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Refinements of the integral form of Jensen's and the Lah–Ribarič inequalities and applications for Csiszár divergence

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Abstract

In this paper, we give refinements of the integral form of Jensen's inequality and the Lah–Ribarič inequality. Using these results, we obtain a refinement of the Hölder inequality and a refinement of some inequalities for integral power means and quasiarithmetic means. We also give applications in information theory, namely, we give some interesting estimates for the integral Csiszár divergence and its important particular cases.

MSC: 26D15; 94A15

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1 Introduction

Let I be an interval in $\mathbb{R}$, and let $f: I \to \mathbb{R}$ a convex function. If $\boldsymbol{x} = (x_1, \dots, x_n)$ is any n-tuple in I^n and $\boldsymbol{p} = (p_1, \dots, p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$, then the well-known *Jensen inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \tag{1.1}$$

holds (see [6] or, e.g., [16, p. 43]). If f is strictly convex, then (1.1) is strict unless $x_i = c$ for all $i \in \{j : p_j > 0\}$.

Jensen's inequality is probably the most important inequality: it has many applications in mathematics and statistics, and some other well-known inequalities are its particular cases (such as Cauchy's inequality, Hölder's inequality, A–G–H inequality, etc.).

One of many generalizations of the Jensen inequality is its integral form (see [1, 7], or, e.g., [8]).

Theorem 1.1 (Integral form of Jensen's inequality) *Let $g: [a,b] \to \mathbb{R}$ be an integrable function, and let $p: [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an* *interval I that includes the image of g, then*

$$f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)\,dt\right) \le \frac{1}{P(b)}\int_a^b p(t)f\bigl(g(t)\bigr)\,dt, \tag{1.2}$$

where

$$P(t) = \int_a^t p(x)\,dx.$$

Our first main result is a refinement of inequality (1.2).

Strongly related to Jensen's inequality is the Lah–Ribarič inequality (see [11] or, e.g., [13, p. 9]). Its integral form is given in the following theorem.

Theorem 1.2 (Integral form of the Lah–Ribarič inequality) *Let $g\colon [a,b] \to \mathbb{R}$ be an integrable function such that $m \le g(t) \le M$ for $t \in [a,b]$, $m < M$, and let $p\colon [a,b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I such that $[m,M] \subseteq I$, then*

$$\frac{1}{P(b)}\int_a^b p(t)f\bigl(g(t)\bigr)\,dt \le \frac{M-\bar{g}}{M-m}f(m) + \frac{\bar{g}-m}{M-m}f(M), \tag{1.3}$$

where P is given as before, and

$$\bar{g} = \frac{\int_a^b p(t)g(t)\,dt}{P(b)}.$$

Our second main result is a refinement of inequality (1.3).

Another famous inequality established for the class of convex functions is the Hermite–Hadamard inequality. This double inequality, which was first discovered by Hermite in 1881, is stated as follows (see, e.g., [16, p. 137]). Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

This result was later incorrectly attributed to Hadamard, who apparently was not aware of Hermite's discovery, and today, when relating to (1.4), we use both names.

This result can be improved by applying (1.4) on each subinterval $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b]$, and the following result is obtained (see [14, p. 37]):

$$f\left(\frac{a+b}{2}\right) \le l \le \frac{1}{b-a}\int_a^b f(x)\,dx \le L \le \frac{f(a)+f(b)}{2}, \tag{1.5}$$

where $l = \frac{1}{2}(f(\frac{3b+a}{4}) + f(\frac{b+3a}{2}))$ and $L = \frac{1}{2}(f(\frac{b+a}{2}) + \frac{f(a)+f(b)}{2})$.

The following improvement of (1.5) is given in [2].

Theorem 1.3 *Let $f\colon I \to \mathbb{R}$ be a convex function on I. Then for all $\lambda \in [0,1]$ and $a, b \in I$, we have*

$$f\left(\frac{a+b}{2}\right) \le l(\lambda) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le L(\lambda) \le \frac{f(a)+f(b)}{2}, \tag{1.6}$$

where

$$l(\lambda) = \lambda f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}\bigl(f\bigl(\lambda b + (1-\lambda)a\bigr) + \lambda f(a) + (1-\lambda)f(b)\bigr).$$

Inequality (1.6) for $\lambda = \frac{1}{2}$ gives inequality (1.5). Further improvement was given in [3].

Theorem 1.4 *Let $I \subseteq \mathbb{R}$ be an interval, and let $f: I \to \mathbb{R}$ be a convex function. Let $\Phi: [a,b] \to I$ be such that $f \circ \Phi$ is also convex, where $a < b$. Then for $n \in \mathbb{N}$, $\lambda_0 = 0$, $\lambda_{n+1} = 1$, and arbitrary $0 \le \lambda_1 \le \cdots \le \lambda_n \le 1$, we have*

$$f\left(\frac{1}{b-a}\int_a^b \Phi(x)\,dx\right) \le l(\lambda_1,\dots,\lambda_n) \le \frac{1}{b-a}\int_a^b f\circ\Phi(x)\,dx \tag{1.7}$$

$$\le L(\lambda_1,\dots,\lambda_n) \le \frac{f\circ\Phi(a) + f\circ\Phi(b)}{2}, \tag{1.8}$$

where

$$l(\lambda_1,\dots,\lambda_n) = \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k) f\left(\frac{1}{(\lambda_{k+1}-\lambda_k)(b-a)}\int_{(1-\lambda_k)a+\lambda_k b}^{(1-\lambda_{k+1})a+\lambda_{k+1}b} \Phi(x)\,dx\right)$$

and

$$L(\lambda_1,\dots,\lambda_n) = \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\frac{f\circ\Phi((1-\lambda_k)a+\lambda_k b) + f\circ\Phi((1-\lambda_{k+1})a+\lambda_{k+1}b)}{2}.$$

Applying the previous theorem to $\Phi(x) = x$ and $n = 1$, we get inequality (1.6).

