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Generalized fractional integral inequalities for exponentially (s,m)-convex functions

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Abstract

In this paper we have derived the fractional integral inequalities by defining exponentially (s,m)-convex functions. These inequalities provide upper bounds, boundedness, continuity, and Hadamard type inequality for fractional integrals containing an extended Mittag-Leffler function. The results about fractional integral operators for s-convex, m-convex, (s,m)-convex, exponentially convex, exponentially s-convex, and convex functions are direct consequences of presented results.

Keywords: Convex function; (s,m)-convex function; Mittag-Leffler function; Fractional integral operators; Boundedness

1 Introduction

Convex functions are very useful in mathematical analysis due to their fascinating properties and convenient characterizations.

Definition 1 A function $f: I \to \mathbb{R}$ is said to be convex function if the following inequality holds:

$$f\bigl(ta + (1-t)b\bigr) \leq tf(a) + (1-t)f(b) \tag{1.1}$$

for all $a, b \in I$ and $t \in [0,1]$. If inequality (1.1) holds in reverse order, then the function f is called concave function.

A graphical interpretation of a convex function f over an interval $[a,b]$ provides at a glance the following well-known Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This inequality has been studied extensively, and a lot of its versions have been published by defining new functions obtained from inequality (1.1). Next we define some of these definitions.

Definition 2 ([10]) Let $s \in [0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex function in the second sense if

$$f\big(ta + (1-t)b\big) \le t^s f(a) + (1-t)^s f(b)$$

holds for all $a, b \in [0,\infty)$ and $t \in [0,1]$.

In [22], Toader gave the following definition of m-convex function.

Definition 3 A function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex if

$$f\big(tx + m(1-t)y\big) \le tf(x) + m(1-t)f(y)$$

holds, where $m \in [0,1]$, $x, y \in [0,b]$, and $t \in [0,1]$.

In [4], Awan et al. gave the following definition of exponentially convex function.

Definition 4 A function $f : K \to \mathbb{R}$, where K is an interval, is said to be an exponentially convex function if

$$f\big(ta + (1-t)b\big) \le t\frac{f(a)}{e^{\alpha a}} + (1-t)\frac{f(b)}{e^{\alpha b}} \tag{1.3}$$

holds for all $a, b \in K$, $t \in [0,1]$, and $\alpha \in \mathbb{R}$. If the inequality in (1.3) is reversed, then f is called exponentially concave.

In [12], Mehreen and Anwar gave the following definition of exponentially s-convex function.

Definition 5 ([12]) Let $s \in (0,1]$ and $K \subseteq [0,\infty)$ be an interval. A function $f : K \to \mathbb{R}$ is said to be exponentially s-convex in the second sense if

$$f\big(ta + (1-t)b\big) \le t^s\frac{f(a)}{e^{\alpha a}} + (1-t)^s\frac{f(b)}{e^{\alpha b}} \tag{1.4}$$

holds for all $a, b \in K$, $t \in [0,1]$, and $\alpha \in \mathbb{R}$. If the inequality in (1.4) is reversed, then f is called exponentially s-concave function.

In [1], Anastassiou gave the following definition of (s,m)-convex function.

Definition 6 ([1]) A function $f : [0,b] \to \mathbb{R}$ is said to be an (s,m)-convex function, where $(s,m) \in [0,1]^2$ and $b > 0$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have

$$f\big(ta + m(1-t)b\big) \le t^s f(a) + m\big(1-t^s\big)f(b).$$

The aim of this paper is to define a further generalization named exponentially (s,m)-convex function (Definition 9) and explore the bounds of generalized fractional integral operators containing Mittag-Leffler functions in their kernels. The Mittag-Leffler function $E_\sigma(t)$ was introduced by Gosta [13] in 1903:

$$E_\sigma(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\sigma n + 1)},$$

where $t, \sigma \in \mathbb{C}, \Re(\sigma) > 0$ and $\Gamma(\cdot)$ is the gamma function.

The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces for $\sigma = 1$. In the solution of fractional integral equations and fractional differential equations, the Mittag-Leffler function arises naturally. Due to its importance, the Mittag-Leffler function has been further generalized and extended by many researchers, we refer the reader to [3, 9, 19, 20]. Recently in [2], Andrić et al. introduced a generalized Mittag-Leffler function defined as follows.

