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# On convex combinations of harmonic mappings

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**Abstract**

Let $\psi_{\mu,\nu}(z) = (1 - 2\cos\nu e^{i\mu} z + e^{2i\mu} z^2)^{-1}$, $\mu, \nu \in [0, 2\pi)$ and $p$ be an analytic mapping with $\operatorname{Re} p > 0$ on the open unit disk. We consider the sense-preserving planar harmonic mappings $f = h + \overline{g}$, which are shears of the mapping $\int_0^z \psi_{\mu,\nu}(\xi) p(\xi)\, d\xi$ in the direction $\mu$. These mappings include the harmonic right half-plan mappings, vertical strip mappings, and their rotations. For various choices of dilatations $g'/h'$ of $f$, sufficient conditions are found for the convex combinations of these mappings to be univalent and convex in the direction $\mu$.

**MSC:** 31A05; 30C45

**Keywords:** Harmonic mappings; Convex combination; Directional convexity

## 1 Introduction

On a simply connected domain $\Omega \subset \mathbb{C}$ a complex-valued harmonic mapping $f$ can be written as $f = h + \overline{g}$, where $h$ and $g$ are analytic mappings. By Lewy [8], it is locally univalent sense-preserving if and only if its Jacobian $\mathcal{J}_f = |h'|^2 - |g'|^2$ is positive or, equivalently, its dilatation $\omega_f := g'/h'$ lies in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}$ denote the class of all locally univalent sense-preserving harmonic mappings $f = h + \overline{g}$ defined on $\mathbb{D}$. Also, let $\mathcal{S}_H$ denote the subclass of $\mathcal{H}$ consisting of univalent mappings with normalization $f(0) = 0 = f_z(0) - 1$. Moreover, let $\mathcal{S}_H^0$ be the subclass of $\mathcal{S}_H$ that contains all mappings $f = h + \overline{g}$ such that $f_{\overline{z}}(0) = 0$. For $0 \le \nu < \pi$, a mapping $\varphi$ is called convex in the direction $\nu$ if $\varphi(\mathbb{D})$ has connected intersection with every line that is parallel to the line joining $e^{i\nu}$ to the origin. Such a mapping is also called a directional convex mapping. If $\nu = 0$ (or $\pi/2$), then $\varphi$ is known as convex in the real (or imaginary) direction. A harmonic mapping $f = h + \overline{g} \in \mathcal{S}_H^0$ is said to be a right half-plane or a vertical strip mapping if it maps $\mathbb{D}$ onto the right half-plane

$$R = \{w \in \mathbb{C} : \operatorname{Re}(w) > -1/2\}$$

or the vertical strip

$$V_\alpha := \left\{ w \in \mathbb{C} : \frac{\alpha - \pi}{2\sin\alpha} < \operatorname{Re} w < \frac{\alpha}{2\sin\alpha} \right\}, \quad \frac{\pi}{2} < \alpha < \pi,$$

respectively. It is well known [1, 4] that if $f = h + \overline{g}$ is a right half-plane harmonic mapping then $h'(z) + g'(z) = (1-z)^{-2}$, and if it is a vertical strip harmonic mapping then $h'(z) + g'(z) = (1 + 2z\cos\alpha + z^2)^{-1}$. In this article, we find some sufficient conditions for the convex combination of the right half-plane mappings, the vertical strip mappings, their rotations, and some other harmonic mappings to be univalent and convex in a particular direction. Generally, the convex combination of two analytic/harmonic mappings does not carry the univalency or other geometric properties of individual mappings. One can refer to the survey article by Campbell [2] and the references therein for the univalency and other geometric properties of the convex combination of analytic mappings. However, recently, a convex combination of some harmonic mappings has been studied in [5, 7, 11–13]. In particular, Wang *et al.* [13] and Kumar *et al.* [7] respectively studied the directional convexity of convex combination of harmonic mappings, which are shears of the analytic mappings $z/(1-z)$ and $z(1-\alpha z)/(1-z^2)$, $-1 \le \alpha \le 1$. Motivated by the work carried out in [7, 13], we study the convex combination of harmonic mappings which are shears of the analytic mapping $\psi_{\mu,\nu}p_k$, where $p_k$ is analytic with positive real part on $\mathbb{D}$ and

$$\psi_{\mu,\nu}(z) = \frac{1}{1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}}, \quad \mu,\nu \in [0, 2\pi). \tag{1.1}$$

In particular, we show that the combination $f = tf_1 + (1-t)f_2$, $0 \le t \le 1$ of the mappings $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$, $k = 1, 2$, satisfying $h_k' - e^{2i\mu}g_k' = \psi_{\mu,\nu}p_k$ is univalent and convex in the direction $\mu$ for some specific dilatations of $f_1$ and $f_2$. The following result by Royster and Ziegler [10] is used to check the convexity in a particular direction of analytic mappings.

**Lemma 1.1** *Let $\phi$ be a non-constant analytic mapping in $\mathbb{D}$. Then $\phi$ maps $\mathbb{D}$ onto a domain convex in the direction $\gamma$ $(0 \le \gamma < \pi)$ if and only if there are real numbers $\mu$ and $\nu$ $(0 \le \nu < 2\pi)$ such that*

$$\operatorname{Re}\bigl(e^{i(\mu-\gamma)}\bigl(1 - 2ze^{-i\mu}\cos\nu + z^2e^{-2i\mu}\bigr)\phi'(z)\bigr) \ge 0, \quad z \in \mathbb{D}. \tag{1.2}$$

*Remark* 1.2 By taking $\gamma$ or $\gamma + \pi$ equal to $\mu$ in Lemma 1.1, we see that a non-constant analytic mapping $\phi$ is convex in the direction $\mu$ if, for some real number $\nu$ $(0 \le \nu < 2\pi)$, the real part of the mapping $\phi'/\psi_{\mu,\nu}$, where $\psi_{\mu,\nu}$ is given by (1.1), is either non-negative or non-positive on $\mathbb{D}$.

