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On Copson's inequalities for $0 < p < 1$

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Abstract

Let $(\lambda_n)_{n\geq 1}$ be a positive sequence and let $\Lambda_n = \sum_{i=1}^n \lambda_i$. We study the following Copson inequality for $0 < p < 1$, $L > p$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\Lambda_n}\sum_{k=n}^{\infty}\lambda_k x_k\right)^p \geq \left(\frac{p}{L-p}\right)^p \sum_{n=1}^{\infty} x_n^p.$$

We find conditions on λ_n such that the above inequality is valid with the constant being the best possible.

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1 Introduction

Let $p > 0$ and $\mathbf{x} = (x_n)_{n\geq 1}$ be a positive sequence Let $(\lambda_n)_{n\geq 1}$ be a positive sequence and let $\Lambda_n = \sum_{i=1}^n \lambda_i$. The well-known Copson inequalities [4, Theorem 1.1, 2.1] state that

$$\sum_{n=1}^{\infty}\lambda_n\Lambda_n^{-c}\left(\sum_{k=1}^{n}\lambda_k x_k\right)^p \leq \left(\frac{p}{c-1}\right)^p \sum_{n=1}^{\infty}\lambda_n\Lambda_n^{p-c}x_n^p, \quad 1 < c \leq p; \tag{1.1}$$

$$\sum_{n=1}^{\infty}\lambda_n\Lambda_n^{-c}\left(\sum_{k=n}^{\infty}\lambda_k x_k\right)^p \leq \left(\frac{p}{1-c}\right)^p \sum_{n=1}^{\infty}\lambda_n\Lambda_n^{p-c}x_n^p, \quad 0 \leq c < 1 < p. \tag{1.2}$$

The above two inequalities are equivalent (see [10]) and the constants are best possible. When $\lambda_k = 1$, $k \geq 1$ and $c = p$, inequality (1.1) becomes the following celebrated Hardy inequality ([12, Theorem 326]):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} x_n^p. \tag{1.3}$$

Note that the reversed inequality of (1.2) holds when $c \leq 0 < p < 1$ ([4, Theorem 2.3]) with the constant being best possible and as pointed out in [1, p. 390], the reversed inequality of (1.2) continues to hold with constant p^p when $c > 0$. The particular case of

$c = p$, $\lambda_k = 1$, $k \geq 1$ becomes the following one given in [12, Theorem 345]:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=n}^{\infty}x_k\right)^p \geq p^p\sum_{n=1}^{\infty}x_n^p. \tag{1.4}$$

It is noted in [12] that the constant p^p in (1.4) may not be best possible and the best constant for $0 < p \leq 1/3$ was shown by Levin and Stečkin [13, Theorem 61] to be indeed $(p/(1-p))^p$. In [8], it is shown that the constant $(p/(1-p))^p$ stays best possible for all $0 < p \leq 0.346$. It is further shown in [11] that the constant $(p/(1-p))^p$ is best possible when $p = 0.35$.

There exists an extensive and rich literature on extensions and generalizations of Copson's inequalities and Hardy's inequality (1.3) for $p > 1$. For recent developments in this direction, we refer the reader to the articles in [6–11] and the references therein. On the contrary, the case $0 < p < 1$ is less known as can be seen by comparing inequalities (1.3) and (1.4). On one hand, the constant in (1.3) is shown to be best possible for all $p > 1$. On the other hand, though it is known the best constant that makes inequality (1.4) valid is $(p/(1-p))^p$ when $0 < p \leq 0.346$, it is shown in [8] that the constant $(p/(1-p))^p$ fails to be best possible when $1/2 \leq p < 1$ and the best constant in these cases remains unknown.

Our goal in this paper is to study the following variation of Copson's inequalities for $0 < p < 1$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\Lambda_n}\sum_{k=n}^{\infty}\lambda_k x_k\right)^p \geq \left(\frac{p}{L-p}\right)^p\sum_{n=1}^{\infty}x_n^p, \tag{1.5}$$

where $L > p$ is a constant.

It is an open problem to determine the best possible constant to make inequality (1.5) valid in general. Our choice for presenting the constant in the form $(p/(L-p))^p$ in (1.5) is motivated by the study on the analogue case of inequality (1.5) when $p > 1$. We define

$$L_\lambda := \sup_n\left(\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n}\right). \tag{1.6}$$

A result of Cartlidge [3] shows that when $L_\lambda < p$ for $p > 1$, then the following inequality holds for all non-negative sequences $\mathbf{x}$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k x_k\right)^p \leq \left(\frac{p}{p-L_\lambda}\right)^p\sum_{n=1}^{\infty}x_n^p, \quad p > 1. \tag{1.7}$$

We shall see in Theorem 1.3 below that the constant given in inequality (1.5) is indeed best possible for certain sequences (λ_n) and certain ranges of p when one replaces L by L_λ in (1.5). This includes case concerning the classical inequality (1.4) (with p^p replaced by $(p/(1-p))^p$ there).