Definition 7 Let $\mu, \sigma, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function is defined by

$$E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \sigma)} \frac{t^n}{(l)_{n\delta}}, \tag{1.5}$$

where β_p is the generalized beta function defined as follows:

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt$$

and $(c)_{nk}$ is the Pochhammer symbol defined by $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

Remark 1 The function given in (1.5) is a generalization of the following Mittag-Leffler functions:

(i) If $p = 0$ in (1.5), then it reduces to the Salim–Faraj function defined in [19].
(ii) If $l = \delta = 1$ in (1.5), then it reduces to the function defined by Rahman et al. in [15].
(iii) If $p = 0$ and $l = \delta = 1$ in (1.5), then it reduces to the Shukla–Prajapati function defined in [20], see also [21].
(iv) If $p = 0$ and $l = \delta = k = 1$ in (1.5), then it reduces to the Prabhakar function defined in [14].

Derivative property of the generalized Mittag-Leffler function is given in following lemma.

Lemma 1 ([2]) *If* $m \in \mathbb{N}, \omega, \mu, \sigma, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\sigma), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0, \delta > 0$ *and* $0 < k < \delta + \Re(\mu)$, *then*

$$\left(\frac{d}{dt}\right)^m \left[t^{\sigma-1} E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\omega t^\mu; p)\right] = t^{\sigma-m-1} E_{\mu,\sigma-m,l}^{\gamma,\delta,k,c}(\omega t^\mu; p), \quad \Re(\sigma) > m. \tag{1.6}$$

Fractional integral operators are very useful in advancement of mathematical inequalities. Many researchers have established fractional integral inequalities due to different kinds of fractional and conformable integral operators, see [1, 2, 5, 6, 8, 11, 16–18, 23].

The Mittag-Leffler function is used to define generalized fractional integral operators. The left-sided and right-sided fractional integral operators containing Mittag-Leffler function (1.5) are defined as follows.

Definition 8 ([2]) Let $\omega,\mu,\sigma,l,\gamma,c\in\mathbb{C}$, $\Re(\mu),\Re(\sigma),\Re(l)>0$, $\Re(c)>\Re(\gamma)>0$ with $p\geq 0$, $\delta>0$ and $0<k\leq\delta+\Re(\mu)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then the generalized fractional integral operators containing Mittag-Leffler function are defined by

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\big)(x;p)=\int_a^x(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-t)^{\mu};p\big)f(t)\,dt, \tag{1.7}$$

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,b^-}f\big)(x;p)=\int_x^b(t-x)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(t-x)^{\mu};p\big)f(t)\,dt, \tag{1.8}$$

where $E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\cdot)$ is the Mittag-Leffler function given in (1.5).

Remark 2 Integral operators given in (1.7) and (1.8) are the generalization of the following fractional integral operators containing Mittag-Leffler function:

(i) If we take $p=0$, it reduces to the fractional integral operators defined by Salim and Faraj in [19].
(ii) If we take $l=\delta=1$, it reduces to the fractional integral operators defined by Rahman et al. in [15].
(iii) If we take $p=0$ and $l=\delta=1$, it reduces to the fractional integral operators defined by Srivastava and Tomovski in [21].
(iv) If we take $p=0$ and $l=\delta=k=1$, it reduces to the fractional integral operators defined by Prabhakar in [14].
(v) If we take $p=\omega=0$, it reduces to the right-sided and left-sided Riemann–Liouville fractional integrals.

In [8], Farid et al. proved that

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}1\big)(x;p)=(x-a)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma+1,l}\big(w(x-a)^{\mu};p\big) \tag{1.9}$$

and

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}1\big)(x;p)=(b-x)^{\tau}E^{\gamma,\delta,k,c}_{\mu,\tau+1,l}\big(w(b-x)^{\mu};p\big). \tag{1.10}$$

We will follow the upcoming notations in the main results:

$$D_{\sigma,\omega,a^+}(x;p)=\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a}1\big)(x;p), \tag{1.11}$$

$$D_{\tau,\omega,b^-}(x;p)=\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}1\big)(x;p). \tag{1.12}$$

In the upcoming section we define a new definition named exponentially (s,m)-convex function which generalizes convex, s-convex, m-convex, exponentially convex, and exponentially s-convex functions. Further this definition is used to establish the upper bounds of left-sided and right-sided generalized fractional integral operators (1.7) and (1.8). The upper bounds provide the continuity of these operators. A modulus inequality is obtained for differentiable functions which in absolute value are exponentially (s,m)-convex. Furthermore a fractional version of the Hadamard inequality is proved.