Lemma 1.1 along with the following result due to Clunie and Sheil-Small [3], known as *shear construction*, is used to check the convexity in a particular direction of harmonic mappings.

**Lemma 1.3** *A locally univalent and sense-preserving harmonic mapping $f = h + \overline{g}$ on $\mathbb{D}$ is univalent and maps $\mathbb{D}$ onto a domain convex in the direction $\gamma$ $(0 \le \gamma < \pi)$ if and only if the analytic mapping $h - e^{2i\gamma}g$ is univalent and maps $\mathbb{D}$ onto a domain convex in the direction $\gamma$.*

## 2 Main results

**Theorem 2.1** *For $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ such that*

$$h_k(z) - e^{2i\mu}g_k(z) = \int_0^z \psi_{\mu,\nu}(\xi)p_k(\xi)\,d\xi, \quad \mu,\nu \in [0, 2\pi), \tag{2.1}$$

*where $p_k$ is an analytic mapping with* $\operatorname{Re} p_k > 0$ *on* $\mathbb{D}$ *and* $\psi_{\mu,\nu}$ *is given by* (1.1). *Then the mapping* $f = tf_1 + (1-t)f_2$ *is univalent and is convex in the direction* $\mu$ *for* $0 \le t \le 1$ *if it is locally univalent and sense-preserving.*

*Proof* Let $f = h + \overline{g}$, then

$$h = th_1 + (1-t)h_2 \quad \text{and} \quad g = tg_1 + (1-t)g_2,$$

and thus

$$h - e^{2i\mu}g = t\bigl(h_1 - e^{2i\mu}g_1\bigr) + (1-t)\bigl(h_2 - e^{2i\mu}g_2\bigr).$$

Therefore, in view of (2.1), it follows that

$$\operatorname{Re}\left(\frac{h' - e^{2i\mu}g'}{\psi_{\mu,\nu}}\right) = t \operatorname{Re} p_1 + (1-t)\operatorname{Re} p_2 > 0$$

on $\mathbb{D}$ for $0 \le t \le 1$. Hence, by Lemma 1.1, it follows that the mapping $h - e^{2i\mu}g$ is convex in the direction $\mu$. The result now follows by Lemma 1.3. □

Theorem 2.1 has the following obvious extension to $n$ mappings.

**Theorem 2.2** *For* $k = 1, 2, \ldots, n$, *let* $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ *satisfy* (2.1), *where* $p_k$ *is an analytic mapping with* $\operatorname{Re} p_k > 0$ *on* $\mathbb{D}$ *and* $\psi_{\mu,\nu}$ *is given by* (1.1). *If* $\sum_{t=1}^{n} t_k = 1$, $0 \le t_k \le 1$, *then the mapping* $f = \sum_{t=1}^{n} t_k f_k$ *is univalent and is convex in the direction* $\mu$ *provided it is locally univalent and sense-preserving.*

In Theorems 2.1 and 2.2 we assumed $f$ to be locally univalent and sense-preserving on $\mathbb{D}$. Next, we will study some cases where this assumption can be relaxed.

**Theorem 2.3** *For* $k = 1, 2$, *let* $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ *satisfy* (2.1), *where* $p_k$ *is an analytic mapping with* $\operatorname{Re} p_k > 0$ *on* $\mathbb{D}$ *and* $\psi_{\mu,\nu}$ *is given by* (1.1). *Let* $\omega_{f_k}$ *be the dilatation of* $f_k$, *then the mapping* $f = tf_1 + (1-t)f_2$ *is univalent and is convex in the direction* $\mu$ *for* $0 \le t \le 1$ *if* $\omega_{f_k}$ *and* $p_k$ *satisfy one of the following*:

(i) $\omega_{f_1} = \omega_{f_2}$,
(ii) $p_1/(1 - e^{2i\mu}\omega_{f_1}) = p_2/(1 - e^{2i\mu}\omega_{f_2})$,
(iii) $p_1 = p_2$,
(iv) $\omega_{f_2} = -\omega_{f_1}$ *and* $\operatorname{Re}(p_2(1 - e^{2i\mu}\omega_{f_1})/(p_1(1 + e^{2i\mu}\omega_{f_1}))) > 0$.

*Proof* In view of Theorem 2.1, it is enough to show that $f$ is locally univalent and sense-preserving or, equivalently, $|\omega_f| < 1$ on $\mathbb{D}$, where $\omega_f$ is the dilatation of $f$. Since for $t = 0$ and 1 the result is obvious, we consider $0 < t < 1$. On differentiation (2.1) gives

$$h_k' - e^{2i\mu}g_k' = \psi_{\mu,\nu}p_k.$$

The above equation along with $g_k' = \omega_{f_k}h_k'$ gives

$$h_k' = \frac{\psi_{\mu,\nu}p_k}{1 - e^{2i\mu}\omega_{f_k}}. \tag{2.2}$$

Since $f = h + \overline{g} := th_1 + (1-t)h_2 + \overline{tg_1 + (1-t)g_2}$, in view of (2.2), $\omega_f$ is given by

$$\begin{aligned}\omega_f &= \frac{g'}{h'} = \frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'} \\ &= \frac{t\omega_{f_1}h_1' + (1-t)\omega_{f_2}h_2'}{th_1' + (1-t)h_2'} \\ &= \frac{t\omega_{f_1}(1-e^{2i\mu}\omega_{f_2})p_1 + (1-t)\omega_{f_2}(1-e^{2i\mu}\omega_{f_1})p_2}{t(1-e^{2i\mu}\omega_{f_2})p_1 + (1-t)(1-e^{2i\mu}\omega_{f_1})p_2}. \end{aligned} \tag{2.3}$$