We also give a refinement of the Hermite–Hadamard inequality. In the last section, we give some interesting estimates for the integral Csiszár divergence and for its important particular cases.

2 New refinements

Our first result is a refinement of the integral form of the Jensen inequality (1.2).

Theorem 2.1 *Let g be an integrable function defined on an interval $[a,b]$, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. If f is a convex function given on an interval I that includes the image of g, then*

$$f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)\,dt\right) \le \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} p(t)\,dt\right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)\,dt}{\int_{a_{i-1}}^{a_i} p(t)\,dt}\right)$$

$$\le \frac{1}{P(b)}\int_a^b p(t)f\bigl(g(t)\bigr)\,dt, \tag{2.1}$$

where $p\colon [a,b] \to \mathbb{R}$ is a nonnegative function, and

$$P(t) = \int_a^t p(x)\,dx.$$

Proof Let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then we have (using Jensen's inequality)

$$\begin{aligned} f\left(\frac{1}{P(b)}\int_a^b p(t)g(t)\,dt\right) &= f\left(\frac{1}{P(b)}\sum_{i=1}^n \int_{a_{i-1}}^{a_i} p(t)g(t)\,dt\right) \\ &= f\left(\frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(t)\,dt\right)\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)\,dt}{\int_{a_{i-1}}^{a_i} p(t)\,dt}\right) \\ &\le \frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(t)\,dt\right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)g(t)\,dt}{\int_{a_{i-1}}^{a_i} p(t)\,dt}\right), \end{aligned}$$

which is the left-hand side of (2.1).

Now we will use inequality (1.2) on each subinterval $[a_{i-1}, a_i]$:

$$\begin{aligned} &\frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(t)\,dt\right) f\left(\frac{1}{\int_{a_{i-1}}^{a_i} p(t)\,dt}\int_{a_{i-1}}^{a_i} p(t)g(t)\,dt\right) \\ &\quad \le \frac{1}{P(b)}\sum_{i=1}^n \left(\int_{a_{i-1}}^{a_i} p(t)\,dt\right)\frac{1}{\int_{a_{i-1}}^{a_i} p(t)\,dt}\int_{a_{i-1}}^{a_i} p(t)f\big(g(t)\big)\,dt, \end{aligned}$$

which is the right-hand side of (2.1). □

The next result is a refinement of the integral form of the Lah–Ribarič inequality (1.3). We need the following lemma.

Lemma 2.2 *If f is a convex function on an interval I, then for $a, b, u, c, d \in I$ such that $a \le b \le u \le c \le d$, $b < c$, we have*

$$\frac{c-u}{c-b}f(b) + \frac{u-b}{c-b}f(c) \le \frac{d-u}{d-a}f(a) + \frac{u-a}{d-a}f(d).$$

Proof We can write

$$\begin{aligned} b &= \frac{d-b}{d-a}a + \frac{b-a}{d-a}d, \\ c &= \frac{d-c}{d-a}a + \frac{c-a}{d-a}d, \end{aligned}$$

and since f is convex,

$$\begin{aligned} f(b) &\le \frac{d-b}{d-a}f(a) + \frac{b-a}{d-a}f(d), \\ f(c) &\le \frac{d-c}{d-a}f(a) + \frac{c-a}{d-a}f(d). \end{aligned}$$

Now we have

$$\begin{aligned}
&\frac{c-u}{c-b}f(b)+\frac{u-b}{c-b}f(c)\\
&\quad\le\frac{c-u}{c-b}\left[\frac{d-b}{d-a}f(a)+\frac{b-a}{d-a}f(d)\right]+\frac{u-b}{c-b}\left[\frac{d-c}{d-a}f(a)+\frac{c-a}{d-a}f(d)\right]\\
&\quad=\frac{d-u}{d-a}f(a)+\frac{u-a}{d-a}f(d).
\end{aligned}$$

□

Theorem 2.3 *Let g be an integrable function defined on an interval $[a,b]$, and let $a_0,a_1,\dots,a_{n-1},a_n$ be such that $a=a_0<a_1<\cdots<a_{n-1}<a_n=b$ and $m_i\le g(t)\le M_i$ for $t\in[a_{i-1},a_i]$, $m_i<M_i$, $i=1,\dots,n$, $m=\min_{i\in\{1,\dots,n\}}m_i$, and $M=\max_{i\in\{1,\dots,n\}}M_i$. If f is a convex function given on an interval I that includes the image of g, then*

$$\begin{aligned}
&\frac{1}{P(b)}\int_a^b p(t)f\bigl(g(t)\bigr)\,dt\\
&\quad\le\frac{1}{P(b)}\sum_{i=1}^{n}p_i\left[\frac{M_i-\bar{g}_i}{M_i-m_i}f(m_i)+\frac{\bar{g}_i-m_i}{M_i-m_i}f(M_i)\right]\\
&\quad\le\frac{M-\bar{g}}{M-m}f(m)+\frac{\bar{g}-m}{M-m}f(M),
\end{aligned}\tag{2.2}$$

where $p\colon[a,b]\to\mathbb{R}$ is nonnegative function,

$$P(t)=\int_a^t p(x)\,dx,$$

and $\bar{g},\bar{g}_i,p_i$ are defined as

$$\bar{g}=\frac{\int_a^b p(t)g(t)\,dt}{P(b)},\qquad\bar{g}_i=\frac{\int_{a_{i-1}}^{a_i}p(t)g(t)\,dt}{\int_{a_{i-1}}^{a_i}p(t)\,dt},\qquad p_i=\int_{a_{i-1}}^{a_i}p(t)\,dt.$$