2 Main results

Definition 9 Let $s \in [0,1]$ and $K \subseteq [0,\infty)$ be an interval. A function $f : K \to \mathbb{R}$ is said to be exponentially (s,m)-convex function in the second sense if

$$f\big(ta + m(1-t)b\big) \leq t^s \frac{f(a)}{e^{\alpha a}} + m(1-t)^s \frac{f(b)}{e^{\alpha b}}$$

holds for all $a, b \in K$, $m \in [0,1]$, and $\alpha \in \mathbb{R}$.

Remark 3

(i) For $m = 1$, one can get an exponentially s-convex function.
(ii) For $\alpha = 0$, one can get an (s,m)-convex function.
(iii) For $\alpha = 0$, $m = 1$, one can get an s-convex function in the second sense.
(iv) For $\alpha = 0$, $s = 1$, $m = 1$, one can get a convex function.

Theorem 1 *Let $f : K \subseteq [0,\infty) \longrightarrow \mathbb{R}$ be a real-valued function. If f is positive and exponentially (s,m)-convex, then for $a, b \in K$, $a < b$, and $\sigma, \tau \geq 1$, the following fractional integral inequality for generalized integral operators (1.7) and (1.8) holds:*

$$\begin{aligned} &\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+} f\big)(x;p) + \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-} f\big)(x;p) \\ &\quad \leq \left(\frac{f(a)}{e^{\alpha a}} + \frac{mf(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\right) \frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1} \\ &\qquad + \left(\frac{f(b)}{e^{\beta b}} + \frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\right) \frac{(b-x)D_{\tau-1,b^-}(x;p)}{s+1}, \quad x \in [a,b] \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Proof Let $x \in [a,b]$. Then, for $t \in [a,x)$ and $\sigma \geq 1$, one can have the following inequality:

$$(x-t)^{\sigma-1} E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-t)^\mu;p\big) \leq (x-a)^{\sigma-1} E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-a)^\mu;p\big). \tag{2.2}$$

As f is exponentially (s,m)-convex, therefore one can obtain

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s \frac{f(a)}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{f(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}, \quad \alpha \in \mathbb{R}. \tag{2.3}$$

By multiplying (2.2) and (2.3) and then integrating over $[a,x]$, we get

$$\begin{aligned} &\int_a^x (x-t)^{\sigma-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}\big(\omega(x-t)^\mu;p\big) f(t)\,dt \\ &\quad \leq \frac{(x-a)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-a)^\mu;p)}{(x-a)^s} \\ &\qquad \times \left(\frac{f(a)}{e^{\alpha a}} \int_a^x (x-t)^s\,dt + \frac{mf(\frac{x}{m})}{e^{\frac{\alpha x}{m}}} \int_a^x (t-a)^s\,dt\right), \end{aligned}$$

that is, the left integral operator satisfies the following inequality:

$$\big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+} f\big)(x;p) \leq \frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1} \left(\frac{f(a)}{e^{\alpha a}} + m\frac{f(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\right). \tag{2.4}$$

On the other hand, for $t \in (x,b]$ and $\tau \geq 1$, one can have the following inequality:

$$(t-x)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,k,c}\big(\omega(t-x)^{\mu};p\big) \leq (b-x)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,k,c}\big(\omega(b-x)^{\mu};p\big). \tag{2.5}$$

Again from exponential (s,m)-convexity of f, we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^{s}\frac{f(b)}{e^{\beta b}} + m\left(\frac{b-t}{b-x}\right)^{s}\frac{f(\frac{x}{m})}{e^{\frac{\beta x}{m}}}, \quad \beta \in \mathbb{R}. \tag{2.6}$$