Let $\omega_{f_1} = \omega_{f_2}$, then (2.3) gives that $\omega_f = \omega_{f_1}$ and hence $|\omega_f| < 1$. Also, let $p_k$ and $\omega_{f_k}$ be given by (ii), then (2.3) gives that $\omega_f = t\omega_{f_1} + (1-t)\omega_{f_2}$. Hence, $|\omega_{f_k}| < 1$ follows that $|\omega_f| < 1$. Moreover, let $p_1 = p_2$, then (2.3) shows that

$$\omega_f = \frac{t\omega_{f_1}(1-e^{2i\mu}\omega_{f_2}) + (1-t)\omega_{f_2}(1-e^{2i\mu}\omega_{f_1})}{t(1-e^{2i\mu}\omega_{f_2}) + (1-t)(1-e^{2i\mu}\omega_{f_1})}.$$

Therefore, $|\omega_{f_k}| < 1$ implies that

$$\operatorname{Re}\left(\frac{1+e^{2i\mu}\omega_f}{1-e^{2i\mu}\omega_f}\right) = t\operatorname{Re}\left(\frac{1+e^{2i\mu}\omega_{f_1}}{1-e^{2i\mu}\omega_{f_1}}\right) + (1-t)\operatorname{Re}\left(\frac{1+e^{2i\mu}\omega_{f_2}}{1-e^{2i\mu}\omega_{f_2}}\right) > 0.$$

Hence, $|\omega_f| < 1$. Lastly, let $\omega_{f_2} = -\omega_{f_1}$, then from (2.3) we have

$$\omega_f = \omega_{f_1}\frac{t(1+e^{2i\mu}\omega_{f_1})p_1 - (1-t)(1-e^{2i\mu}\omega_{f_1})p_2}{t(1+e^{2i\mu}\omega_{f_1})p_1 + (1-t)(1-e^{2i\mu}\omega_{f_1})p_2} =: \omega_{f_1}\varphi.$$

Therefore, $|\omega_f| < 1$ if $|\varphi| < 1$. Now, by the assumption in (iv), we have

$$\operatorname{Re}\left(\frac{1+\varphi}{1-\varphi}\right) = \operatorname{Re}\left(\frac{t(1+e^{2i\mu}\omega_{f_1})p_1}{(1-t)(1-e^{2i\mu}\omega_{f_1})p_2}\right) > 0.$$

Hence, $|\varphi| < 1$. This proves the result when $\omega_{f_k}$ and $p_k$ satisfy condition (iv). This completes the proof. □

From its proof, it is easily seen that Theorem 2.3, except case (iv), has a natural extension to $n$ mappings as follows.

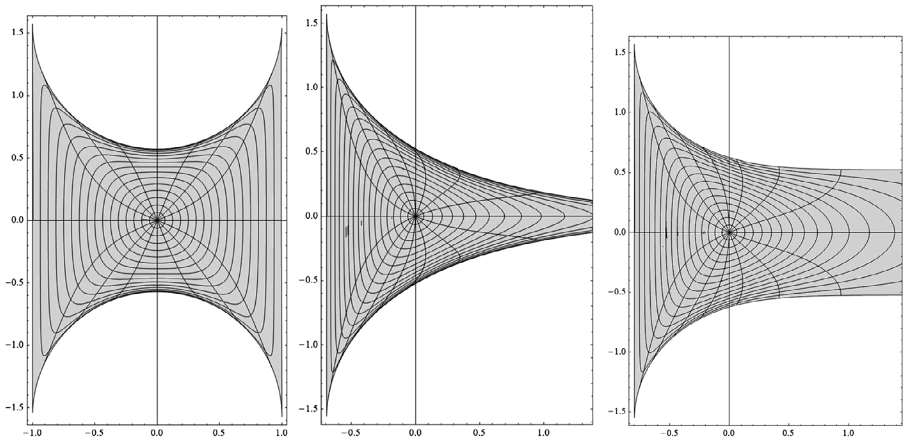
**Theorem 2.4** *For $k = 1, 2, \ldots, n$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ have dilatation $\omega_{f_k}$ and satisfy* (2.1), *where $p_k$ is an analytic mapping with $\operatorname{Re} p_k > 0$ on $\mathbb{D}$ and $\psi_{\mu,\nu}$ is given by* (1.1). *If $\sum_{t=1}^{n} t_k = 1$, $0 \le t_k \le 1$, then the mapping $f = \sum_{t=1}^{n} t_k f_k$ is univalent and is convex in the direction $\mu$ provided $\omega_{f_k}$ and $p_k$ satisfy one of the following*:

(i) $\omega_{f_1} = \omega_{f_2} = \cdots = \omega_{f_n}$,

(ii) $p_1/(1-e^{2i\mu}\omega_{f_1}) = p_2/(1-e^{2i\mu}\omega_{f_2}) = \cdots = p_k/(1-e^{2i\mu}\omega_{f_k})$,

(iii) $p_1 = p_2 = \cdots = p_n$.

The following example gives an illustration of Theorem 2.3.



(A) t=1 (B) t=0 (C) t=1/3

**Figure 1** Images of $\mathbb{D}$ under $f$ at different values of $t$

*Example* 2.5 For $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ be given by

$$f_1(z) = h_1(z) + \overline{g_1(z)} = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + \overline{z - \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)}$$

and

$$f_2(z) = h_2(z) + \overline{g_2(z)} = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + \overline{\frac{1}{2}\log\frac{1}{1-z^2}}.$$

Then, $\omega_{f_k}$, the dilatation of $f_k$, is given by $\omega_{f_1}(z) = -z^2$ and $\omega_{f_2}(z) = z$. Also, we can see that

$$h_k'(z) + g_k'(z) = \frac{1 + \omega_{f_k}(z)}{1 - z^2}.$$

Thus, $f_k$ satisfies (2.1) with $\mu = \pi/2$, $\nu = \pi/2$ and $p_k = 1 + \omega_{f_k}$, where $\operatorname{Re} p_k > 0$ on $\mathbb{D}$. Therefore, it follows from Theorem 2.3 that the mapping $f = tf_1 + (1-t)f_2$ is univalent and convex in the imaginary direction for $0 \le t \le 1$. Images of $\mathbb{D}$ under $f$ at $t = 1$, $t = 0$, and $t = 1/3$ are shown in Fig. 1.

We will use the following lemma to prove our next results.

