subspace is defined as

$$\mathbb{U}_h = \{v_h \in H_0^1(\Theta) \cap C(\bar{\Theta}) : v_h|_K \in \mathbb{P}_l(K), K \in \mathfrak{S}_h\}, \tag{6}$$

where $\mathbb{P}_l(K)$ is formed by l th-degree polynomials on $K \in \mathfrak{S}_h$.

For integer $N > 0$, let $\Delta t = T/N$, and let V_h^n be the FE approximations to the solutions V in Problem 2 at $t_n = n\Delta t$ ($0 \leq n \leq N$), $\bar{\partial} V_h^n = (V_h^{n+1} - V_h^{n-1})/(2\Delta t)$, $\bar{\partial}^2 V_h^n = (V_h^{n+1} - 2V_h^n + V_h^{n-1})/\Delta t^2$, and $f^n = g(x, y, t_n)$. Then we can establish the following CNFE model for the 2D uniform transmission line equation.

Problem 2 Find $V_h^n \in \mathbb{U}_h$ such that

$$\begin{cases} (V_h^{n+1} - 2V_h^n + V_h^{n-1}, \vartheta_h) + \frac{\Delta t^2}{2} (\nabla V_h^{n+1} + \nabla V_h^{n-1}, \nabla \vartheta_h) \\ \quad + \frac{\sigma \Delta t}{2} (V_h^{n+1} - V_h^{n-1}, \vartheta_h) + \frac{\delta \Delta t^2}{2} (V_h^{n+1} + V_h^{n-1}, \vartheta_h) \\ = \Delta t^2 (F(t_n), \vartheta_h), \quad \vartheta_h \in \mathbb{U}_h, 1 \leq n \leq N - 1, \\ V_h^0(x, y) = R_h G(x, y), \quad V_h^1(x, y) = V_h^0 + 2\Delta t R_h H(x, y), \quad (x, y) \in \Theta, \end{cases} \tag{7}$$

where $F(t_n) = F(x, y, t_n)$, and $P_h : \mathbb{U} \rightarrow \mathbb{U}_h$ is the Ritz projection (see [16]).

When $g \in H^{l+1}(\Theta) \cap H_0^1(\Theta)$, the Ritz projection has the following boundedness and error estimates (see, e.g., [16]):

$$\|P_h g\|_0 \leq C \|g\|_0 \quad \text{and} \quad \|g - P_h g\|_0 \leq Ch^{l+1} \|g\|_{l+1}, \tag{8}$$

where $\|\cdot\|_0$ and $\|\cdot\|_l$ represent, respectively, the norms in $L^2(\Theta)$ and $H^l(\Theta)$, and $C > 0$ stands for a generic constant independent of Δt and h , different at different places.

Moreover, when $g \in H_0^1(\Omega)$, it satisfies the Poincaré inequality

$$C_p \|g\|_1 \leq |g|_1 \leq \|g\|_1$$

for some $C_p > 0$.

For Problem 2, we have the existence, uniqueness, and stability of CNFE solutions.

Theorem 2 *If $F \in L^2(0, T; L^2(\Theta))$, $H_0 \in H^1(\Theta)$, and $H_1 \in H^1(\Theta)$, then Problem 2 has a unique sequence of solutions $V_h^n \in \mathbb{U}_h$ ($n = 1, 2, \dots, K$) satisfying the following unconditionally stability:*

$$\begin{aligned} \|V_h^n\|_1 &\leq \left(\frac{8 + C_p^2 + \delta}{C_p^2 \min\{1, \delta\}}\right)^{1/2} (\|\nabla H_0\|_0 + \|\nabla H_1\|_0) \\ &\quad + \left(\frac{\Delta t}{\sigma \min\{1, \delta\}} \sum_{j=1}^n \|F(t_j)\|_0^2\right)^{1/2}. \end{aligned} \tag{9}$$

Proof Since scheme (7) is a linear equation system with respect to V_h^{n+1} , to prove that Problem 2 has a unique sequence of solutions, it suffices to check that it has merely zero solutions as $H_0(x, y) = H_1(x, y) = F(x, y, t) = 0$.