By multiplying (2.5) and (2.6) and then integrating over $[x,b]$, we get

$$\begin{aligned}
&\int_{x}^{b}(t-x)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,k,c}\big(\omega(t-x)^{\mu};p\big)f(t)\,dt \\
&\quad\leq \frac{(b-x)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,k,c}(\omega(b-x)^{\mu};p)}{(b-x)^{s}} \\
&\qquad\times\left(\frac{f(b)}{e^{\beta b}}\int_{x}^{b}(t-x)^{s}\,dt + \frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\int_{x}^{b}(b-t)^{s}\,dt\right),
\end{aligned}$$

that is, the right integral operator satisfies the following inequality:

$$\big(\epsilon_{\mu,\tau,l,\omega,b^{-}}^{\gamma,\delta,k,c}f\big)(x;p) \leq \frac{(b-x)D_{\tau-1,b^{-}}(x;p)}{s+1}\left(\frac{f(b)}{e^{\beta b}} + \frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\right). \tag{2.7}$$

By adding (2.4) and (2.7), the required inequality (2.1) can be obtained. □

The following special cases are considered.

Corollary 1 *If we set $\sigma = \tau$ in* (2.1), *then the following inequality is obtained*:

$$\begin{aligned}
&\big(\epsilon_{\mu,\sigma,l,\omega,a^{+}}^{\gamma,\delta,k,c}f\big)(x;p) + \big(\epsilon_{\mu,\sigma,l,\omega,b^{-}}^{\gamma,\delta,k,c}f\big)(x;p) \\
&\quad\leq \left(\frac{f(a)}{e^{\alpha a}} + \frac{mf(\frac{x}{m})}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)D_{\sigma-1,a^{+}}(x;p)}{s+1} \\
&\qquad+\left(\frac{f(b)}{e^{\beta b}} + \frac{mf(\frac{x}{m})}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)D_{\sigma-1,b^{-}}(x;p)}{s+1}, \quad x \in [a,b].
\end{aligned} \tag{2.8}$$

Corollary 2 *Along with the assumption of Theorem* 1, *if $f \in L_{\infty}[a,b]$, then the following inequality is obtained*:

$$\begin{aligned}
&\big(\epsilon_{\mu,\sigma,l,\omega,a^{+}}^{\gamma,\delta,k,c}f\big)(x;p) + \big(\epsilon_{\mu,\tau,l,\omega,b^{-}}^{\gamma,\delta,k,c}f\big)(x;p) \\
&\quad\leq \frac{\|f\|_{\infty}}{s+1}\left(\left(\frac{1}{e^{\alpha a}} + \frac{m}{e^{\frac{\alpha x}{m}}}\right)(x-a)D_{\sigma-1,a^{+}}(x;p)\right. \\
&\qquad\left.+\left(\frac{1}{e^{\beta b}} + \frac{m}{e^{\frac{\beta x}{m}}}\right)(b-x)D_{\tau-1,b^{-}}(x;p)\right).
\end{aligned} \tag{2.9}$$

Corollary 3 *For $\sigma = \tau$ in* (2.9), *we get the following result*:

$$\big(\epsilon_{\mu,\sigma,l,\omega,a^{+}}^{\gamma,\delta,k,c}f\big)(x;p) + \big(\epsilon_{\mu,\sigma,l,\omega,b^{-}}^{\gamma,\delta,k,c}f\big)(x;p)$$

$$\leq \frac{\|f\|_\infty}{s+1}\left(\left(\frac{1}{e^{\alpha a}}+\frac{m}{e^{\frac{\alpha x}{m}}}\right)(x-a)D_{\sigma-1,a^+}(x;p) + \left(\frac{1}{e^{\beta b}}+\frac{m}{e^{\frac{\beta x}{m}}}\right)(b-x)D_{\sigma-1,b^-}(x;p)\right). \tag{2.10}$$

Corollary 4 *For $s=1$ in (2.9), we get the following result:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f\right)(x;p)\\ &\quad\leq \frac{\|f\|_\infty}{2}\left(\left(\frac{1}{e^{\alpha a}}+\frac{m}{e^{\frac{\alpha x}{m}}}\right)(x-a)D_{\alpha-1,a^+}(x;p) + \left(\frac{1}{e^{\beta b}}+\frac{m}{e^{\frac{\beta x}{m}}}\right)(b-x)D_{\beta-1,b^-}(x;p)\right).\end{aligned} \tag{2.11}$$