Further, let $q < 0$ be the number satisfying $1/p + 1/q = 1$, we note that inequality (1.5) is equivalent to its dual version (assuming that $x_n > 0$ for all n):

$$\sum_{n=1}^{\infty}\left(\lambda_n\sum_{k=1}^{n}\frac{x_k}{\Lambda_k}\right)^q \leq \left(\frac{p}{L-p}\right)^q\sum_{n=1}^{\infty}x_n^q. \tag{1.8}$$

The equivalence of the above two inequalities can be easily established following the discussions in [8, Sect. 1].

Our main result gives a condition on λ_n and L such that inequalities (1.5) and (1.8) hold. For this purpose, we define, for constants p and L,

$$\begin{aligned} a_1(L,p) &= \left(\frac{L}{p}-2\right)^2\left(1+L\frac{2-p}{1-p}\right) \\ &\quad - \left(1+\left(\frac{L}{p}-2\right)\frac{1-2p}{1-p}\right) \\ &\quad \times \bigl(L^2(L-1)^2+2L(L-1)(L-p-1)+L^2-2(L-1)(p+1)\bigr) \\ &= \frac{2p-1}{p(1-p)}L^5 - \frac{3p-1}{1-p}L^4 - \frac{4p^3+2p^2-p-2}{p^2(1-p)}L^3 \\ &\quad + \frac{6p^4+4p^3+2p^2-9p+1}{p^2(1-p)}L^2 + \frac{14p-6}{p(1-p)}L - \frac{6p^2+8p-6}{1-p}, \\ a_2(L,p) &= \left(\frac{L}{p}-1\right)L^4 + \frac{(1-p)(1-2p)}{p}L^3 \\ &\quad - (3-p)(1-p)L^2 - \bigl(p^2-p+2\bigr)L + 2p(1+p). \end{aligned} \tag{1.9}$$

Our main result is the following statement.

Theorem 1.1 *Let $0 < p < 1$ be fixed. Let $\lambda = (\lambda_n)_{n\ge 1}$ be a positive sequence and let $\Lambda_n = \sum_{i=1}^n \lambda_i$. If there exists a positive constant $L > p$ such that, for any integer $n \ge 1$,*

$$\frac{L-p}{p}\cdot\frac{\lambda_n}{\Lambda_n} \le \left(1+\left(\frac{L}{p}-2\right)\frac{\lambda_n}{\Lambda_n}\right)^{1/(1-p)} - \left(\frac{\lambda_n}{\lambda_{n+1}}\right)^{1/(1-p)}\left(\frac{\Lambda_n}{\Lambda_{n+1}}\right)^{p/(1-p)}, \tag{1.10}$$

then inequality (1.5) holds for all non-negative sequences $\mathbf{x}$. In particular, let L_λ be defined by (1.6) and let $a_i(L,p)$, $i = 1,2$ be defined by (1.9), then inequality (1.5) holds with L replaced by L_λ there for all non-negative sequences $\mathbf{x}$ when $L_\lambda \ge 1$, $0 < p \le 1/3$ and $a_1(L_\lambda,p) \ge 0$ or when $0 < L_\lambda < 1$, $0 < p \le L_\lambda/4$ and $a_2(L_\lambda,p) \ge 0$.

We note that when $0 < p < 1/2$, we have $\lim_{L\to\infty} a_1(L,p) < 0$, $\lim_{p\to 0^+} a_2(4p,p)/p < 0$. This implies that the values of $a_i(L,p)$, $i = 1,2$ do give restrictions on the validity of inequality (1.5). We note that when $\lambda_n = 1$, then $L_\lambda = 1$, $a_1(1,p) = 3(\frac{1}{p}-2)^2 - 1$ and Theorem 1.1 implies the above-mentioned result of Levin and Stečkin.

In [2, 5–8], two special cases of inequality (1.7) corresponding to $\lambda_n = n^\alpha - (n-1)^\alpha$ and $\lambda_n = n^{\alpha-1}$ for $p > 1$, $\alpha p > 1$ were studied. It follows from this work that inequality (1.7) holds in either case with best possible constant $(\alpha p/(\alpha p - 1))^p$ except for the case when $\lambda_n = n^{\alpha-1}$, $1 < p \le 4/3$, $1 \le \alpha < 1 + 1/p$ or $4/3 < p < 2$, $1 \le \alpha < 2$.

It is now interesting to study the following $0 < p < 1$, $\alpha < 1/p$ analogues of the above inequalities:

$$\sum_{n=1}^\infty\left(\frac{1}{n^\alpha}\sum_{k=n}^\infty\bigl(k^\alpha-(k-1)^\alpha\bigr)x_k\right)^p \ge \left(\frac{\alpha p}{1-\alpha p}\right)^p\sum_{n=1}^\infty x_n^p, \tag{1.11}$$

$$\sum_{n=1}^\infty\left(\frac{1}{\sum_{i=1}^n i^{\alpha-1}}\sum_{k=n}^\infty k^{\alpha-1}x_k\right)^p \ge \left(\frac{\alpha p}{1-\alpha p}\right)^p\sum_{n=1}^\infty x_n^p. \tag{1.12}$$

Note that when $\alpha = 1$, the above two inequalities become inequality (1.4) (with the constant p^p there being replaced by the best possible constant $(p/(1-p))^p$). We note that it is shown in [8, (4.14)] that inequality (1.12) holds when $0 < \alpha < 1$, $0 < p < 1/(\alpha + 2)$. It is also shown in [9, Theorem 1.1] that inequality (1.11) holds when $\alpha > 0$, $0 < p < 1/(\alpha + 2)$ when one replaces $k^\alpha - (k-1)^\alpha$ by $(k+1)^\alpha - k^\alpha$. As $(k+1)^\alpha - k^\alpha \le k^\alpha - (k-1)^\alpha$ when $0 < \alpha \le 1$, this implies that inequality (1.11) holds when $0 < \alpha \le 1$, $0 < p < 1/(\alpha + 2)$.