Choosing $\vartheta_h = V_h^{n+1} - V_h^{n-1}$ in the first equation in (7) and applying the Hölder and Cauchy inequalities, we get

$$\begin{aligned} &\|V_h^{n+1} - V_h^n\|_0^2 - \|V_h^n - V_h^{n-1}\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla V_h^{n+1}\|_0^2 - \|\nabla V_h^{n-1}\|_0^2) \\ &\quad + \frac{\sigma \Delta t}{2} \|V_h^{n+1} - V_h^{n-1}\|_0^2 + \frac{\delta \Delta t^2}{2} (\|V_h^{n+1}\|_0^2 - \|V_h^{n-1}\|_0^2) \\ &= \Delta t^2 (F(t_n), V_h^{n+1} - V_h^{n-1})_0 \\ &\leq \frac{\Delta t^3}{2\sigma} \|F(t_n)\|_0^2 + \frac{\sigma \Delta t}{2} \|V_h^{n+1} - V_h^{n-1}\|_0^2. \end{aligned} \tag{10}$$

Furthermore, we get

$$\begin{aligned} &\|V_h^{n+1} - V_h^n\|_0^2 - \|V_h^n - V_h^{n-1}\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla V_h^{n+1}\|_0^2 - \|\nabla V_h^{n-1}\|_0^2) \\ &\quad + \frac{\delta \Delta t^2}{2} (\|V_h^{n+1}\|_0^2 - \|V_h^{n-1}\|_0^2) \leq \frac{\Delta t^3}{2\sigma} \|F(t_n)\|_0^2. \end{aligned} \tag{11}$$

Summing (11) from 1 to n , by the second formula in (7) and the Poincaré inequality we get

$$\begin{aligned} &\|V_h^{n+1} - V_h^n\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla V_h^{n+1}\|_0^2 + \|\nabla V_h^n\|_0^2) + \frac{\delta \Delta t^2}{2} (\|V_h^{n+1}\|_0^2 + \|V_h^n\|_0^2) \\ &\leq \|V_h^1 - V_h^0\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla V_h^1\|_0^2 + \|\nabla V_h^0\|_0^2) \\ &\quad + \frac{\delta \Delta t^2}{2} (\|V_h^1\|_0^2 + \|V_h^0\|_0^2) + \frac{\Delta t^3}{2\sigma} \sum_{j=1}^n \|F(t_j)\|_0^2 \\ &\leq \frac{4\Delta t^2}{C_p^2} \|\nabla R_h H_0\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla R_h H_0\|_0^2 + \|\nabla R_h H_1\|_0^2) \\ &\quad + \frac{\delta \Delta t^2}{2C_p^2} (\|\nabla R_h H_0\|_0^2 + \|\nabla R_h H_1\|_0^2) + \frac{\Delta t^3}{2\sigma} \sum_{j=1}^n \|F(t_j)\|_0^2 \\ &\leq \left(\frac{4\Delta t^2}{C_p^2} + \frac{\Delta t^2}{2} + \frac{\delta \Delta t^2}{2C_p^2}\right) (\|\nabla H\|_0^2 + \|\nabla G\|_0^2) + \frac{\Delta t^3}{2\sigma} \sum_{j=1}^n \|F(t_j)\|_0^2 \end{aligned} \tag{12}$$

for $n = 1, 2, \dots, N - 1$. Thereupon, by (12) we have $\|\nabla V_h^n\| = \|V_h^n\| = 0$ ($n = 1, \dots, N$) as $H_0(x, y) = H_1(x, y) = F(x, y, t) = 0$, which shows that $V_h^n = 0$ ($n = 1, 2, \dots, N$). Hence Problem 2 has a unique set of solutions. By (12) we directly get (9), which finishes the proof of Theorem 2. \square

The set of solutions for Problem 2 has the following error estimates.