Theorem 2 *With the assumptions of Theorem 1, if $f\in L_\infty[a,b]$, then operators defined in (1.7) and (1.8) are continuous.*

Proof If $f\in L_\infty[a,b]$, then from (2.4) we have

$$\begin{aligned}\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f\right)(x;p)\right| &\leq \frac{2\|f\|_\infty(x-a)D_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{1}{e^{\alpha a}}+\frac{m}{e^{\frac{\alpha x}{m}}}\right)\\ &\leq \frac{2(b-a)D_{\sigma-1,a^+}(b;p)}{s+1}\left(\frac{1}{e^{\alpha a}}+\frac{m}{e^{\frac{\alpha a}{m}}}\right)\|f\|_\infty,\end{aligned} \tag{2.12}$$

that is, $|(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f)(x;p)|\leq M\|f\|_\infty$, where $M=\frac{2(b-a)D_{\sigma-1,a^+}(b;p)}{s+1}(\frac{1}{e^{\alpha a}}+\frac{m}{e^{\frac{\alpha a}{m}}})$. Therefore $(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma,l,\omega,a^+}f)(x;p)$ is bounded, also it is easy to see that it is linear, hence this is a continuous operator. On the other hand, from (2.7) one can obtain

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f\right)(x;p)\right|\leq K\|f\|_\infty,$$

where $K=\frac{2(b-a)D_{\tau-1,b^-}(a;p)}{s+1}(\frac{1}{e^{\beta a}}+\frac{m}{e^{\frac{\beta a}{m}}})$. Therefore $(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau,l,\omega,b^-}f)(x;p)$ is bounded, also it is linear, hence continuous. □

The next result provides the boundedness of a sum of left and right integrals at an arbitrary point for functions whose derivatives in absolute values are exponentially (s,m)-convex.

Theorem 3 *Let $f:K\subseteq[0,\infty)\longrightarrow\mathbb{R}$ be a real-valued function. If f is differentiable and $|f'|$ is exponentially (s,m)-convex, then for $a,b\in K$, $a<b$, and $\sigma,\tau\geq 1$, the following fractional integral inequality for generalized integral operators (1.7) and (1.8) holds:*

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(x;p) - \left(D_{\sigma-1,a^+}(x;p)f(a)+D_{\tau-1,b^-}(x;p)f(b)\right)\right|\\ &\quad\leq \left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1}\\ &\qquad+\left(\frac{|f'(b)|}{e^{\beta b}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)D_{\tau-1,b^-}(x;p)}{s+1},\quad x\in[a,b], \alpha,\beta\in\mathbb{R}.\end{aligned} \tag{2.13}$$

Proof Let $x \in [a,b]$ and $t \in [a,x)$, by using exponential (s,m)-convexity of $|f'|$, we have

$$\left|f'(t)\right| \leq \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}. \tag{2.14}$$

From (2.14), one can have

$$f'(t) \leq \left(\frac{x-t}{x-a}\right)^s \frac{|f'(a)|}{e^{\alpha a}} + m\left(\frac{t-a}{x-a}\right)^s \frac{|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}. \tag{2.15}$$

The product of (2.2) and (2.15) gives the following inequality:

$$\begin{aligned}
&(x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-t)^{\mu};p\big)f'(t)\,dt \\
&\quad\leq (x-a)^{\sigma-1-s}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-a)^{\mu};p\big)\left(\frac{|f'(a)|}{e^{\alpha a}}(x-t)^s + \frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}(t-a)^s\right).
\end{aligned} \tag{2.16}$$

After integrating the above inequality over $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-t)^{\mu};p\big)f'(t)\,dt \\
&\quad\leq (x-a)^{\sigma-1-s}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-a)^{\mu};p\big) \\
&\qquad\times\left(\frac{|f'(a)|}{e^{\alpha a}}\int_a^x (x-t)^s\,dt + \frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\int_a^x (t-a)^s\,dt\right) \\
&\quad= \frac{(x-a)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right).
\end{aligned} \tag{2.17}$$

The left-hand side of (2.17) is calculated as follows:

$$\int_a^x (x-t)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-t)^{\mu};p\big)f'(t)\,dt. \tag{2.18}$$