Note that the values of p are not given explicitly in (1.10), nor by the conditions $a_i(L_\lambda, p) \ge 0$, $i = 1, 2$. Thus, Theorem 1.1 is not readily applied in practice. For this reason, and with future applications in mind, we develop the following result.

Theorem 1.2 *Let $0 < p < 1$ be fixed. Let $\lambda = (\lambda_n)_{n \ge 1}$ be a positive sequence and let $\Lambda_n = \sum_{i=1}^n \lambda_i$. Let L_λ be defined by (1.6) such that $0 < L_\lambda < 1$ and that*

$$p \le \frac{L_\lambda^2}{4} := p_{L_\lambda}, \tag{1.13}$$

then inequality (1.5) holds with L replaced by L_λ there for all non-negative sequences $\mathbf{x}$.

Suppose that there exist positive constants $1/2 < L < 1$, $0 < M < 1$, $L + 2M < 1$ such that, for any integer $n \ge 1$,

$$\frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n} \le L + M \frac{\lambda_n}{\Lambda_n}. \tag{1.14}$$

Then inequality (1.5) holds for all non-negative sequences $\mathbf{x}$ *when*

$$p \le \min\left\{ \frac{L(2L-1)}{2(4L+M)}, \frac{L(1-L-2M)}{2(1-L-M)} \right\}. \tag{1.15}$$

We remark here that it is easy to see that the minimum on the right-hand side of (1.15) can take either values. We now apply Theorem 1.2 to study inequalities (1.11)–(1.12). As the case $\alpha = 1$ yields the classical inequality (1.4), we concentrate on the case $\alpha > 1$ and we deduce readily from Theorem 1.2 the following result.

Theorem 1.3 *Let $\alpha \ge 1$ and $p_{1/\alpha}$ be defined as in (1.13). Then inequality (1.11) holds for all non-negative sequences* $\mathbf{x}$ *when $\alpha > 1$, $0 < p \le p_{1/\alpha}$ and inequality (1.12) holds for all non-negative sequences* $\mathbf{x}$ *when $\alpha \ge 2$, $0 < p \le p_{1/\alpha}$. The constants are best possible in both cases.*

In fact, note that [5, Lemma 2.1] implies that (1.6) is satisfied for $\lambda_n = n^\alpha - (n-1)^\alpha$ with $L_\lambda = 1/\alpha$ when $\alpha \ge 1$ and (1.6) is satisfied for $\lambda_n = n^{\alpha-1}$ with $L_\lambda = 1/\alpha$ when $\alpha \ge 2$. That the constant is best possible can be seen by setting $x_n = n^{-1/p-\epsilon}$ and letting $\epsilon \to 0^+$.

2 Proof of Theorem 1.1

For the first assertion of Theorem 1.1, our goal is to find conditions on the λ_n such that the following inequality holds for $0 < p < 1$, $L > p$:

$$\sum_{n=1}^\infty \left(\frac{1}{\Lambda_n} \sum_{k=n}^\infty \lambda_k x_k \right)^p \ge \left(\frac{p}{L-p} \right)^p \sum_{n=1}^\infty x_n^p.$$

It suffices to prove the above inequality by replacing the infinite sums by finite sums from $n = 1$ to N (and $k = n$ to N) for any integer $N \geq 1$. Note that, as in [6, Sect. 3], we have, for any positive sequence $\mathbf{w} = (w_n)$,

$$\sum_{n=1}^{N} x_n^p = \sum_{n=1}^{N} \frac{x_n^p}{\sum_{i=1}^{n} w_i} \sum_{k=1}^{n} w_k = \sum_{n=1}^{N} w_n \sum_{k=n}^{N} \frac{x_k^p}{\sum_{i=1}^{k} w_i}.$$

By Hölder's inequality, we have

$$\sum_{k=n}^{N} \frac{x_k^p}{\sum_{i=1}^{k} w_i} \leq \left(\sum_{k=n}^{N} \left(\lambda_k^p \sum_{i=1}^{k} w_i \right)^{-1/(1-p)} \right)^{1-p} \left(\sum_{k=n}^{N} \lambda_k x_k \right)^p.$$

It follows that

$$\sum_{n=1}^{N} x_n^p \leq \sum_{n=1}^{N} w_n \left(\sum_{k=n}^{N} \left(\lambda_k^p \sum_{i=1}^{k} w_i \right)^{-1/(1-p)} \right)^{1-p} \left(\sum_{k=n}^{N} \lambda_k x_k \right)^p.$$