**Lemma 2.6** *For $n \in \mathbb{N}$ and $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ such that*

$$h_k(z) - g_k(z) = \left(1 + (-1)^k a\right) \int_0^z \frac{q(\xi)\,d\xi}{\psi_{\mu,\nu_k}(\xi^n)}, \quad \mu, \nu_k \in [0, 2\pi), \tag{2.4}$$

*where $q$ is an analytic mapping and $\psi_{\mu,\nu}$ is defined by (1.1). Let $\omega_{f_k}$ be the dilatation of $f_k$. If*

$$\omega_{f_1}(z) = -\omega_{f_2}(z) = \frac{a + e^{i(\theta-\mu)}z^n}{1 + ae^{i(\theta-\mu)}z^n}, \quad a \in (-1, 1), \theta \in [0, 2\pi), \tag{2.5}$$

*then the mapping* $f = tf_1 + (1-t)f_2$ *is locally univalent and sense-preserving for* $0 \leq t \leq 1$ *provided*:

(i) $\cos\theta > \max\{\cos\nu_1, -\cos\nu_2\}$ *and* $\cos\nu_1 > \cos\nu_2$, *or*

(ii) $\cos\theta < \min\{\cos\nu_1, -\cos\nu_2\}$ *and* $\cos\nu_2 > \cos\nu_1$.

To prove the above lemma, we will use the following result commonly known as *Cohn's rule* [9].

**Theorem 2.7** *Let* $r(z) = a_0 + a_1 z + \cdots + a_n z^n$ *be a polynomial of degree* $n$ *and*

$$r^*(z) = z^n \overline{r(1/\overline{z})} = \overline{a}_n + \overline{a}_{n-1} z + \cdots + \overline{a}_0 z^n.$$

*Let* $s$ *and* $s_1$ *be the number of zeros of* $r$ *inside and on the unit circle* $|z| = 1$, *respectively. If* $|a_0| < |a_n|$, *then*

$$r_1(z) = \frac{\overline{a}_n r(z) - a_0 r^*(z)}{z}$$

*is a polynomial of degree* $n-1$ *and has* $s-1$ *and* $s_1$ *number of zeros inside and on the unit circle* $|z| = 1$, *respectively.*

*Proof of Theorem* 2.1 Since $f_k \in \mathcal{S}_H$, we need to prove the result only for $0 < t < 1$. First of all, we will show that both conditions (i) and (ii) imply

$$\left|1 - (\cos\nu_1 - \cos\nu_2)e^{-i\theta}\right| < 1 \tag{2.6}$$

and

$$\left|\frac{\cos\nu_1 + \cos\nu_2}{\cos\nu_1 - \cos\nu_2 + 2\cos\theta}\right| < 1. \tag{2.7}$$

We see that $(\cos\nu_1 - \cos\nu_2)(\cos\nu_1 - \cos\nu_2 - 2\cos\theta) < 0$ if condition (i) or (ii) is satisfied. Therefore,

$$\begin{aligned} \left|1 - (\cos\nu_1 - \cos\nu_2)e^{-i\theta}\right|^2 - 1 &= \bigl(1 - (\cos\nu_1 - \cos\nu_2)\cos\theta\bigr)^2 + \bigl((\cos\nu_1 - \cos\nu_2)\sin\theta\bigr)^2 \\ &= (\cos\nu_1 - \cos\nu_2)(\cos\nu_1 - \cos\nu_2 - 2\cos\theta) < 0. \end{aligned}$$

Hence, both (i) and (ii) imply (2.6). Next, let condition (i) be satisfied. Then $\cos\theta > \cos\nu_1$ and $\cos\theta > -\cos\nu_2$, and hence

$$\cos\nu_1 - \cos\nu_2 - 2\cos\theta < \cos\nu_1 + \cos\nu_2 < -\cos\nu_1 + \cos\nu_2 + 2\cos\theta. \tag{2.8}$$

Similarly, if condition (ii) is satisfied, then

$$-\cos\nu_1 + \cos\nu_2 + 2\cos\theta < \cos\nu_1 + \cos\nu_2 < \cos\nu_1 - \cos\nu_2 - 2\cos\theta. \tag{2.9}$$

Therefore, (2.8) and (2.9) show that both (i) and (ii) imply (2.7).

Now, differentiating (2.4), we have

$$\big(h_k'(z)-g_k'(z)\big)\psi_{\mu,\nu_k}\big(z^n\big)=\big(1+(-1)^k a\big)q(z).$$

The above equation along with $g_k'=\omega_{f_k}h_k'$ gives

$$h_k'(z)=\frac{(1+(-1)^k a)q(z)}{\psi_{\mu,\nu_k}(z^n)(1-\omega_{f_k}(z))}.$$

Therefore $\omega_f$, the dilatation of $f=tf_1+(1-t)f_2$, is given by

$$\begin{aligned}\omega_f(z)&=\frac{tg_1'(z)+(1-t)g_2'(z)}{th_1'(z)+(1-t)h_2'(z)}\\&=\frac{t\omega_{f_1}(z)h_1'(z)+(1-t)\omega_{f_2}(z)h_2'(z)}{th_1'(z)+(1-t)h_2'(z)}\\&=\frac{t\omega_{f_1}(z)\psi_{\mu,\nu_2}(z^n)(1-\omega_{f_2}(z))(1-a)+(1-t)\omega_{f_2}(z)\psi_{\mu,\nu_1}(z^n)(1-\omega_{f_1}(z))(1+a)}{t\psi_{\mu,\nu_2}(z^n)(1-\omega_{f_2}(z))(1-a)+(1-t)\psi_{\mu,\nu_1}(z^n)(1-\omega_{f_1}(z))(1+a)}.\end{aligned}\tag{2.10}$$