Proof We will use (1.3) on each subinterval $[a_{i-1},a_i]$:

$$\begin{aligned}
&\frac{1}{P(b)}\int_a^b p(t)f\bigl(g(t)\bigr)\,dt\\
&\quad=\frac{1}{P(b)}\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i}p(t)f\bigl(g(t)\bigr)\,dt\\
&\quad\le\frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i}p(t)\,dt\right)\left[\frac{M_i-\frac{\int_{a_{i-1}}^{a_i}p(t)g(t)\,dt}{\int_{a_{i-1}}^{a_i}p(t)\,dt}}{M_i-m_i}f(m_i)+\frac{\frac{\int_{a_{i-1}}^{a_i}p(t)g(t)\,dt}{\int_{a_{i-1}}^{a_i}p(t)\,dt}-m_i}{M_i-m_i}f(M_i)\right],
\end{aligned}$$

which is the left-hand side of inequality (2.2).

Using $m\le m_i\le\bar{g}_i\le M_i\le M, m<M, m_i<M_i$, and Lemma 2.2, we get

$$\begin{aligned}
&\frac{1}{P(b)}\sum_{i=1}^{n}p_i\left[\frac{M_i-\bar{g}_i}{M_i-m_i}f(m_i)+\frac{\bar{g}_i-m_i}{M_i-m_i}f(M_i)\right]\\
&\quad\le\frac{1}{P(b)}\sum_{i=1}^{n}\left[\frac{p_iM-\int_{a_{i-1}}^{a_i}p(t)g(t)\,dt}{M-m}f(m)+\frac{\int_{a_{i-1}}^{a_i}p(t)g(t)\,dt-p_im}{M-m}f(M)\right]
\end{aligned}$$

$$= \frac{1}{P(b)}\left[\frac{\sum_{i=1}^{n} p_i M - \sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)g(t)\,dt}{M-m} f(m) + \frac{\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} p(t)g(t)\,dt - \sum_{i=1}^{n} p_i m}{M-m} f(M)\right]$$

$$= \frac{M - \frac{\int_a^b p(t)g(t)\,dt}{P(b)}}{M-m} f(m) + \frac{\frac{\int_a^b p(t)g(t)\,dt}{P(b)} - m}{M-m} f(M),$$

which is the right-hand side of (2.2). □

Remark 2.4 If we set $p(t) = 1$ in Theorem 2.1, then we get (1.7) in the form

$$f\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \le \frac{1}{b-a}\sum_{i=1}^{n}(a_i - a_{i-1}) f\left(\frac{\int_{a_{i-1}}^{a_i} g(t)\,dt}{a_i - a_{i-1}}\right) \le \frac{1}{b-a}\int_a^b f\bigl(g(t)\bigr)\,dt.$$

In particular, for $g(t) = t$, this gives

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\sum_{i=1}^{n}(a_i - a_{i-1}) f\left(\frac{a_{i-1}+a_i}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt,$$

which is a refinement of the left-hand side of (1.6).

Analogously, from Theorem 2.3 we have (for $p(t) = 1$)

$$\frac{1}{b-a}\int_a^b f\bigl(g(t)\bigr)\,dt \le \frac{1}{b-a}\sum_{i=1}^{n}(a_i - a_{i-1})\left[\frac{M_i - \frac{\int_{a_{i-1}}^{a_i} g(t)\,dt}{a_i - a_{i-1}}}{M_i - m_i} f(m_i) + \frac{\frac{\int_{a_{i-1}}^{a_i} g(t)\,dt}{a_i - a_{i-1}} - m_i}{M_i - m_i} f(M_i)\right] \le \frac{M - \frac{\int_a^b g(t)\,dt}{b-a}}{M-m} f(m) + \frac{\frac{\int_a^b g(t)\,dt}{b-a} - m}{M-m} f(M),$$

and for $g(t) = t, m_i = a_{i-1}, M_i = a_i$, we get

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{b-a}\sum_{i=1}^{n}(a_i - a_{i-1})\left[\frac{a_i - \frac{a_{i-1}+a_i}{2}}{a_i - a_{i-1}} f(a_{i-1}) + \frac{\frac{a_{i-1}+a_i}{2} - a_{i-1}}{a_i - a_{i-1}} f(a_i)\right] = \frac{1}{b-a}\sum_{i=1}^{n}(a_i - a_{i-1})\frac{f(a_{i-1}) + f(a_i)}{2} \le \frac{f(a)+f(b)}{2},$$

which is a refinement of the right-hand side of (1.6).

Using our main result, we give a refinement of the Hölder inequality (more about the Hölder inequality see [16]).