Put $x-t=z$, that is, $t=x-z$, also using the derivative property (1.6) of Mittag-Leffler function, we have

$$\begin{aligned}
&\int_0^{x-a} z^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega z^{\mu};p\big)f'(x-z)\,dz \\
&\quad= (x-a)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-a)^{\mu};p\big)f(a) - \int_0^{x-a} z^{\sigma-2}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega z^{\mu};p\big)f(x-z)\,dz.
\end{aligned}$$

Now putting $x-z=t$ in the second term of the right-hand side of the above equation and then using (1.7), we get

$$\begin{aligned}
&\int_0^{x-a} z^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega z^{\mu};p\big)f'(x-z)\,dz \\
&\quad= (x-a)^{\sigma-1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\big(\omega(x-a)^{\mu};p\big)f(a) - \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\big)(x;p).
\end{aligned}$$

Therefore (2.17) takes the following form:

$$\big(D_{\sigma-1,a^+}(x;p)\big)f(a) - \big(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\big)(x;p)$$

$$\leq \frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right). \tag{2.19}$$

Also from (2.14) one can have

$$f'(t) \geq -\left(\left(\frac{x-t}{x-a}\right)^s\frac{|f'(a)|}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^s\frac{|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right). \tag{2.20}$$

Following the same procedure as we did for (2.15), one can obtain

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(x;p) - D_{\sigma-1,a^+}(x;p)f(a)\\&\quad\leq \frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right).\end{aligned} \tag{2.21}$$

From (2.19) and (2.21), we get

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(x;p) - D_{\sigma-1,a^+}(x;p)f(a)\right|\\&\quad\leq \frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right).\end{aligned} \tag{2.22}$$

Now we let $x \in [a,b]$ and $t \in (x,b]$. Then, by exponential (s,m)-convexity of $|f'|$, we have

$$\left|f'(t)\right| \leq \left(\frac{t-x}{b-x}\right)^s\frac{|f'(b)|}{e^{\beta b}}+m\left(\frac{b-t}{b-x}\right)^s\frac{|f'(\frac{x}{m})|}{e^{\frac{\beta x}{m}}}, \quad \beta \in \mathbb{R}. \tag{2.23}$$

On the same lines as we have done for (2.2), (2.15), and (2.20), one can get from (2.5) and (2.23) the following inequality:

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(x;p) - D_{\tau-1,b^-}(x;p)f(b)\right|\\&\quad\leq \frac{(b-x)D_{\tau-1,b^-}(x;p)}{s+1}\left(\frac{|f'(b)|}{e^{\beta b}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\beta x}{m}}}\right).\end{aligned} \tag{2.24}$$

From inequalities (2.22) and (2.24) via the triangular inequality, (2.13) can be obtained. □

Corollary 5 *If we put $\sigma = \tau$ in (2.13), then the following inequality is obtained*:

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,b^-}f\right)(x;p)\right.\\&\qquad\left.-\left(D_{\sigma-1,a^+}(x;p)f(a)+D_{\sigma-1,b^-}(x;p)f(b)\right)\right|\\&\quad\leq \left(\frac{|f'(a)|}{e^{\alpha a}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\alpha x}{m}}}\right)\frac{(x-a)D_{\sigma-1,a^+}(x;p)}{s+1}\\&\qquad+\left(\frac{|f'(b)|}{e^{\beta b}}+\frac{m|f'(\frac{x}{m})|}{e^{\frac{\beta x}{m}}}\right)\frac{(b-x)D_{\sigma-1,b^-}(x;p)}{s+1}, \quad x \in [a,b], \alpha,\beta \in \mathbb{R}.\end{aligned} \tag{2.25}$$

Definition 10 Let $f : [a,b] \to \mathbb{R}$ be a function, we will say that f is exponentially m-symmetric about $\frac{a+b}{2}$ if

$$\frac{f(x)}{e^{\alpha x}} = \frac{f(\frac{a+b-x}{m})}{e^{\alpha(\frac{a+b-x}{m})}}, \quad \alpha \in \mathbb{R}. \tag{2.26}$$

It is required to give the following lemma which will be helpful to produce Hadamard type estimations for the generalized fractional integral operators.