Suppose we can find a sequence $\mathbf{w} = (w_n)$ of positive terms, such that, for any integer $n \geq 1$,

$$\left(\sum_{i=1}^{n} w_i \right)^{1/(p-1)} \leq \left(\frac{p}{L-p} \right)^{p/(p-1)} \lambda_n^{p/(1-p)} \left(\frac{w_n^{1/(p-1)}}{\Lambda_n^{p/(1-p)}} - \frac{w_{n+1}^{1/(p-1)}}{\Lambda_{n+1}^{p/(1-p)}} \right).$$

Then the desired inequality follows. Now we make a change of variables: $w_i \mapsto \lambda_i w_i$ to recast the above inequality as

$$\left(\sum_{i=1}^{n} \lambda_i w_i \right)^{1/(p-1)} \leq \left(\frac{p}{L-p} \right)^{p/(p-1)} \lambda_n^{p/(1-p)} \left(\frac{\lambda_n^{1/(p-1)} w_n^{1/(p-1)}}{\Lambda_n^{p/(1-p)}} - \frac{\lambda_{n+1}^{1/(p-1)} w_{n+1}^{1/(p-1)}}{\Lambda_{n+1}^{p/(1-p)}} \right). \tag{2.1}$$

We now define our sequence $\mathbf{w} = (w_n)$ to satisfy $w_1 = 1$ and we inductively see that, for $n \geq 2$,

$$\frac{1}{\Lambda_n} \sum_{i=1}^{n} \lambda_i w_i = \frac{p}{L-p} w_{n+1}. \tag{2.2}$$

This allows us to deduce that, for $n \geq 1$,

$$w_{n+1} = \left(1 + \left(\frac{L}{p} - 2 \right) \frac{\lambda_n}{\Lambda_n} \right) w_n. \tag{2.3}$$

Applying (2.2), (2.3) in (2.1), we see that inequality (2.1) becomes (1.10). This completes the proof for the first assertion of Theorem 1.1.

We now prove the second assertion of Theorem 1.1. We set $x = \lambda_n / \Lambda_n$, $y = \lambda_{n+1} / \Lambda_{n+1}$ to recast inequality (1.10) as

$$\frac{L-p}{p} x \leq \left(1 + \left(\frac{L}{p} - 2 \right) x \right)^{1/(1-p)} - \left(\frac{1}{y} - 1 \right)^{(1+p)/(1-p)} x^{1/(1-p)} y^{p/(1-p)}.$$

To facilitate also the proof of Theorem 1.2 below, we proceed by taking the condition (1.14) into consideration to assume that L is a constant such that $1/y \le 1/x + L + Mx$, where $M \ge 0$ is another constant. As the function $t \mapsto (t-1)^{(1+p)/(1-p)} t^{-p/(1-p)}$ is an increasing function of $t \ge 1$, we deduce that it suffices to prove the above inequality for $1/y = 1/x + L + Mx$, which is equivalent to showing that $f_{L,M,p}(x) \ge 0$ for $0 < x \le 1$, where

$$\begin{aligned} f_{L,M,p}(x) &:= \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{1/(1-p)} \\ &\quad - \left(1 + (L-1)x + Mx^2\right)^{(1+p)/(1-p)} \left(1 + Lx + Mx^2\right)^{-p/(1-p)} - \frac{L-p}{p}x. \end{aligned}$$

Suppose that (1.6) is valid and $L_\lambda \ge 1$. In this case we set $L = L_\lambda$ and $M = 0$ so that it suffices to show that $f_{L,0,p}(x) \ge 0$. Calculation shows that

$$\begin{aligned} \frac{(1-p)^2}{p} f''_{L,0,p}(x) &= \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{(2p-1)/(1-p)} \\ &\quad \times \left(1 + (L-1)x\right)^{2p/(1-p)-1} (1 + Lx)^{-1/(1-p)-1} g_{L,p}(x), \end{aligned}$$

where

$$\begin{aligned} g_{L,p}(x) &= \left(\frac{L}{p} - 2\right)^2 \left(1 + (L-1)x\right)^{\frac{1-3p}{1-p}} (1 + Lx)^{\frac{2-p}{1-p}} \\ &\quad - \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{\frac{1-2p}{1-p}} \\ &\quad \times \left(L^2(L-1)^2x^2 + 2L(L-1)(L-p-1)x + L^2 - 2(L-1)(p+1)\right). \end{aligned}$$

Suppose that $0 < p \le 1/3$. We want to show that $g_{L,p}(x) \ge 0$ for $0 < x \le 1$. We first note that we have

$$\begin{aligned} g_{L,p}(x) &\ge \left(\frac{L}{p} - 2\right)^2 \left(1 + (L-1)x\right)^{\frac{1-3p}{1-p}} (1 + Lx)^{\frac{2-p}{1-p}} \\ &\quad - \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{\frac{1-2p}{1-p}} \\ &\quad \times \left(L^2(L-1)^2x + 2L(L-1)(L-p-1)x + L^2 - 2(L-1)(p+1)\right). \end{aligned}$$