Corollary 2.5 *Let $p,q\in\mathbb{R}$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Let w,g_1, and g_2 be nonnegative functions defined on $[a,b]$ such that $wg_1^p, wg_2^q, wg_1g_2\in L^1([a,b])$.*

(i) *If $p>1$, then*

$$\begin{aligned}
&\int_a^b w(t)g_1(t)g_2(t)\,dt \\
&\quad\le \left(\int_a^b w(t)g_2^q(t)\,dt\right)^{\frac{1}{q}} \\
&\qquad\times\left(\sum_{i=1}^n\left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\right)^{1-p}\left(\int_{a_{i-1}}^{a_i} w(t)g_1(t)g_2(t)\,dt\right)^p\right)^{\frac{1}{p}} \\
&\quad\le \left(\int_a^b w(t)g_1^p(t)\,dt\right)^{\frac{1}{p}}\left(\int w(t)g_2^q(t)\,dt\right)^{\frac{1}{q}}.
\end{aligned}$$

(ii) *If $p<1, p\neq 0$, then*

$$\begin{aligned}
&\left(\int_a^b w(t)g_1^p(t)\,dt\right)^{\frac{1}{p}}\left(\int w(t)g_2^q(t)\,dt\right)^{\frac{1}{q}} \\
&\quad\le \sum_{i=1}^n\left(\int_{a_{i-1}}^{a_i} w(t)g_1^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\right)^{\frac{1}{q}} \\
&\quad\le \int_a^b w(t)g_1(t)g_2(t)\,dt.
\end{aligned}$$

Proof For the case $p>1$, we use Theorem 2.1 with $p(t)=w(t)g_2^q(t)$, $g(t)=g_1(t)g_2^{-\frac{q}{p}}$, and the function $f(x)=x^p$, which is convex for $x>0, p>1$. From (2.1) we get

$$\begin{aligned}
&\left(\frac{1}{\int_a^b w(t)g_2^q(t)\,dt}\int_a^b w(t)g_2^q(t)g_1(t)g_2^{-\frac{q}{p}}\,dt\right)^p \\
&\quad\le \frac{1}{\int_a^b w(t)g_2^q(t)\,dt}\sum_{i=1}^n\left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\right)\left(\frac{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)g_1(t)g_2^{-\frac{q}{p}}(t)\,dt}{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt}\right)^p \\
&\quad\le \frac{1}{\int_a^b w(t)g_2^q(t)\,dt}\int_a^b w(t)g_2^q(t)\big(g_1(t)g_2^{-\frac{q}{p}}(t)\big)^p\,dt.
\end{aligned}$$

Using $q-\frac{q}{p}=1$, multiplying by $\int_a^b w(t)g_2^q(t)\,dt$, and taking the power $\frac{1}{p}$, we have

$$\begin{aligned}
&\left(\int_a^b w(t)g_2^q(t)\,dt\right)^{\frac{1}{p}-1}\left(\int_a^b w(t)g_1(t)g_2(t)\,dt\right) \\
&\quad\le \left(\sum_{i=1}^n\left(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\right)^{1-p}\left(\int_{a_{i-1}}^{a_i} w(t)g_1(t)g_2(t)\,dt\right)^p\right)^{\frac{1}{p}} \\
&\quad\le \left(\int_a^b w(t)g_1^p(t)\,dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Now multiplying by $(\int_a^b w(t)g_2^q(t)\,dt)^{\frac{1}{q}}$, we get

$$\begin{aligned}
&\int_a^b w(t)g_1(t)g_2(t)\,dt \\
&\quad\le \biggl(\int_a^b w(t)g_2^q(t)\,dt\biggr)^{\frac{1}{q}} \Biggl(\sum_{i=1}^n \biggl(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\biggr)^{1-p} \biggl(\int_{a_{i-1}}^{a_i} w(t)g_1(t)g_2(t)\,dt\biggr)^p\Biggr)^{\frac{1}{p}} \\
&\quad\le \biggl(\int_a^b w(t)g_2^q(t)\,dt\biggr)^{\frac{1}{q}} \biggl(\int_a^b w(t)g_1^p(t)\,dt\biggr)^{\frac{1}{p}}.
\end{aligned}$$

For $0 < p < 1$, we use Theorem 2.1 with $p(t) = w(t)g_2^q(t)$, $g(t) = g_1^p(t)g_2^{-q}(t)$, and the function $f(x) = x^{\frac{1}{p}}$, which is convex for $x > 0, 0 < p < 1$. From (2.1) we get

$$\begin{aligned}
&\biggl(\frac{1}{\int_a^b w(t)g_2^q(t)\,dt}\int_a^b w(t)g_2^q(t)g_1^p(t)g_2^{-q}\,dt\biggr)^{\frac{1}{p}} \\
&\quad\le \frac{1}{\int_a^b w(t)g_2^q(t)\,dt}\sum_{i=1}^n \biggl(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\biggr)\biggl(\frac{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)g_1^p(t)g_2^{-q}\,dt}{\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt}\biggr)^{\frac{1}{p}} \\
&\quad\le \frac{1}{\int_a^b w(t)g_2^q(t)\,dt}\int_a^b w(t)g_2^q(t)\bigl(g_1^p(t)g_2^{-q}\bigr)^{\frac{1}{p}}\,dt.
\end{aligned}$$