Theorem 3 *Under the hypotheses of Theorem 2, we have the following error estimates between the weak solution to Problem 1 and the CNFE solutions to Problem 2:*

$$\|V(t_n) - V_h^n\|_1 \leq C(\Delta t^2 + h^l), \quad 1 \leq n \leq N. \tag{13}$$

Proof As V_i and V_{tt} are, respectively, approximated by $(V^{n+1} - V^{n-1})/2\Delta t$ and $(V^{n+1} - 2V^n + V^{n-1})/\Delta t^2$, we get the following time semidiscrete scheme for Problem 1:

$$\begin{cases} (V^{n+1} - 2V^n + V^{n-1}, \vartheta) + \frac{\Delta t^2}{2}(\nabla V^{n+1} + \nabla V^{n-1}, \nabla \vartheta) \\ \quad + \frac{\sigma \Delta t}{2}(V^{n+1} - V^{n-1}, \vartheta) + \frac{\delta \Delta t^2}{2}(V^{n+1} + V^{n-1}, \vartheta) \\ = \Delta t^2(F(t_n), \vartheta), \quad v \in U, 1 \leq n \leq K - 1, \\ V^0(x, y) = H_0(x, y), \quad V^1(x, y) = H_1(x, y), \quad (x, y) \in \Theta. \end{cases} \tag{14}$$

Let $E_1^n = V(t_n) - V^n$, $E_2^n = V^n - R_h V^n$, and $e_3^n = R_h V^n - V_h^n$.

(1) Estimate for E_1^n .

Using Taylor’s formula to (2) at $t = t_n$, subtracting (14), and choosing $\vartheta = E_1^{n+1} - E_1^{n-1}$, by Green’s formula and the Hölder and Cauchy inequalities we get

$$\begin{aligned} & \|E_1^{n+1} - E_1^n\|_0^2 - \|E_1^n - E_1^{n-1}\|_0^2 + \frac{\Delta t^2}{2}(\|\nabla E_1^{n+1}\|_0^2 - \|\nabla E_1^{n-1}\|_0^2) \\ & \quad + \sigma \Delta t \|E_1^{n+1} - E_1^{n-1}\|_0^2 + \delta \Delta t^2(\|E_1^{n+1}\|_0^2 - \|E_1^{n-1}\|_0^2) \\ & = \frac{\Delta t^4}{12}(V_{ttt}(\xi_1^n), E_1^{n+1} - E_1^{n-1}) - \frac{\Delta t^4}{2}(\Delta V_{tt}(\xi_2^n), E_1^{n+1} - E_1^{n-1}) \\ & \quad + \frac{\sigma \Delta t^4}{6}(V_{ttt}(\xi_3^n), E_1^{n+1} - E_1^n) + \frac{\delta \Delta t^4}{2}(V_{tt}(\xi_2^n), E_1^{n+1} - E_1^{n-1}) \\ & \leq \sigma \Delta t \|E_1^{n+1} - E_1^{n-1}\|_0^2 + \frac{\Delta t^7}{144\sigma} \|V_{ttt}(\xi_1^n)\|_0^2 + \frac{\Delta t^7}{4\sigma} \|\Delta V_{tt}(\xi_2^n)\|_0^2 \\ & \quad + \frac{\sigma \Delta t^7}{36} \|V_{ttt}(\xi_3^n)\|_0^2 + \frac{\delta^2 \Delta t^7}{4\sigma} \|V_{tt}(\xi_2^n)\|_0^2, \end{aligned} \tag{15}$$

where $t_n \leq \xi_1^n, \xi_2^n, \xi_3^n \leq t_{n+1}$. Since $E_1^1 = E_1^0 = 0$, simplifying and summing (15) from 1 to n , we get

$$\begin{aligned} & 2\|E_1^{n+1} - E_1^n\|_0^2 + \Delta t^2(\|\nabla E_1^{n+1}\|_0^2 + \|\nabla E_1^n\|_0^2) + 2\delta \Delta t^2(\|E_1^{n+1}\|_0^2 + \|E_1^n\|_0^2) \\ & \leq C^2(V) \min\{1, 2\delta\} \Delta t^6, \end{aligned} \tag{16}$$

where $C^2(V) = \frac{1}{72\sigma \min\{1, 2\delta\}} [\|V_{ttt}(\xi_1^n)\|_0^2 + 18\|\Delta V_{tt}(\xi_2^n)\|_0^2 + 4\sigma^2 \|V_{ttt}(\xi_3^n)\|_0^2 + 36\delta^2 \|V_{tt}(\xi_2^n)\|_0^2]$. Furthermore, we get

$$\|E_1^n\|_1 \leq C(V) \Delta t^2. \tag{17}$$

(2) Estimate for E_2 .