Now, on substituting the values of $\omega_{f_k}$, given by (2.5), in (2.10), we obtain

$$\begin{aligned}\omega_f(z)&=\omega_{f_1}(z)\\&\quad\times\left(\frac{t(1+ae^{i(\theta-\mu)}z^n+a+e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_2}(z^n)(1-a)-(1-t)(1+ae^{i(\theta-\mu)}z^n-a-e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_1}(z^n)(1+a)}{t(1+ae^{i(\theta-\mu)}z^n+a+e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_2}(z^n)(1-a)+(1-t)(1+ae^{i(\theta-\mu)}z^n-a-e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_1}(z^n)(1+a)}\right)\\&=\omega_{f_1}(z)\frac{t(1+e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_2}(z^n)-(1-t)(1-e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_1}(z^n)}{t(1+e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_2}(z^n)+(1-t)(1-e^{i(\theta-\mu)}z^n)\psi_{\mu,\nu_1}(z^n)}.\end{aligned}$$

The above equation, after substituting the values of $\psi_{\mu,\nu_k}$ and then putting $e^{-i\mu}z^n=w$, is equivalent to

$$\begin{aligned}\omega_f\big(\big(e^{i\mu}w\big)^{1/n}\big)&=\omega_{f_1}\big(\big(e^{i\mu}w\big)^{1/n}\big)\\&\quad\times\left(\frac{t(1+e^{i\theta}w)(1-2w\cos\nu_1+w^2)-(1-t)(1-e^{i\theta}w)(1-2w\cos\nu_2+w^2)}{t(1+e^{i\theta}w)(1-2w\cos\nu_1+w^2)+(1-t)(1-e^{i\theta}w)(1-2w\cos\nu_2+w^2)}\right)\\&=:\omega_{f_1}\big(\big(e^{i\mu}w\big)^{1/n}\big)W(w).\end{aligned}\tag{2.11}$$

To prove our result, we have to show $|\omega_f|<1$ on $\mathbb{D}$. Since $|\omega_{f_1}|<1$, in view of (2.11), it is enough to show that $|W|<1$ on $\mathbb{D}$. Let

$$W(w)=e^{-i\theta}\frac{\mathfrak{p}(w)}{\mathfrak{q}(w)},$$

where, after a simplification,

$$\begin{aligned}\mathfrak{p}(w)&=e^{i\theta}w^3+\big(2t-1-2te^{i\theta}\cos\nu_1-2(1-t)e^{i\theta}\cos\nu_2\big)w^2\\&\quad+\big(e^{i\theta}-2t\cos\nu_1+2(1-t)\cos\nu_2\big)w+2t-1\end{aligned}$$

and

$$\begin{aligned}\mathfrak{q}(w) &= (2t-1)w^3 + \big(e^{-i\theta} - 2t\cos\nu_1 + 2(1-t)\cos\nu_2\big)w^2 \\ &\quad + \big(2t - 1 - 2te^{-i\theta}\cos\nu_1 - 2(1-t)e^{-i\theta}\cos\nu_2\big)w + e^{-i\theta}.\end{aligned}$$

Clearly $\mathfrak{q}(w) = w^3\overline{\mathfrak{p}(1/\overline{w})}$. Hence, we can write $W$ as follows:

$$W(w) = \frac{\mathfrak{p}(w)}{w^3\overline{\mathfrak{p}(1/\overline{w})}} = e^{i\theta}\prod_{i=1}^{3}\frac{w-w_i}{1-\overline{w_i}w},$$

where $w_1$, $w_2$, and $w_3$ are the zeros of $k$. Thus, to show $|W| < 1$, it is enough to show $w_1, w_2, w_3 \in \mathbb{D}$. We will discuss it for the cases $t = 1/2$ and $t \neq 1/2$ separately. For $t \neq 1/2$, we have $0 < |2t-1| < |e^{i\theta}| = 1$. Define a polynomial $\mathfrak{p}_1$ by

$$\mathfrak{p}_1(w) = \frac{e^{-i\theta}\mathfrak{p}(w) - (2t-1)\mathfrak{q}(w)}{w}.$$

A calculation gives

$$\begin{aligned}\mathfrak{p}_1(w) &= 4t(1-t)w^2 - 4t(1-t)(\cos\nu_1 + \cos\nu_2)w + 4t(1-t)\big(1 - (\cos\nu_1 - \cos\nu_2)e^{-i\theta}\big) \\ &= 4t(1-t)\tilde{\mathfrak{p}}_1(w),\end{aligned}$$

where

$$\tilde{\mathfrak{p}}_1(w) = w^2 - (\cos\nu_1 + \cos\nu_2)w + 1 - (\cos\nu_1 - \cos\nu_2)e^{-i\theta}.$$

Recall that inequality (2.6) holds. Again, define a polynomial $\mathfrak{p}_2$ by

$$\mathfrak{p}_2(w) = \frac{\tilde{\mathfrak{p}}_1(w) - (1 - (\cos\nu_1 - \cos\nu_2)e^{-i\theta})\tilde{\mathfrak{p}}_1^*(w)}{w},$$

where $\tilde{\mathfrak{p}}_1^*(w) = w^2\overline{\tilde{\mathfrak{p}}_1(1/\overline{w})}$. Furthermore, we see that

$$\begin{aligned}\mathfrak{p}_2(w) &= \big(1 - \big|1 - (\cos\nu_1 - \cos\nu_2)e^{-i\theta}\big|^2\big)w - \big(\cos^2\nu_1 - \cos^2\nu_2\big)e^{-i\theta} \\ &= -(\cos\nu_1 - \cos\nu_2)\big((\cos\nu_1 - \cos\nu_2 - 2\cos\theta)w + (\cos\nu_1 + \cos\nu_2)e^{-i\theta}\big).\end{aligned}$$

Since $\cos\nu_1 \neq \cos\nu_2$, it follows from (2.7) that the only zero of $\mathfrak{p}_2$ lies in $\mathbb{D}$. Thus, by Theorem 2.7 both the zeros of $\mathfrak{p}_1$ and hence all the three zeros of $\mathfrak{p}$ lie in $\mathbb{D}$. This completes the proof for $t \neq 1/2$. Now, for $t = 1/2$, we have

$$\mathfrak{p}(w) = e^{i\theta}w\mathfrak{p}_1(w). \tag{2.12}$$

Since $\mathfrak{p}_1$ has two zeros and both of them lie in $\mathbb{D}$, by (2.12), all the three zeros of $\mathfrak{p}$ lie in $\mathbb{D}$. This completes the proof of Theorem 2.1. □