Now using $q - \frac{q}{p} = 1$ and multiplying by $\int_a^b w(t)g_2^q(t)\,dt$, we have

$$\begin{aligned}
&\biggl(\int_a^b w(t)g_1^p(t)\,dt\biggr)^{\frac{1}{p}}\biggl(\int w(t)g_2^q(t)\,dt\biggr)^{\frac{1}{q}} \\
&\quad\le \sum_{i=1}^n \biggl(\int_{a_{i-1}}^{a_i} w(t)g_2^q(t)\,dt\biggr)^{\frac{1}{q}}\biggl(\int_{a_{i-1}}^{a_i} w(t)g_1^p(t)\,dt\biggr)^{\frac{1}{p}} \\
&\quad\le \int_a^b w(t)g_1(t)g_2(t)\,dt.
\end{aligned}$$

If $p < 0$, then $0 < q < 1$, and we have the same result by symmetry. □

Let p and g be positive integrable functions defined on $[a, b]$. Then the integral power means of order $r \in \mathbb{R}$ are defined as follows:

$$M_r(g;p;a,b) = \begin{cases} (\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)g^r(x)\,dx)^{\frac{1}{r}}, & r \neq 0, \\ \exp(\frac{\int_a^b p(x)\log g(x)\,dx}{\int_a^b p(x)\,dx}), & r = 0. \end{cases}$$

Let $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{w} = (w_1,\dots,w_n)$ be positive n-tuples. The weighted power mean (of the n-tuple $\mathbf{x}$ with weight $\mathbf{w}$) of order $r \in \mathbb{R}$ is defined as

$$M_r(\mathbf{x};\mathbf{w}) = \begin{cases} (\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^n w_i x_i^r)^{\frac{1}{r}}, & r \neq 0, \\ e^{\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^n w_i \log x_i} = (\prod_{i=1}^n x_i^{w_i})^{\frac{1}{\sum_{i=1}^n w_i}}, & r = 0. \end{cases}$$

In this paper, it is more suitable to use the notation $M_r(x_i;w_i;\overline{1,n})$.

Using our main result, we obtain following inequalities for integral power means.

Corollary 2.6 *Let p and g be positive integrable functions defined on $[a,b]$, and let $a_0,a_1,\dots,a_{n-1},a_n$ be such that $a=a_0<a_1<\cdots<a_{n-1}<a_n=b$. Let $s,t\in\mathbb{R}$ be such that $s\le t$. Then*

$$\begin{aligned} M_s(g;p;a,b) &\le M_t\left(M_s(g;p;a_{i-1},a_i);\int_{a_{i-1}}^{a_i}p(x)\,dx;\overline{1,n}\right)\\ &\le M_t(g;p;a.b), \end{aligned}\tag{2.3}$$

$$\begin{aligned} M_t(g;p;a,b) &\ge M_s\left(M_t(g;p;a_{i-1},a_i);\int_{a_{i-1}}^{a_i}p(x)\,dx;\overline{1,n}\right)\\ &\ge M_s(g;p;a.b). \end{aligned}\tag{2.4}$$

Proof We use Theorem 2.1 with $f(x)=x^{\frac{t}{s}}$ for $x>0, s,t\in\mathbb{R}, s,t\neq 0, s\le t$ (convex on $\langle 0,+\infty\rangle$). From (2.1) we get

$$\begin{aligned} \left(\frac{1}{P(b)}\int_a^b p(x)g(x)\,dx\right)^{\frac{t}{s}} &\le \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i}p(x)\,dx\right)\left(\frac{1}{\int_{a_{i-1}}^{a_i}p(x)\,dx}\int_{a_{i-1}}^{a_i}p(x)g(x)\,dx\right)^{\frac{t}{s}}\\ &\le \frac{1}{P(b)}\int_a^b p(x)g^{\frac{t}{s}}(x)\,dx. \end{aligned}$$

Substituting g with g^s and taking the power $\frac{1}{t}$, we get the result.

Similarly, we use Theorem 2.1 with $f(x)=x^{\frac{s}{t}}$ for $x>0, s,t\in\mathbb{R}, s,t\neq 0, s\le t$ (concave on $\langle 0,+\infty\rangle$). From (2.1) we get

$$\begin{aligned} \left(\frac{1}{P(b)}\int_a^b p(x)g(x)\,dx\right)^{\frac{s}{t}} &\ge \frac{1}{P(b)}\sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i}p(x)\,dx\right)\left(\frac{1}{\int_{a_{i-1}}^{a_i}p(x)\,dx}\int_{a_{i-1}}^{a_i}p(x)g(x)\,dx\right)^{\frac{s}{t}}\\ &\ge \frac{1}{P(b)}\int_a^b p(x)g^{\frac{s}{t}}(x)\,dx. \end{aligned}$$

Substituting g with g^t and takingthe power $\frac{1}{s}$, we get the result.

The cases $t=0$ and $s=0$ follow from inequalities (2.3) and (2.4) by simple limiting process. □

Means of the type

$$M_t\left(M_s(g;p;a_{i-1},a_i);\int_{a_{i-1}}^{a_i}p(x)\,dx;\overline{1,n}\right)$$

can be regarded as mixed means.