We may now assume that

$$L^2(L-1)^2x + 2L(L-1)(L-p-1)x + L^2 - 2(L-1)(p+1) \ge 0.$$

Otherwise, we have trivially $g_{L,p}(x) \ge 0$. We then estimate $(1 + (L-1)x)^{\frac{1-3p}{1-p}}$ trivially by $(1 + (L-1)x)^{\frac{1-3p}{1-p}} \ge 1$ and we apply a Taylor expansion to see that

$$\begin{aligned} &(1 + Lx)^{\frac{2-p}{1-p}} \ge 1 + L\frac{2-p}{1-p}x, \\ &\left(1 + \left(\frac{L}{p} - 2\right)x\right)^{\frac{1-2p}{1-p}} \le 1 + \left(\frac{L}{p} - 2\right)\frac{1-2p}{1-p}x. \end{aligned}$$

It then follows that $g_{L,p}(x) \geq u_{L,p}(x)$, where

$$\begin{aligned} u_{L,p}(x) &= \left(\frac{L}{p}-2\right)^2\left(1+L\frac{2-p}{1-p}x\right) \\ &\quad - \left(1+\left(\frac{L}{p}-2\right)\frac{1-2p}{1-p}x\right) \\ &\quad \times \left(L^2(L-1)^2x + 2L(L-1)(L-p-1)x + L^2 - 2(L-1)(p+1)\right). \end{aligned}$$

Suppose first that

$$L^2(L-1)^2 + 2L(L-1)(L-p-1) \leq 0. \tag{2.4}$$

We then deduce that

$$\begin{aligned} u_{L,p}(x) &= \left(\frac{L}{p}-2\right)^2\left(1+L\frac{2-p}{1-p}x\right) - \left(1+\left(\frac{L}{p}-2\right)\frac{1-2p}{1-p}x\right)\left(L^2-2(L-1)(p+1)\right) \\ &\quad - \left(1+\left(\frac{L}{p}-2\right)\frac{1-2p}{1-p}x\right)\left(L^2(L-1)^2+2L(L-1)(L-p-1)\right)x \\ &\geq \left(\frac{L}{p}-2\right)^2\left(1+L\frac{2-p}{1-p}x\right) - \left(1+\left(\frac{L}{p}-2\right)\frac{1-2p}{1-p}x\right)\left(L^2-2(L-1)(p+1)\right). \end{aligned} \tag{2.5}$$

We regard the last expression above as a linear function of x to see that its derivative with respect to x is

$$\begin{aligned} &\left(\frac{L}{p}-2\right)^2\left(\frac{2-p}{1-p}\right)L - \left(\frac{L}{p}-2\right)\left(\frac{1-2p}{1-p}\right)\left(L^2-2(L-1)(p+1)\right) \\ &\quad \geq \left(\frac{L}{p}-2\right)^2\left(\frac{2-p}{1-p}\right)L - \left(\frac{L}{p}-2\right)\left(\frac{1-2p}{1-p}\right)L^2 \\ &\quad = \left(\frac{L}{p}-2\right)\left(\frac{L}{1-p}\right)\left(\left(\frac{L}{p}-2\right)(2-p) - L(1-2p)\right). \end{aligned}$$

As we have $0 < p \leq 1/3$ and $L \geq 1$, we deduce that

$$\left(\frac{L}{p}-2\right)(2-p) \geq 3L-2 \geq L \geq L(1-2p).$$

It follows from this that the last expression in (2.5) is minimized at $x = 0$ with corresponding value being $u_{L,p}(0)$. On the other hand, if the inequality in (2.4) does not hold, then one checks that $u_{L,p}(x)$ is a quadratic function of x with negative leading coefficient, hence is minimized at $x = 0$ or $x = 1$. Thus, in either case, we conclude that $u_{L,p}(x) \geq \min\{u_{L,p}(0), u_{L,p}(1)\}$ for $0 \leq x \leq 1$. One checks that

$$u_{L,p}(0) = \left(\frac{L}{p}-2\right)^2 - L^2 + 2(L-1)(p+1). \tag{2.6}$$

When we regard the above expression as a function of L, it is readily seen that $u_{L,p}(0)$ is convex in L such that

$$u_{1,p}(0) \geq 0, \qquad \frac{\partial u_{L,p}(0)}{\partial L}\bigg|_{L=1} \geq 0.$$

We thus deduce that $u_{L,p}(0) \geq 0$ when $0 < p \leq 1/3$. On the other hand, we have $u_{L,p}(1) = a_1(L,p) \geq 0$ by our assumption, where $a_1(L,p)$ is defined in (1.9). It follows that $g_{L,p}(x) \geq u_{L,p}(x) \geq 0$, hence $f''_{L,0,p}(x) \geq 0$ for $0 < x \leq 1$. As $f_{L,0,p}(0) = f'_{L,0,p}(0) = 0$, we then deduce that $f_{L,0,p}(x) \geq 0$ and this completes the proof for the case $L_\lambda \geq 1$ of the second assertion of Theorem 1.1.