Lemma 2 *Let $f: K \subseteq [0,\infty) \longrightarrow \mathbb{R}, a,b \in K, a<b$, be an exponentially (s,m)-convex function. If f is exponentially m-symmetric about $\frac{a+b}{2}$, then the following inequality holds*:

$$f\left(\frac{a+b}{2}\right) \leq (1+m)\frac{f(x)}{2^s e^{\alpha x}}, \quad \alpha \in \mathbb{R}. \tag{2.27}$$

Proof Since f is exponentially (s,m)-convex, so

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(at+(1-t)b)}{2^s e^{\alpha(at+(1-t)b)}} + \frac{mf(\frac{a(1-t)+bt}{m})}{2^s e^{\alpha(\frac{a(1-t)+bt}{m})}}, \quad t \in [0,1]. \tag{2.28}$$

Let $x = at+(1-t)b$, where $x \in [a,b]$. Then we have $a+b-x = bt+(1-t)a$, and we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(x)}{2^s e^{\alpha x}} + m\frac{f(\frac{a+b-x}{m})}{2^s e^{\alpha(\frac{a+b-x}{m})}}. \tag{2.29}$$

Now, using that f is exponentially m-symmetric, we will get (2.27). □

Theorem 4 *Let $f: K \subseteq [0,\infty) \longrightarrow \mathbb{R}, a,b \in K, a<b$, be a real-valued function. If f is positive, exponentially (s,m)-convex and exponentially m-symmetric about $\frac{a+b}{2}$, then for $\sigma,\tau > 0$, the following fractional integral inequality for generalized integral operators* (1.7) *and* (1.8) *holds*:

$$\begin{aligned} &\frac{2^s h(\alpha)}{1+m} f\left(\frac{a+b}{2}\right)\left[D_{\tau+1,b^-}(a;p) + D_{\sigma+1,a^+}(b;p)\right] \\ &\quad \leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+} f\right)(b;p) \\ &\quad \leq \left[D_{\tau-1,b^-}(a;p) + D_{\sigma-1,a^+}(b;p)\right]\frac{(b-a)^2}{s+1}\left(\frac{f(\frac{a}{m})}{e^{\frac{\alpha a}{m}}} + \frac{f(b)}{e^{\beta b}}\right), \quad \alpha,\beta \in \mathbb{R}, \end{aligned} \tag{2.30}$$

where $h(\alpha) = e^{\alpha b}$ for $\alpha < 0$ and $h(\alpha) = e^{\alpha a}$ for $\alpha \geq 0$.

Proof For $x \in [a,b]$, we have

$$(x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}\left(\omega(x-a)^\mu;p\right) \leq (b-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}\left(\omega(b-a)^\mu;p\right), \quad \tau > 0. \tag{2.31}$$

As f is exponentially (s,m)-convex, so for $x \in [a,b]$, we have

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^s \frac{f(b)}{e^{\alpha b}} + m\left(\frac{b-x}{b-a}\right)^s \frac{f(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}, \quad \alpha \in \mathbb{R}. \tag{2.32}$$

By multiplying (2.31) and (2.32) and then integrating over $[a,b]$, we get

$$\begin{aligned} &\int_a^b (x-a)^\tau E^{\gamma,\delta,k,c}_{\mu,\tau,l}\left(\omega(x-a)^\mu;p\right) f(x)\,dx \\ &\quad \leq (b-a)^{\tau-s} E^{\gamma,\delta,k,c}_{\mu,\tau,l}\left(\omega(b-a)^\mu;p\right)\left(\frac{f(b)}{e^{\alpha b}}\int_a^b (x-a)^s\,dx + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\int_a^b (b-x)^s\,dx\right), \end{aligned}$$

from which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p) \le \frac{(b-a)^{\tau+1}E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(b-a)^{\mu};p)}{s+1}\left(\frac{f(b)}{e^{\alpha b}} + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right), \tag{2.33}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p) \le \frac{(b-a)^2}{s+1}D_{\tau-1,b^-}(a;p)\left(\frac{f(b)}{e^{\alpha b}} + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right). \tag{2.34}$$

On the other hand, for $x \in [a,b]$, we have

$$(b-x)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\left(\omega(b-x)^{\mu};p\right) \le (b-a)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\left(\omega(b-a)^{\mu};p\right), \quad \alpha > 0. \tag{2.35}$$