Let p be positive integrable function defined on $[a,b]$, and let g be any integrable function defined on $[a,b]$. Then for a strictly monotone continuous function h with domain belonging to the image of g, the quasiarithmetic mean is defined as follows:

$$M_h(g;p;a,b)=h^{-1}\left(\frac{1}{\int_a^b p(x)\,dx}\int_a^b p(x)h\bigl(g(x)\bigr)\,dx\right).$$

Using our main result, we obtain following inequalities for quasiarithmetic means.

Corollary 2.7 *Let p be positive integrable function defined on $[a,b]$, let g be an integrable function defined on $[a,b]$, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Also, assume that h is a strictly monotone continuous function with domain belonging to the image of g. If $f \circ h^{-1}$ is a convex function, then*

$$f\big(M_h(g;p;a,b)\big) \leq \frac{1}{P(b)} \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} p(x)\,dx \right) f\big(M_h(g;p;a_{i-1},a_i)\big) \leq \frac{1}{P(b)} \int_a^b p(x) f\big(g(x)\big)\,dx.$$

Proof We use Theorem 2.1 with $f \to f \circ h^{-1}$ and $g \to h \circ g$. □

3 Applications in information theory

In this section, we give some interesting estimates for the integral Csiszár divergence and for its important particular cases (see, e.g., [4, 5, 9, 10, 12, 15]).

Definition 3.1 (Csiszár divergence) Let $f: I \to \mathbb{R}$ be a function defined on some positive interval I, and let $p, q: [a,b] \to \mathbb{R}^+$ be two probability density functions such that $\frac{p(t)}{q(t)} \in I$ for $t \in [a,b]$. The Csiszár divergence is defined as

$$C_d(p,q) = \int_a^b q(t) f\left(\frac{p(t)}{q(t)}\right) dt.$$

Theorem 3.2 *Let $f: I \to \mathbb{R}$ be a convex function defined on a positive interval I, let $p, q: [a,b] \to \mathbb{R}^+$ be probability density functions such that $\frac{p(t)}{q(t)} \in I$ for $t \in [a,b]$, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$f(1) \leq \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)\,dt \right) f\left(\frac{\int_{a_{i-1}}^{a_i} p(t)\,dt}{\int_{a_{i-1}}^{a_i} q(t)\,dt} \right) \leq C_d(p,q).$$

Proof Using Theorem 2.1 with $p \to q$ and $g \to \frac{p}{q}$, we obtain the result.

The condition $\frac{p(t)}{q(t)} \in I$ for $t \in [a,b]$ obviously implies that $1 \in I$ and $\frac{\int_{a_{i-1}}^{a_i} p(t)\,dt}{\int_{a_{i-1}}^{a_i} q(t)\,dt} \in I$ for $i = 1, \dots, n$. □

Theorem 3.3 *Let $f: I \to \mathbb{R}$ be a convex function defined on a positive interval I, let $p, q: [a,b] \to \mathbb{R}^+$ be probability density functions such that $\frac{p(t)}{q(t)} \in I$ for $t \in [a,b]$, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Let $m_i \leq \frac{p(t)}{q(t)} \leq M_i$ for $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \dots, n$, $m = \min_{i=1,\dots,n} m_i$, and $M = \max_{i=1,\dots,n} M_i$. Then*

$$\begin{aligned} &C_d(p,q) \\ &\quad \leq \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)\,dt \right) \left[\frac{M_i - \frac{\int_{a_{i-1}}^{a_i} p(t)\,dt}{\int_{a_{i-1}}^{a_i} q(t)\,dt}}{M_i - m_i} f(m_i) + \frac{\frac{\int_{a_{i-1}}^{a_i} p(t)\,dt}{\int_{a_{i-1}}^{a_i} q(t)\,dt} - m_i}{M_i - m_i} f(M_i) \right] \\ &\quad \leq \frac{M-1}{M-m} f(m) + \frac{1-m}{M-m} f(M). \end{aligned}$$

Proof Using Theorem 2.3 with $p \to q$ and $g \to \frac{p}{q}$, we obtain the result. □

Definition 3.4 (Shannon entropy) Let $p\colon [a,b] \to \mathbb{R}^+$ be a probability density function. The Shannon entropy is defined as

$$\mathrm{SE}(p) = -\int_a^b p(t) \log p(t)\, dt.$$

Corollary 3.5 *Let $q\colon [a,b] \to \mathbb{R}^+$ be a probability density function, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$-\log(b-a) \le \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)\, dt \right) \log\left(\frac{\int_{a_{i-1}}^{a_i} q(t)\, dt}{a_i - a_{i-1}} \right) \le -\mathrm{SE}(q).$$

Proof Using Theorem 3.2 with $f(t) = -\log t$, $t \in \mathbb{R}^+$, and $p(t) = \frac{1}{b-a}$, $t \in [a,b]$, we obtain the result. □

Corollary 3.6 *Let $q\colon [a,b] \to \mathbb{R}^+$ be a probability density function, let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{1}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \dots, n$, $m = \min_{i=1,\dots,n} m_i$ and $M = \max_{i=1,\dots,n} M_i$. Then*

$$\begin{aligned} &-\mathrm{SE}(q) + \log(b-a) \\ &\quad \le \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} q(t)\, dt \right) \left[\frac{\frac{a_i - a_{i-1}}{(b-a)\int_{a_{i-1}}^{a_i} q(t)\, dt} - M_i}{M_i - m_i} \log m_i + \frac{m_i - \frac{a_i - a_{i-1}}{(b-a)\int_{a_{i-1}}^{a_i} q(t)\, dt}}{M_i - m_i} \log M_i \right] \\ &\quad \le \frac{1-M}{M-m} \log m + \frac{m-1}{M-m} \log M. \end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = -\log t$, $t \in \mathbb{R}^+$, and $p(t) = \frac{1}{b-a}$, $t \in [a,b]$, we obtain the result. □