To prove the case $0 < L_\lambda < 1$ of the second assertion of Theorem 1.1, we first note that

$$\begin{aligned}
&\frac{p}{L-p} f'_{L,M,p}(x) + 1 \\
&\quad = \left(\frac{L}{p} - 2\right) \frac{p}{(1-p)(L-p)} \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{p/(1-p)} \\
&\qquad + \frac{p(1+p)(1-L)}{(1-p)(L-p)} \left(1 - \frac{2Mx}{1-L}\right) \left(1 + (L-1)x + Mx^2\right)^{(2p)/(1-p)} \left(1 + Lx + Mx^2\right)^{-p/(1-p)} \\
&\qquad + \frac{Lp^2}{(1-p)(L-p)} \left(1 + \frac{2Mx}{L}\right) \left(1 + (L-1)x + Mx^2\right)^{(1+p)/(1-p)} \left(1 + Lx + Mx^2\right)^{-1/(1-p)}.
\end{aligned}$$

One checks that

$$\left(\frac{L}{p} - 2\right) \frac{p}{(1-p)(L-p)} + \frac{p(1+p)(1-L)}{(1-p)(L-p)} + \frac{Lp^2}{(1-p)(L-p)} = 1.$$

It follows from this and the arithmetic–geometric mean inequality that

$$\begin{aligned}
&\frac{p}{L-p} f'_{L,M,p}(x) + 1 \\
&\quad \geq \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{\frac{p}{1-p} \cdot (\frac{L}{p} - 2) \frac{p}{(1-p)(L-p)}} \\
&\qquad \cdot \left(\left(1 - \frac{2Mx}{1-L}\right) \left(1 + (L-1)x + Mx^2\right)^{(2p)/(1-p)} \left(1 + Lx + Mx^2\right)^{-p/(1-p)}\right)^{\frac{p(1+p)(1-L)}{(1-p)(L-p)}} \\
&\qquad \cdot \left(\left(1 + \frac{2Mx}{L}\right) \left(1 + (L-1)x + Mx^2\right)^{(1+p)/(1-p)} \left(1 + Lx + Mx^2\right)^{-1/(1-p)}\right)^{\frac{Lp^2}{(1-p)(L-p)}}.
\end{aligned} \tag{2.7}$$

Suppose that (1.6) is valid and $0 < L_\lambda < 1$. In this case we can also set $L = L_\lambda$ and $M = 0$ to see that inequality (2.7) becomes

$$\frac{p}{L-p} f'_{L,0,p}(x) + 1 \geq h_{L,p}(x)^{\frac{p}{(1-p)^2(L-p)}}, \tag{2.8}$$

where

$$h_{L,p}(x) := \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{L-2p} \left(1 + (L-1)x\right)^{p(1+p)(2-L)} (1 + Lx)^{-p(1+p-pL)}.$$

Calculation shows that

$$h'_{L,p}(x) = \left(1 + \left(\frac{L}{p} - 2\right)x\right)^{L-2p-1} \left(1 + (L-1)x\right)^{p(1+p)(2-L)-1} (1 + Lx)^{-p(1+p-pL)-1} v_{L,p}(x),$$

where

$$\begin{aligned} v_{L,p}(x) &= (L - 2p)\left(\frac{L}{p} - 2\right)\left(1 + (L-1)x\right)(1 + Lx) \\ &\quad - p(1+p)(2-L)(1-L)\left(1 + \left(\frac{L}{p} - 2\right)x\right)(1 + Lx) \\ &\quad - p(1+p-pL)L\left(1 + \left(\frac{L}{p} - 2\right)x\right)\left(1 + (L-1)x\right). \end{aligned}$$

One checks that $v_{L,p}(x)$ is a quadratic polynomial of x with a negative leading coefficient when $L \geq 2p$. It follows that $v_{L,p}(x) \geq \min\{v_{L,p}(0), v_{L,p}(1)\}$ for $0 < x \leq 1$ and one checks that $v_{L,p}(0) = pu_{L,p}(0)$, where $u_{L,p}(0)$ is defined in (2.6). Similar to our discussions for the case $L > 1$, one checks that $u_{L,p}(0)$ is convex in L such that

$$u_{4p,p}(0) \geq 0, \qquad \left.\frac{\partial u_{L,p}(0)}{\partial L}\right|_{L=4p} \geq 0,$$

with $u_{L,p}(0) \geq 0$ when $L \geq 4p$. On the other hand, we have $v_{L,p}(1) = a_2(L,p) \geq 0$ by our assumption, where $a_2(L,p)$ is defined in (1.9). It follows that $h'_{L,p}(x) \geq 0$, hence $h_{L,p}(x) \geq h_{L,p}(0) \geq 1$. It follows from this and (2.8) that $f'_{L,0,p}(x) \geq 1$ for $0 < x \leq 1$. Now, since $f_{L,0,p}(x) = 0$, this implies that $f_{L,0,p}(x) \geq 0$ for $0 < x \leq 1$, which then completes the proof for the case $0 < L_\lambda < 1$ of the second assertion of Theorem 1.1.