**Corollary 2.8** *For $n \in \mathbb{N}$ and $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ such that*

$$h_k(z) - g_k(z) = \left(1 + (-1)^k a\right) \int_0^z q(\xi)\psi_{\mu,\nu_k}\left(\xi^n\right) d\xi, \quad \mu, \nu \in [0, 2\pi), \tag{2.13}$$

*where $q$ is an analytic mapping and $\psi_{\mu,\nu}$ is defined by (1.1). Let $\omega_{f_k}$ be the dilatation of $f_k$. If*

$$\omega_{f_1}(z) = -\omega_{f_2}(z) = \frac{a + e^{i(\theta-\mu)}z^n}{1 + ae^{i(\theta-\mu)}z^n}, \quad a \in (-1, 1), \theta \in [0, 2\pi),$$

*then the mapping $f = tf_1 + (1-t)f_2$ is locally univalent and sense-preserving for $0 \le t \le 1$ provided*:

(i) $\cos\theta > \max\{\cos\nu_2, -\cos\nu_1\}$ *and* $\cos\nu_2 > \cos\nu_1$, *or*
(ii) $\cos\theta < \min\{\cos\nu_2, -\cos\nu_1\}$ *and* $\cos\nu_1 > \cos\nu_2$.

*Proof* Following similarly as in Lemma 2.6, we find the expression for the dilatation $\omega_f$ of $f = tf_1 + (1-t)f_2$ as follows:

$$\omega_f(z) = \frac{t\omega_{f_1}(z)\psi_{\mu,\nu_1}(z^n)(1 - \omega_{f_2}(z))(1-a) + (1-t)\omega_{f_2}(z)\psi_{\mu,\nu_2}(z^n)(1 - \omega_{f_1}(z))(1+a)}{t\psi_{\mu,\nu_1}(z^n)(1 - \omega_{f_2}(z))(1-a) + (1-t)\psi_{\mu,\nu_2}(z^n)(1 - \omega_{f_1}(z))(1+a)}.$$

The above equation is identical with (2.10) except that $\cos\nu_1$ and $\cos\nu_2$ are interchanged. Hence, the result follows by Lemma 2.6. □

By using Lemma 2.6, we now examine the local univalence of $f$ in Theorem 2.1 for some specific values of $p_k$.

**Theorem 2.9** *For $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ such that*

$$h_k(z) + e^{2i\mu}g_k(z) = \left(1 + (-1)^k a\right) \int_0^z \psi_{\mu,\nu_k}(\xi)\, d\xi, \quad -1 < a < 1, \tag{2.14}$$

*where $\psi_{\mu,\nu_k}$ is defined by (1.1). Let $\omega_{f_k}$ be the dilatation of $f_k$. If*

$$\omega_{f_1}(z) = -\omega_{f_2}(z) = -e^{-2i\mu}\frac{a + e^{i(\theta-\mu)}z}{1 + ae^{i(\theta-\mu)}z}, \quad 0 \le \theta < 2\pi, \tag{2.15}$$

*then the mapping $f = tf_1 + (1-t)f_2$ is univalent and convex in the direction $\mu + \pi/2$ for $0 \le t \le 1$ provided $\theta$ and $\nu_k$ are given as in Corollary 2.8.*

*Proof* Let $F_k = H_k + \overline{G_k}$, where $H_k = h_k$ and $G_k = -e^{2i\mu}g_k$. Then, in view of (2.14) and (2.15), we have

$$H_k(z) - G_k(z) = \left(1 + (-1)^k a\right) \int_0^z \psi_{\mu,\nu_k}(\xi)\, d\xi,$$

and the dilatation of $\omega_{F_k}$ of $F_k$ is given by

$$\omega_{F_1}(z) = -\omega_{F_2}(z) = \frac{a + e^{i(\theta-\mu)}z}{1 + ae^{i(\theta-\mu)}z}.$$

Therefore, by Corollary 2.8, the mapping $F := tF_1 + (1-t)F_2$ is locally univalent and sense-preserving. Thus,

$$\left|\frac{tG_1' + (1-t)G_2'}{tH_1' + (1-t)H_2'}\right| < 1 \quad \text{on } \mathbb{D}.$$

Equivalently,

$$\left|\frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'}\right| < 1 \quad \text{on } \mathbb{D}.$$

Hence, $f$ is locally univalent and sense-preserving. Now, we can write (2.14) as

$$h_k(z) - e^{2i(\mu+\pi/2)}g_k(z) = \int_0^z \psi_{\mu+\pi/2,\pi/2}(\xi)p_k(\xi)\,d\xi, \tag{2.16}$$

where $\psi_{\mu+\pi/2,\pi/2}$ is defined by (1.1) and

$$p_k(z) = \frac{(1+(-1)^k a)\psi_{\mu,\nu_k}(z)}{\psi_{\mu+\pi/2,\pi/2}(z)} =: \bigl(1+(-1)^k a\bigr)\tilde{p}_k\bigl(e^{-i\mu}z\bigr).$$

Therefore, in view of (2.16), Theorem 2.1 follows the result once we show that $\operatorname{Re} p_k$ or, equivalently, $\operatorname{Re}\tilde{p}_k$ is positive on $\mathbb{D}$. Since

$$\tilde{p}_k(z) = \frac{1-z^2}{1-2z\cos\nu_k + z^2}, \tag{2.17}$$

we see that

$$\left|\frac{\tilde{p}_k(z)-1}{\tilde{p}_k(z)+1}\right| = \left|\frac{z(\cos\nu_k - z)}{1 - z\cos\nu_k}\right| < 1,$$

and hence $\operatorname{Re}\tilde{p}_k(z) > 0$ on $\mathbb{D}$. This completes the proof. □