The estimate of E_2 may be directly obtained from (8) as $V^n \in H^{l+1}(\Theta)$:

$$\|E_2^n\|_1 \leq Ch^l, \quad n = 1, 2, \dots, N. \tag{18}$$

(3) Estimate for $E_3 = R_h V^n - V_h^n$.

Subtracting Problem 2 from (14) and choosing $\vartheta = \vartheta_h \in U_h$, we get

$$\begin{aligned} & (V^{n+1} - V_h^{n+1} - 2(V^n - V_h^n) + V^{n-1} - V_h^{n-1}, \vartheta_h) \\ & + \frac{\Delta t^2}{2} (\nabla(V^{n+1} - V_h^{n+1}) + \nabla(V^{n-1} - V_h^{n-1}), \nabla \vartheta_h) \\ & + \frac{\sigma \Delta t}{2} (V^{n+1} - V_h^{n+1} - (V^{n-1} - V_h^{n-1}), \vartheta_h) \\ & + \frac{\delta \Delta t^2}{2} (V^{n+1} - V_h^{n+1} + V^{n-1} - V_h^{n-1}, \vartheta_h) = 0, \quad \vartheta_h \in U_h. \end{aligned} \tag{19}$$

Using the property of R_h , (8), (19), Taylor’s formula, and the Hölder and Cauchy inequalities, we get

$$\begin{aligned} & \|E_3^{n+1} - E_3^n\|_0^2 - \|E_3^n - E_3^{n-1}\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla E_3^{n+1}\|_0^2 - \|\nabla E_3^{n-1}\|_0^2) \\ & + \frac{\sigma \Delta t}{2} \|E_3^{n+1} - E_3^{n-1}\|_0^2 + \delta \Delta t^2 (\|E_3^{n+1}\|_0^2 - \|E_3^{n-1}\|_0^2) \\ & = (V^{n+1} - 2V^n + V^{n-1} - (V_h^{n+1} - 2V_h^n + V_h^{n-1}), E_3^{n+1} - E_3^{n-1}) \\ & + (R_h V^{n+1} - V^{n+1} - 2(R_h V^n - V^n) + (R_h V^{n-1} - V^{n-1}), E_3^{n+1} - E_3^{n-1}) \\ & + \frac{\Delta t^2}{2} (\nabla(V^{n+1} - V_h^{n+1}) + \nabla(V^{n-1} - V_h^{n-1}), \nabla(E_3^{n+1} - E_3^{n-1})) \\ & + \frac{\Delta t^2}{2} (\nabla(R_h V^{n+1} - V^{n+1}) + \nabla(R_h V^{n-1} - V^{n-1}), \nabla(E_3^{n+1} - E_3^{n-1})) \\ & + \frac{\delta \Delta t^2}{2} (V^{n+1} - V_h^{n+1} + V^{n-1} - V_h^{n-1}, E_3^{n+1} - E_3^{n-1}) \\ & + \frac{\delta \Delta t^2}{2} (R_h V^{n+1} - V^{n+1} + R_h V^{n-1} - V^{n-1}, E_3^{n+1} - E_3^{n-1}) \\ & = (R_h V^{n+1} - V^{n+1} - 2(R_h V^n - V^n) + (R_h V^{n-1} - V^{n-1}), E_3^{n+1} - E_3^{n-1}) \\ & + \frac{\delta \Delta t^2}{2} (R_h V^{n+1} - V^{n+1} + R_h V^{n-1} - V^{n-1}, E_3^{n+1} - E_3^{n-1}) \\ & \leq \frac{\sigma \Delta t}{2} \|E_3^{n+1} - E_3^{n-1}\|_0^2 + C \Delta t^3 h^{2(l+1)}, \quad n = 0, 1, \dots, N - 1. \end{aligned} \tag{20}$$