3 Proof of Theorem 1.2

First, we assume that (1.6) is valid and we set $L = L_\lambda$ in this case. It suffices to find a value of p such that $a_2(L,p) \geq 0$ by Theorem 1.1. Note that $\lim_{p \to 0^+} a_2(ap,p)/p < 0$ when $a > 1$, it is therefore not possible to show $a_2(L,p) \geq 0$ by assuming that $p \leq L/a$ for any $a > 1$. We therefore seek to show $a_2(L,p) \geq 0$ for $p \leq L^2/4$. We first note that

$$\begin{aligned} \frac{a_2(L,p)}{1-p} &\geq \frac{1}{p}L^4 + \frac{1-2p}{p}L^3 - (3-p)L^2 + \left(p - \frac{2}{1-p}\right)L + \frac{2p(1+p)}{1-p} \\ &\geq 4L^2 + 4(1-2p)L - (3-p)L^2 + \left(p - \frac{2}{1-p}\right)L + \frac{2p(1+p)}{1-p}L \\ &\geq 4(1-2p)L + \left(p - \frac{2}{1-p}\right)L + 2pL \\ &= \left(4 - 5p - \frac{2}{1-p}\right)L \geq 0, \end{aligned}$$

where the last inequality follows from the observation that the function $p \mapsto 4 - 5p - \frac{2}{1-p}$ is non-negative for $0 < p \leq 1/4$. This completes the proof for the first assertion of Theorem 1.2.

To prove the second assertion of Theorem 1.2, we see from the proof of Theorem 1.1 that it suffices to show the right-hand side expression of (2.7) is ≥ 1 for $0 < x \leq 1$. We simplify it to see that it is equivalent to showing that

$$\begin{aligned} &\left(1+\left(\frac{L}{p}-2\right)x\right)^{L-2p}\left(1-\frac{2Mx}{1-L}\right)^{(1-p^2)(1-L)}\left(1+\frac{2Mx}{L}\right)^{p(1-p)L} \\ &\quad\cdot\left(1+(L-1)x+Mx^2\right)^{p(1+p)(2-L)}\left(1+Lx+Mx^2\right)^{-p(1+p-pL)} \geq 1. \end{aligned} \tag{3.1}$$

We assume that

$$\frac{L}{p}-2 \geq 0. \tag{3.2}$$

This implies that the function

$$\left(1+\left(\frac{L}{p}-2\right)x\right)\left(1-\frac{2Mx}{1-L}\right)$$

is a concave function of x, hence is minimized at $x = 0$ or 1. When $x = 0$, the above function takes the value 1. We further assume that the above function takes a value ≥ 1 when $x = 1$. That is,

$$\left(\frac{L}{p}-1\right)\left(1-\frac{2M}{1-L}\right) \geq 1. \tag{3.3}$$

We then deduce that

$$1-\frac{2Mx}{1-L} \geq \left(1+\left(\frac{L}{p}-2\right)x\right)^{-1}.$$

We apply the above estimation and the estimation that $1 + 2Mx/L \geq 1$ in (3.1) to see that it suffices to show that, for $0 < x \leq 1$,

$$\begin{aligned} h_{L,M,p}(x) :=& \left(1+\left(\frac{L}{p}-2\right)x\right)^{L-2p-(1-p^2)(1-L)} \\ &\times\left(1+(L-1)x+Mx^2\right)^{p(1+p)(2-L)}\left(1+Lx+Mx^2\right)^{-p(1+p-pL)} \\ \geq& 1. \end{aligned}$$

We now assume that

$$L-2p-\left(1-p^2\right)(1-L) > 0. \tag{3.4}$$

Then calculation shows that

$$\begin{aligned} h'_{L,M,p}(x) :=& \left(1+\left(\frac{L}{p}-2\right)x\right)^{L-2p-(1-p^2)(1-L)-1} \\ &\cdot\left(1+(L-1)x+Mx^2\right)^{p(1+p)(2-L)-1}\left(1+Lx+Mx^2\right)^{-p(1+p-pL)-1}u_{L,M,p}(x), \end{aligned}$$

where

$$\begin{aligned}
u_{L,M,p}(x) &= \big(L-2p-\big(1-p^2\big)(1-L)\big)\left(\frac{L}{p}-2\right)\big(1+(L-1)x+Mx^2\big)\big(1+Lx+Mx^2\big)\\
&\quad -p(1+p)(2-L)(1-L)\left(1+\left(\frac{L}{p}-2\right)x\right)\left(1-\frac{2Mx}{1-L}\right)\big(1+Lx+Mx^2\big)\\
&\quad -p(1+p-pL)L\left(1+\left(\frac{L}{p}-2\right)x\right)\left(1+\frac{2Mx}{L}\right)\big(1+(L-1)x+Mx^2\big)\\
&\geq \big(L-2p-\big(1-p^2\big)(1-L)\big)\left(\frac{L}{p}-2\right)\big(1+(L-1)x\big)(1+Lx)\\
&\quad -p(1+p)(2-L)(1-L)\left(1+\left(\frac{L}{p}-2\right)x\right)\big(1+(L+M)x\big)\\
&\quad -p(1+p-pL)L\left(1+\left(\frac{L}{p}-2\right)x\right)\left(1+\frac{2M}{L}\right)\big(1+(L+M-1)x\big)\\
&:= v_{L,M,p}(x).
\end{aligned}$$