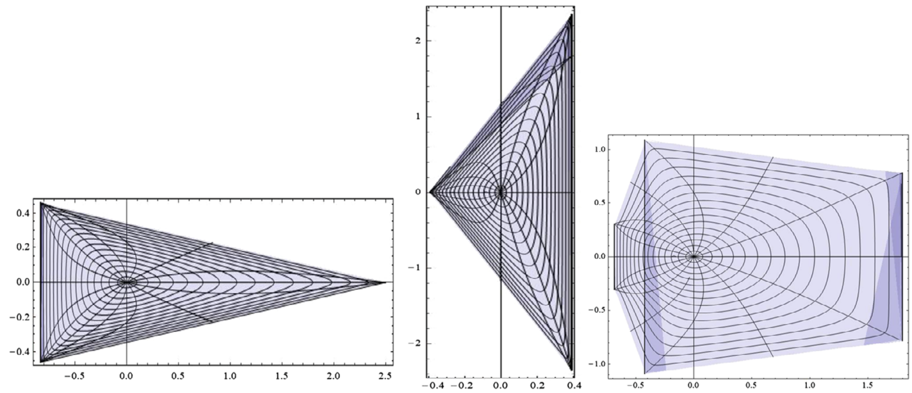
Next, we give an illustration of Theorem 2.9 through an example.

*Example* 2.10 For $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ be such that

$$\begin{aligned}
h_1(z) &= \frac{1}{4}\tan^{-1} z + \frac{3}{8}\log\frac{1+z^2}{(1-z)^2}, \\
g_1(z) &= \frac{1}{4}\tan^{-1} z - \frac{3}{8}\log\frac{1+z^2}{(1-z)^2}, \\
h_2(z) &= \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}z - 1) + \frac{1}{8+4\sqrt{2}}\log\frac{(1+z)^2}{1-\sqrt{2}z+z^2} + \frac{3\pi}{8\sqrt{2}},
\end{aligned}$$

and

$$g_2(z) = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}z - 1) - \frac{1}{8+4\sqrt{2}}\log\frac{(1+z)^2}{1-\sqrt{2}z+z^2} + \frac{3\pi}{8\sqrt{2}}.$$



(A) t=0 (B) t=1 (C) t=1/3

**Figure 2** Images of $\mathbb{D}$ under $f$ at different values of $t$

Then we have

$$h_1(z)+g_1(z)=\frac{1}{2}\tan^{-1}z=\int_0^z\frac{1/2}{1+\xi^2}\,d\xi,$$

$$h_2(z)+g_2(z)=-\frac{3i}{2\sqrt{2}}\log\frac{\sqrt{2}-(1-i)z}{\sqrt{2}-(1+i)z}=\int_0^z\frac{3/2}{1-\sqrt{2}\xi+\xi^2}\,d\xi,$$

and

$$\omega_{f_2}(z)=\frac{g_2'(z)}{h_2'(z)}=\frac{1/2+z}{1+z/2}=-\omega_{f_1}(z),$$

where $\omega_{f_k}$ is the dilatation of $f_k$. Thus, it is seen that $f_k$ satisfy (2.14) and (2.15) with $\mu=0$, $\nu_1=\pi/2$, $\nu_2=\pi/4$, $\theta=0$, and $a=1/2$. Moreover, since $\cos\nu_2=1/\sqrt{2}>0=\cos\nu_1$ and

$$\cos\theta=1>\frac{1}{\sqrt{2}}=\max\{\cos\nu_2,-\cos\nu_1\},$$

condition (i) in Corollary 2.8 holds. Hence, by Theorem 2.9, the mapping $f=tf_1+(1-t)f_2$ is univalent and convex in the imaginary direction for $0\le t\le 1$. Images of $\mathbb{D}$ under $f$ at $t=0$, $t=1$, and $t=1/3$ are shown in Fig. 2.

**Theorem 2.11** *For $k=1,2$, let $f=h_k+\overline{g_k}\in\mathcal{S}_H$ such that*

$$h_k(z)+e^{2i\mu}g_k(z)=\bigl(1+(-1)^k a\bigr)\frac{z(1-ze^{-i\mu}\cos\nu_k)}{1-z^2e^{-2i\mu}},\quad -1<a<1, \tag{2.18}$$

*for $\mu,\nu_k\in[0,2\pi)$. If $\omega_{f_k}$, the dilatation of $f_k$ is given by*

$$\omega_{f_1}(z)=-\omega_{f_2}(z)=-e^{-2i\mu}\frac{a+e^{i(\theta-\mu)}z}{1+ae^{i(\theta-\mu)}z},\quad 0\le\theta<2\pi, \tag{2.19}$$

*then the mapping $f=tf_1+(1-t)f_2$ is univalent and convex in the direction $\mu+\pi/2$ for $0\le t\le 1$ provided $\theta$ and $\nu_k$ are given as in Lemma 2.6.*

*Proof* Differentiating (2.18), we get

$$h_k'(z) + e^{2i\mu} g_k'(z) = \frac{(1 + (-1)^k a)(1 - 2ze^{-i\mu}\cos\nu_k + z^2 e^{-2i\mu})}{(1 - z^2 e^{-2i\mu})^2}.$$

The above equation can be written as

$$h_k(z) + e^{2i\mu} g_k(z) = \bigl(1 + (-1)^k a\bigr) \int_0^z \frac{q(\xi)}{\psi_{\mu,\nu_k}(\xi)}\, d\xi, \tag{2.20}$$

where $q(z) = (1 - z^2 e^{-2i\mu})^{-2}$. Similar to the proof of Theorem 2.9, by Lemma 2.6, we obtain that $f$ is locally univalent and sense-preserving. Also, we can write (2.20) as

$$h_k(z) - e^{2i(\mu+\pi/2)} g_k(z) = \int_0^z p_k(\xi)\psi_{\mu+\pi/2,\pi/2}(\xi)\, d\xi, \tag{2.21}$$

where

$$p_k(z) = \frac{(1 + (-1)^k a)(1 - 2ze^{-i\mu}\cos\nu_k + z^2 e^{-2i\mu})}{1 - z^2 e^{-2i\mu}}.$$

Note that $p_k(z) = (1 + (-1)^k a)/\tilde{p}_k(e^{-i\mu}z)$, where $\tilde{p}_k$ is defined by (2.17) and thus $\operatorname{Re}\tilde{p}_k$ or equivalently $\operatorname{Re} p_k$ is positive on $\mathbb{D}$. Therefore, in view of (2.21), Theorem 2.1 follows the result. □

*Remark* 2.12 If we put $a = \theta = \mu = 0$ in Theorem 2.11, we get Theorem 7 of Kumar *et al.* [7].