Definition 3.7 (Kullback–Leibler divergence) Let $p, q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The Kullback–Leibler divergence is defined as

$$\mathrm{KL}_d(p,q) = \int_a^b p(t) \log\left(\frac{p(t)}{q(t)} \right) dt.$$

Corollary 3.8 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \le \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} p(t)\, dt \right) \log\left(\frac{\int_{a_{i-1}}^{a_i} p(t)\, dt}{\int_{a_{i-1}}^{a_i} q(t)\, dt} \right) \le \mathrm{KL}_d(p,q).$$

Proof Using Theorem 3.2 with $f(t) = t \log t$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 3.9 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{1}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i <$*

M_i, $i = 1,\ldots,n$, $m = \min_{i=1,\ldots,n} m_i$ and $M = \max_{i=1,\ldots,n} M_i$. Then

$$\begin{aligned}
&\mathrm{KL}_d(p,q)\\
&\quad\le \sum_{i=1}^{n}\left(\int_{a_{i-1}}^{a_i} q(t)\,dt\right)\left[\frac{M_i-\frac{\int_{a_{i-1}}^{a_i}p(t)\,dt}{\int_{a_{i-1}}^{a_i}q(t)\,dt}}{M_i-m_i}m_i\log m_i+\frac{\frac{\int_{a_{i-1}}^{a_i}p(t)\,dt}{\int_{a_{i-1}}^{a_i}q(t)\,dt}-m_i}{M_i-m_i}M_i\log M_i\right]\\
&\quad\le \frac{M-1}{M-m}m\log m+\frac{1-m}{M-m}M\log M.
\end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = t\log t$, $t \in \mathbb{R}^+$, we obtain the result. □

Definition 3.10 (Variational distance) Let $p,q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The variational distance is defined by

$$V_d(p,q) = \int_a^b \bigl|p(t)-q(t)\bigr|\,dt.$$

The following corollary can be also proved elementarily using the triangle inequality for integrals.

Corollary 3.11 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be a such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \le \sum_{i=1}^{n}\left|\int_{a_{i-1}}^{a_i} p(t)\,dt-\int_{a_{i-1}}^{a_i} q(t)\,dt\right| \le V_d(p,q).$$

Proof Using Theorem 3.2 with $f(t) = |t-1|$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 3.12 *Let $p,q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{p(t)}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1,\ldots,n$, $m = \min_{i=1,\ldots,n} m_i$ and $M = \max_{i=1,\ldots,n} M_i$. Then*

$$\begin{aligned}
&\int_a^b \bigl|p(t)-q(t)\bigr|\,dt\\
&\quad\le \sum_{i=1}^{n}\left[\frac{M_i\int_{a_{i-1}}^{a_i}q(t)\,dt-\int_{a_{i-1}}^{a_i}p(t)\,dt}{M_i-m_i}|m_i-1|\right.\\
&\qquad\left.+\frac{\int_{a_{i-1}}^{a_i}p(t)\,dt-m_i\int_{a_{i-1}}^{a_i}q(t)\,dt}{M_i-m_i}|M_i-1|\right]\\
&\quad\le \frac{2(M-1)(1-m)}{M-m}.
\end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = |t-1|$, $t \in \mathbb{R}^+$, and $m \le 1 \le M$, we obtain get the result. □

Definition 3.13 (Jeffrey's distance) Let $p,q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The Jeffrey distance is defined as

$$J_d(p,q) = \int_a^b \bigl(p(t)-q(t)\bigr)\log\left(\frac{p(t)}{q(t)}\right)dt.$$

Corollary 3.14 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \le \sum_{i=1}^{n} \left(\int_{a_{i-1}}^{a_i} p(t)\,dt - \int_{a_{i-1}}^{a_i} q(t)\,dt \right) \log\left(\frac{\int_{a_{i-1}}^{a_i} p(t)\,dt}{\int_{a_{i-1}}^{a_i} q(t)\,dt} \right) \le J_d(p,q).$$

Proof Using Theorem 3.2 with $f(t) = (t-1)\log t$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 3.15 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{p(t)}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \dots, n$, $m = \min_{i=1,\dots,n} m_i$, and $M = \max_{i=1,\dots,n} M_i$. Then*

$$\begin{aligned} &J_d(p,q) \\ &\quad\le \sum_{i=1}^{n} \left[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)\,dt - \int_{a_{i-1}}^{a_i} p(t)\,dt}{M_i - m_i} (m_i - 1)\log m_i \right. \\ &\qquad \left. + \frac{\int_{a_{i-1}}^{a_i} p(t)\,dt - m_i \int_{a_{i-1}}^{a_i} q(t)\,dt}{M_i - m_i} (M_i - 1)\log M_i \right] \\ &\quad\le \frac{(M-1)(1-m)}{M-m} \log \frac{M}{m}. \end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = (t-1)\log t$, $t \in \mathbb{R}^+$, we obtain the result. □