It is easy to see that $v_{L,M,p}(x)$ is a quadratic polynomial of x with negative leading coefficient when

$$\begin{aligned}
&\big(L-2p-\big(1-p^2\big)(1-L)\big)\left(\frac{L}{p}-2\right)(L-1)L-p(1+p)(2-L)(1-L)\left(\frac{L}{p}-2\right)(L+M)\\
&\quad -p(1+p-pL)\left(\frac{L}{p}-2\right)(L+2M)(L+M-1)\leq 0.
\end{aligned}$$

The above inequality is certainly valid when $L/p=2$. We may thus assume that $L/p>2$ to see that the above inequality is a consequence of the following inequality:

$$\begin{aligned}
&\big(L-2p-\big(1-p^2\big)(1-L)\big)(1-L)L+p(1+p)(2-L)(1-L)(L+M)\\
&\quad -p(1+p-pL)(L+2M)(1-L-M)\geq 0. \qquad (3.5)
\end{aligned}$$

Assuming the above inequality, we see that $v_{L,M,p}(x)\geq\min\{v_{L,M,p}(0),v_{L,M,p}(1)\}$ for $0<x\leq 1$ and we have

$$\begin{aligned}
v_{L,M,p}(0) &= \big(L-2p-\big(1-p^2\big)(1-L)\big)\left(\frac{L}{p}-2\right)\\
&\quad -p(1+p)(2-L)(1-L)-p(1+p-pL)(L+2M),\\
v_{L,M,p}(1) &= \big(L-2p-\big(1-p^2\big)(1-L)\big)\left(\frac{L}{p}-2\right)L(L+1)\\
&\quad -p(1+p)(2-L)(1-L)\left(\frac{L}{p}-1\right)(1+L+M)\\
&\quad -p(1+p-pL)\left(\frac{L}{p}-1\right)(L+2M)(L+M).
\end{aligned}$$

We note that, when $M < 1$,

$$p(1+p)(2-L)(1-L) + p(1+p-pL)(L+2M) \le 2p(1+p)(1+L+M),$$

$$\begin{aligned}
&p(1+p)(2-L)(1-L)\left(\frac{L}{p}-1\right)(1+L+M) + p(1+p-pL)\left(\frac{L}{p}-1\right)(L+2M)(L+M) \\
&\quad \le (1+p)L(1+L+M) + (1+p)L(L+2M)(L+M) \\
&\quad \le (1+p)L(1+L+M) + (1+p)L(1+L+M)(L+M) = (1+p)L(1+L+M)^2.
\end{aligned}$$

We then deduce that $v_{L,M,p}(0) \ge 0$ when

$$\left(L-2p-\left(1-p^2\right)(1-L)\right)\left(\frac{L}{p}-2\right) \ge 2p(1+p)(1+L+M), \tag{3.6}$$

and that $v_{L,M,p}(1) \ge 0$ when

$$\left(L-2p-\left(1-p^2\right)(1-L)\right)\left(\frac{L}{p}-2\right) \ge 2(1+p)(1+L+M). \tag{3.7}$$

Thus, one just needs to find values of p to satisfy inequalities (3.2)–(3.7). Indeed, if (3.2)–(3.7) hold, then $v_{L,M,p}(x) \ge 0$ for $0 < x \le 1$, which implies that $h'_{L,M,p}(x) \ge 0$ for $0 < x \le 1$. Now, since $h_{L,M,p}(0) = 1$, we get $h_{L,M,p}(x) \ge 1$ for $0 < x \le 1$ as desired.

We note that

$$p(1+p)(2-L)(1-L)(L+M) \ge 0,$$

$$p(1+p-pL)(L+2M)(1-L-M) \le p(1+p)(1-L)(1+L+M).$$

We apply the above estimates to see that inequality (3.5) is a consequence of the following inequality:

$$\left(L-2p-\left(1-p^2\right)(1-L)\right)\frac{L}{p} \ge (1+p)(1+L+M). \tag{3.8}$$

As inequality (3.7) implies inequalities (3.6) and (3.8), we first find values of p so that inequality (3.7) holds. To do so, we first simply inequality (3.7) by noting that

$$L-2p-\left(1-p^2\right)(1-L) \ge 2L-1-2p.$$

Using this, we see that it suffices to find values of p that satisfy

$$(2L-1-2p)\left(\frac{L}{p}-2\right) \ge 2(1+p)(1+L+M).$$

We recast the above as

$$(2L-1)L - 2(4L+M)p + 2(1-L-M)p^2 \ge 0.$$

Note as $1-L-M > 0$, we have $(1-L-M)p^2 \ge 0$, so that the above inequality follows from

$$(2L-1)L - 2(4L+M)p \ge 0,$$

which is equivalent to

$$p \le \frac{L(2L-1)}{2(4L+M)}. \tag{3.9}$$

One checks that the above inequality implies inequalities (3.2) and (3.4). One further notes that inequality (3.3) is equivalent to

$$p \le \frac{L(1-L-2M)}{2(1-L-M)}. \tag{3.10}$$

Combining inequalities (3.9) and (3.10), one readily deduces the second assertion of Theorem 1.2 and this completes the proof of Theorem 1.2.

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