Noting that $E_3^1 = E_3^0 = 0$ and summing (20) from 1 to n , we get

$$\begin{aligned} & \|E_3^{n+1} - E_3^n\|_0^2 + \frac{\Delta t^2}{2} (\|\nabla E_3^{n+1}\|_0^2 + \|\nabla E_3^n\|_0^2) + \delta \Delta t^2 (\|E_3^{n+1}\|_0^2 + \|E_3^n\|_0^2) \\ & \leq C \Delta t^2 h^{2(l+1)}, \quad n = 1, 2, \dots, N. \end{aligned} \tag{21}$$

Thereupon, we get

$$\|E_3^n\|_1 \leq Ch^{l+1}, \quad n = 1, 2, \dots, N. \tag{22}$$

Uniting (17)–(18) and (22), we gain (13), which completes the proof of Theorem 3. \square

Remark 1 The order of error estimates in Theorem 3 is optimal. Theorems 2 explains that the CNFE method (Problem 2) for the 2D uniform transmission line equation has a unique set of solutions that is unconditionally stable, so that it is unconditionally convergent and continuously depends on the initial value and source term. This shows that Problem 2 is theoretically reliable and effective for settling the 2D uniform transmission line equation.

3 Numerical tests

We employ some numerical tests to verify the correctness of the theoretical consequences of the CNFE method (Problem 2) of the 2D uniform transmission line equation, which can settle out the genuine solution, but usually it has no genuine solution.

In the 2D uniform transmission line equation (1), we choose $\Theta = [-1, 1] \times [-1, 1]$, $\sigma = 2$, $\delta = 1$, $V_0(\pm 1, y, t) = \exp(-t)(1 - \cos 2\pi y)$ ($-1 \leq y \leq 1$ and $t \in [0, T)$), $V_0(x, \pm 1, t) = \exp(-t)(1 - \cos 2\pi x)$ ($-1 \leq x \leq 1$ and $t \in [0, T)$), $H_0(x, y) = 1 - \cos 2\pi y \cos 2\pi x$, $H_1(x, y) = \cos 2\pi y \cos 2\pi x - 1$, and $F(x, y, t) = 8\pi^2 \exp(-t) \cos 2\pi y \cos 2\pi x$. Thus we can find the following genuine solution for the uniform transmission line equation (1):

$$V(x, y, t) = \exp(-t)(1 - \cos 2\pi y \cos 2\pi x), \quad (x, y) \in [-1, 1] \times [-1, 1], t \in [0, T).$$

When choosing time step $\Delta t = 0.01$ and adopting the uniformly regular triangulation with $h = 0.01$ and the $\mathbb{P}_2(K)$ with 2nd-degree polynomials on every triangular element, by Theorem 3 the theoretical errors between the genuine solution and the CNFE solutions V_h^k ($k = 1, 2, \dots, N$) should be $O(10^{-4})$.

Employing the CNFE method (Problem 3), we can compute out the CNFE solutions at $t = 0, 0.3, 0.6, 0.9$, depicted in Figs. 1, 2, 3, 4(a), respectively. The genuine solutions at the same time nodes are depicted in Figs. 1, 2, 3, 4(b), respectively. Each pair of plots in Figs. 1, 2, 3, 4 is almost the same. The CNFE solutions after $t = 0.9$ tend to be stable, which agree with cases of voltage or current in actual uniform transmission line.

Table 1 shows the errors between the genuine solutions and the CNFE solutions and CPU run time for $t = 0.3$, $t = 0.6$, and $t = 0.9$, respectively, which verifies that the correctness of the theoretical results due to both theoretical and numerical errors reaching