For $\mu, \nu \in [0, 2\pi)$, define $\Phi_{\mu,\nu}$ by

$$\Phi_{\mu,\nu}(z) = \frac{1 - \cos\nu}{4e^{-i\mu}} \log\left(\frac{1 + e^{-i\mu}z}{1 - e^{-i\mu}z}\right) + \frac{(1 + \cos\nu)z}{2(1 + e^{-2i\mu}z^2)}. \tag{2.22}$$

The mapping $\Phi_{0,\nu}$ maps $\mathbb{D}$ onto a domain with parallel slits along the real direction and its harmonic shears along the real direction were studied in [6]. In the next result we find sufficient conditions for the directional convexity of the convex combination of harmonic shears of $\Phi_{\mu,\nu}$.

**Theorem 2.13** *For $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ such that*

$$h_k(z) - e^{2i\mu} g_k(z) = \bigl(1 + (-1)^k a\bigr)\Phi_{\mu,\nu_k}(z), \quad a \in (-1, 1), \mu, \nu_k \in [0, 2\pi), \tag{2.23}$$

*where $\Phi_{\mu,\nu_k}$ is defined by (2.22). If $\omega_{f_k}$, the dilatation of $f_k$ is given by*

$$\omega_{f_1}(z) = -\omega_{f_2}(z) = e^{-2i\mu} \frac{a + e^{i(\theta-2\mu)}z^2}{1 + ae^{i(\theta-2\mu)}z^2}, \quad 0 \le \theta < 2\pi,$$

*then the mapping $f = tf_1 + (1 - t)f_2$ is univalent and convex in the direction $\mu$ for $0 \le t \le 1$ provided $\theta$ and $\nu_k$ are given as in Lemma 2.6.*

*Proof* On differentiating (2.23), we have

$$\begin{aligned} h_k'(z) - e^{2i\mu} g_k'(z) &= \bigl(1 + (-1)^k a\bigr) \left( \frac{1 - \cos \nu_k}{2(1 - e^{-2i\mu} z^2)} + \frac{(1 + \cos \nu_k)(1 - e^{-2i\mu} z^2)}{2(1 + e^{-2i\mu} z^2)^2} \right) \\ &= \bigl(1 + (-1)^k a\bigr) \frac{1 - 2\cos \nu_k e^{-2i\mu} z^2 + e^{-4i\mu} z^4}{(1 - e^{-2i\mu} z^2)(1 + e^{-2i\mu} z^2)^2}. \end{aligned}$$

Therefore,

$$h_k(z) - e^{2i\mu} g_k(z) = \bigl(1 + (-1)^k a\bigr) \int_0^z \frac{q(\xi)}{\psi_{2\mu,\nu_k}(\xi^2)}\, d\xi, \tag{2.24}$$

where

$$q(z) = \frac{1}{(1 - e^{-2i\mu} z^2)(1 + e^{-2i\mu} z^2)^2}.$$

Hence, following similarly as in the proof of Theorem 2.9, we see by using Lemma 2.6 that $f$ is locally univalent and sense-preserving. Moreover, (2.24) can also be written as

$$h_k(z) - e^{2i\mu} g_k(z) = \int_0^z p_k(\xi) \psi_{\mu,\pi/2}(\xi)\, d\xi, \tag{2.25}$$

where

$$p_k(z) = \bigl(1 + (-1)^k a\bigr) \frac{1 - 2\cos \nu_k e^{-2i\mu} z^2 + e^{-4i\mu} z^4}{1 - e^{-4i\mu} z^4}.$$

Since $p_k(z) = (1 + (-1)^k a)/\tilde{p}_k(e^{-2i\mu} z^2)$, where $\tilde{p}_k$ is defined by (2.17) and thus $\operatorname{Re} \tilde{p}_k$ or equivalently $\operatorname{Re} p_k$ is positive on $\mathbb{D}$. Therefore, in view of (2.25), the result follows from Theorem 2.1. □

**Theorem 2.14** *For $k = 1, 2$, let $f_k = h_k + \overline{g_k} \in \mathcal{S}_H$ such that*

$$h_k(z) - e^{2i\mu} g_k(z) = \bigl(1 + (-1)^k a\bigr) \int_0^z \Psi_k(\xi)\, d\xi, \quad a \in (-1, 1),$$

*where*

$$\Psi_k(z) = \frac{1 - 2\cos \nu_k e^{-in\mu} z^n + e^{-2in\mu} z^{2n}}{(1 - e^{-2in\mu} z^{2n})(1 - 2\cos \nu e^{-i\mu} z + e^{-2i\mu} z^2)}, \quad n \in \mathbb{N}, \mu, \nu, \nu_k \in [0, 2\pi).$$

*If $\omega_{f_k}$, the dilatation of $f_k$ is given by*

$$\omega_{f_1}(z) = -\omega_{f_2}(z) = e^{-2i\mu} \frac{a + e^{i(\theta - n\mu)} z^n}{1 + a e^{i(\theta - n\mu)} z^n}, \quad 0 \le \theta < 2\pi,$$

*then the mapping $f = t f_1 + (1 - t) f_2$ is univalent and convex in the direction $\mu$ for $0 \le t \le 1$ provided $\theta$ and $\nu_k$ are given as in Lemma 2.6.*

The proof of the above theorem is similar to that of Theorem 2.13 and is thus omitted here.

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**Authors' contributions**
All authors worked in coordination. All authors carried out the proof, read and approved the current version of the manuscript.

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