Definition 3.16 (Bhattacharyya coefficient) Let $p, q\colon [a,b] \to \mathbb{R}^+$ be two probability density functions. The Bhattacharyya distance is defined as

$$B_d(p,q) = \int_a^b \sqrt{p(t)q(t)}\,dt.$$

Corollary 3.17 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, and let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$1 \ge \sum_{i=1}^{n} \sqrt{\int_{a_{i-1}}^{a_i} p(t)\,dt \int_{a_{i-1}}^{a_i} q(t)\,dt} \ge B_d(p,q).$$

Proof Using Theorem 3.2 with $f(t) = -\sqrt{t}$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 3.18 *Let $p, q\colon [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \dots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{p(t)}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \dots, n$, $m = \min_{i=1,\dots,n} m_i$, and $M = \max_{i=1,\dots,n} M_i$. Then*

$$\begin{aligned} &B_d(p,q) \\ &\quad\ge \sum_{i=1}^{n} \left[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)\,dt - \int_{a_{i-1}}^{a_i} p(t)\,dt}{M_i - m_i} \sqrt{m_i} + \frac{\int_{a_{i-1}}^{a_i} p(t)\,dt - m_i \int_{a_{i-1}}^{a_i} q(t)\,dt}{M_i - m_i} \sqrt{M_i} \right] \\ &\quad\ge \frac{1 + \sqrt{mM}}{\sqrt{m} + \sqrt{M}}. \end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = -\sqrt{t}$, $t \in \mathbb{R}^+$, we obtain the result. □

Definition 3.19 (Hellinger distance) Let $p, q: [a,b] \to \mathbb{R}^+$ be two probability density functions. The Hellinger distance is defined as

$$H_d(p,q) = \int_a^b \left(\sqrt{p(t)} - \sqrt{q(t)}\right)^2 dt.$$

Corollary 3.20 *Let $p, q: [a,b] \to \mathbb{R}^+$ be probability density functions, and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \le \sum_{i=1}^n \left(\sqrt{\int_{a_{i-1}}^{a_i} p(t)\,dt} - \sqrt{\int_{a_{i-1}}^{a_i} q(t)\,dt}\right)^2 \le H_d(p,q).$$

Proof Using Theorem 3.2 with $f(t) = (\sqrt{t} - 1)^2$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 3.21 *Let $p, q: [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{p(t)}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i < M_i$, $i = 1, \ldots, n$, $m = \min_{i=1,\ldots,n} m_i$, and $M = \max_{i=1,\ldots,n} M_i$. Then*

$$\begin{aligned} &H_d(p,q) \\ &\quad\le \sum_{i=1}^n \biggl[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)\,dt - \int_{a_{i-1}}^{a_i} p(t)\,dt}{M_i - m_i}(\sqrt{m_i} - 1)^2 \\ &\qquad + \frac{\int_{a_{i-1}}^{a_i} p(t)\,dt - m_i \int_{a_{i-1}}^{a_i} q(t)\,dt}{M_i - m_i}(\sqrt{M_i} - 1)^2\biggr] \\ &\quad\le 2\frac{(\sqrt{M} - 1)(1 - \sqrt{m})}{\sqrt{m} + \sqrt{M}}. \end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = (\sqrt{t} - 1)^2$, $t \in \mathbb{R}^+$, we obtain the result. □

Definition 3.22 (Triangular discrimination) Let $p, q: [a,b] \to \mathbb{R}^+$ be two probability density functions. The triangular discrimination between p and q is defined as

$$T_d(p,q) = \int_a^b \frac{(p(t) - q(t))^2}{p(t) + q(t)}\,dt.$$

Corollary 3.23 *Let $p, q: [a,b] \to \mathbb{R}^+$ be probability density functions, and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$. Then*

$$0 \le \sum_{i=1}^n \frac{(\int_{a_{i-1}}^{a_i} p(t)\,dt - \int_{a_{i-1}}^{a_i} q(t)\,dt)^2}{\int_{a_{i-1}}^{a_i} p(t)\,dt + \int_{a_{i-1}}^{a_i} q(t)\,dt} \le T_d(p,q).$$

Proof Using Theorem 3.2 with $f(t) = \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$, we obtain the result. □

Corollary 3.24 *Let $p, q: [a,b] \to \mathbb{R}^+$ be probability density functions, let $a_0, a_1, \ldots, a_{n-1}, a_n$ be such that $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$, and let $m_i \le \frac{p(t)}{q(t)} \le M_i$ for $t \in [a_{i-1}, a_i]$, $m_i <$*

M_i, $i = 1, \ldots, n$, $m = \min_{i=1,\ldots,n} m_i$, *and* $M = \max_{i=1,\ldots,n} M_i$. *Then*

$$\begin{aligned} &T_d(p,q) \\ &\quad \le \sum_{i=1}^{n} \biggl[\frac{M_i \int_{a_{i-1}}^{a_i} q(t)\,dt - \int_{a_{i-1}}^{a_i} p(t)\,dt}{M_i - m_i} \frac{(m_i - 1)^2}{m_i + 1} \\ &\qquad + \frac{\int_{a_{i-1}}^{a_i} p(t)\,dt - m_i \int_{a_{i-1}}^{a_i} q(t)\,dt}{M_i - m_i} \frac{(M_i - 1)^2}{M_i + 1} \biggr] \\ &\quad \le \frac{2(M-1)(1-m)}{(M+1)(m+1)}. \end{aligned}$$

Proof Using Theorem 3.3 with $f(t) = \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$, we obtain the result. □

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