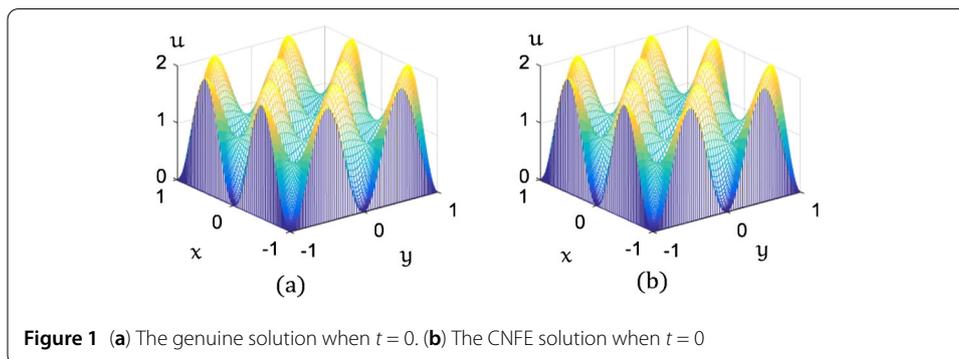
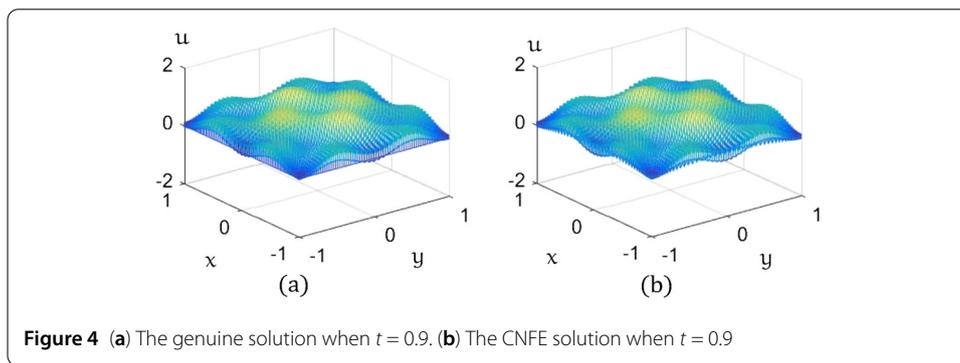
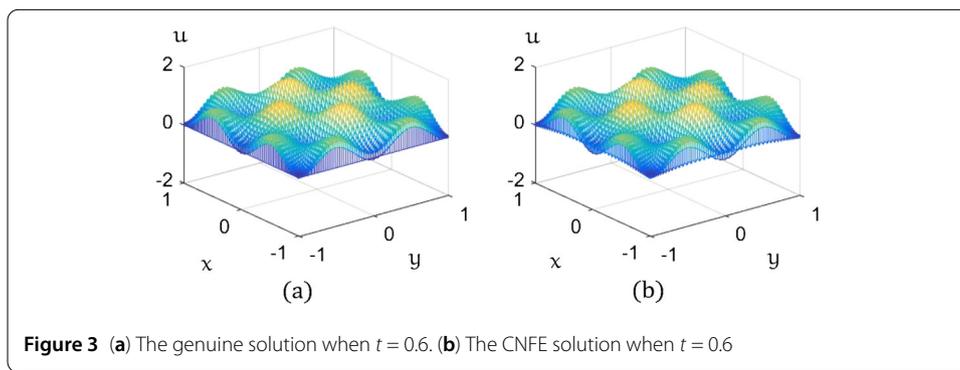
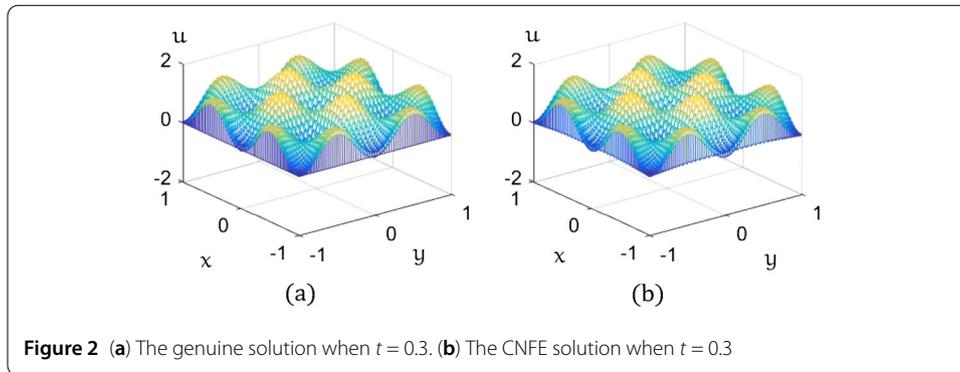


Figure 1 (a) The genuine solution when $t = 0$. (b) The CNFE solution when $t = 0$



$O(10^{-4})$, signifies that the CNFE method is effective and reliable for calculating the 2D uniform transmission line equation. In Table 1, we also give the errors between the genuine and FE solutions and CPU run times when $t = 0.3$, $t = 0.6$, and $t = 0.9$ using the following FE method:

Problem 3 Find $\tilde{V}_h^n \in \mathbb{U}_h$ such that

$$\begin{cases} (\tilde{V}_h^{n+1} - 2\tilde{V}_h^n + \tilde{V}_h^{n-1}, \vartheta_h) + \Delta t^2(\nabla \tilde{V}_h^n, \nabla \vartheta_h) + \sigma \Delta t(\tilde{V}_h^n - \tilde{V}_h^{n-1}, \vartheta_h) \\ \quad + \delta \Delta t^2(\tilde{V}_h^n, \vartheta_h) = \Delta t^2(F(t_n), \vartheta_h), \quad \vartheta_h \in \mathbb{U}_h, 1 \leq n \leq N - 1, \\ \tilde{V}_h^0(x, y) = R_h G(x, y), \quad \tilde{V}_h^1(x, y) = \tilde{V}_h^0 + \Delta t R_h H(x, y), \quad (x, y) \in \Theta, \end{cases} \tag{23}$$

where $F(t_n) = F(x, y, t_n)$ and $P_h : \mathbb{U} \rightarrow \mathbb{U}_h$ is the Ritz projection.

Table 1 The CPU elapsed time and the relative errors of the FE and CNFE solutions

t	n	FE method		CNFE method	
		$\ V(t_n) - \tilde{V}_h^n\ _1$	CPU run time	$\ V(t_n) - V_h^n\ _1$	CPU run time
0.3	300	1.413276e-2	1.693 second	4.051625e-5	1.791 second
0.6	600	3.232292e-2	5.897 second	4.371744e-4	6.127 second
0.9	900	5.482546e-2	11.825 second	4.581633e-4	11.982 second

It is readily proven that Problem 3 is conditionally stable, so that it is conditionally convergent, and its FE solutions only have the first-order accuracy about time, that is, we have the following error estimates:

$$\|V(t_n) - \tilde{V}_h^n\|_1 \leq C(\Delta t + h^l), \quad n = 1, 2, \dots, N. \quad (24)$$

Therefore, if $\Delta t = 0.01$ and $l = 2$ (also adopting second-degree elements), then the theoretical errors are of order $O(10^{-2})$ only, which are consistent with numerical errors; see Table 1. It is further shown that the CNFE method (i.e., Problem 2) has more merits than the FE method (i.e., Problem 3) for settling the 2D uniform transmission line equation.

4 Conclusions and expectation

In our study, we have developed the CNFE method for the 2D uniform transmission line equation and analyzed the existence, uniqueness, stability, and errors for the CNFE solutions. We have also adopted some numeric tests to confirm the correctness for the CNFE method. It has been shown, by comparing with the usual FE method, that the CNFE method has more merits. Furthermore, it is revealed that the CNFE method is very effective for settling the 2D uniform transmission line equation.

In spite of the fact that we have only concerned with the CNFE method for the 2D uniform transmission line equation, the CNFE method may be applied to settle the three-dimensional uniform transmission line equations or the uniform transmission line equation with complicated geometrical regions.

Although the CNFE method has many merits, such as unconditional stability, unconditional convergence, and the second-order accuracy in time, it has unknowns (freedom of degrees). When adopting the $\mathbb{P}_2(K)$ with 2nd-degree polynomials on every triangular element in Sect. 3, we may reckon that the CNFE method has about $4 \times 4 \times 10^4$ unknowns in the FE method (see [16, Lemma 1.30]). Provided that it is applied for settling big data processing in artificial intelligence and/or computational linguistics, there will be more than tens of millions unknown numbers. Fortunately, a proper orthogonal decomposition (POD) technique may be used to reduce the unknowns in the CNFE method, which can be used to reduce many numerical methods (see, e.g., [17–22]). Our future work is reducing the number of unknowns in the CNFE method by the POD technique.

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Availability of data and materials

The authors declare that all data and material in the paper are available and veritable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors wrote, read, and approved the final manuscript.

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