By multiplying (2.32) and (2.35) and then integrating over $[a,b]$, we get

$$\begin{aligned}&\int_a^b (b-x)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\left(\omega(b-x)^{\mu};p\right)f(x)\,dx\\&\quad \le (b-a)^{\sigma-s}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}\left(\omega(b-a)^{\mu};p\right)\left(\frac{f(b)}{e^{\alpha b}}\int_a^b (x-a)^s\,dx + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\int_a^b (b-x)^s\,dx\right),\end{aligned}$$

from which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p) \le \frac{(b-a)^{\sigma+1}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-a)^{\mu};p)}{s+1}\left(\frac{f(b)}{e^{\alpha b}} + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right), \tag{2.36}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p) \le \frac{(b-a)^2}{s+1}D_{\sigma-1,a^+}(b;p)\left(\frac{f(b)}{e^{\alpha b}} + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right). \tag{2.37}$$

Adding (2.34) and (2.37), we get

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p)\\&\quad \le \left[D_{\tau-1,b^-}(a;p) + D_{\sigma-1,a^+}(b;p)\right]\frac{(b-a)^2}{s+1}\left(\frac{f(b)}{e^{\alpha b}} + \frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right).\end{aligned} \tag{2.38}$$

Multiplying (2.27) with $(x-a)^{\tau}E^{\gamma,\delta,k,c}_{\mu,\tau,l}(\omega(x-a)^{\mu};p)$ and integrating over $[a,b]$, we get

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{\tau}E^{\gamma,\delta,k,c}_{\mu,\tau,l}\left(\omega(x-a)^{\mu};p\right)dx\\&\quad \le \frac{m+1}{2^s}\int_a^b (x-a)^{\tau}E^{\gamma,\delta,k,c}_{\mu,\tau,l}\left(\omega(x-a)^{\mu};p\right)\frac{f(x)}{e^{\alpha x}}\,dx.\end{aligned} \tag{2.39}$$

By using (1.8) and (1.12), we get

$$f\left(\frac{a+b}{2}\right)D_{\tau+1,b^-}(a;p) \le \frac{m+1}{2^s e^{\alpha x}}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p). \tag{2.40}$$

Multiplying (2.27) with $(b-x)^{\sigma}E^{\gamma,\delta,k,c}_{\mu,\sigma,l}(\omega(b-x)^{\mu};p)$ and integrating over $[a,b]$, also using (1.7) and(1.11), we get

$$f\left(\frac{a+b}{2}\right)D_{\sigma+1,a^+}(b;p) \le \frac{m+1}{2^s h(\alpha)}\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p). \tag{2.41}$$

By adding (2.40) and (2.41), we get

$$\frac{2^s h(\alpha)}{1+m} f\left(\frac{a+b}{2}\right)\left[D_{\tau+1,b^-}(a;p)+D_{\sigma+1,a^+}(b;p)\right] \leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\tau+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p). \quad (2.42)$$

By combining (2.38) and (2.42), inequality (2.30) can be obtained. □

Corollary 6 *If we put* $\sigma=\tau$ *in* (2.30), *then the following inequality is obtained*:

$$\begin{aligned} &\frac{2^s e^{\alpha x}}{1+m} f\left(\frac{a+b}{2}\right)\left[D_{\sigma+1,b^-}(a;p)+D_{\sigma+1,a^+}(b;p)\right] \\ &\quad\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,b^-}f\right)(a;p)+\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\sigma+1,l,\omega,a^+}f\right)(b;p) \\ &\quad\leq \left(D_{\sigma-1,b^-}(a;p)+D_{\sigma-1,a^+}(b;p)\right)\frac{(b-a)^2}{s+1}\left(\frac{f(b)}{e^{\alpha b}}+\frac{mf(\frac{a}{m})}{e^{\frac{\alpha a}{m}}}\right). \end{aligned} \quad (2.43)$$

3 Concluding remarks

This paper has investigated generalized fractional integral inequalities which provide the bounds of fractional integral operators containing Mittag-Leffler functions in their kernels. By setting different values of parameters involved in the Mittag-Leffler function, the results for various known fractional operators can be obtained. For example, by setting $p=0$, fractional integral inequalities for fractional operators defined by Salim and Faraj in [19] can be obtained; by setting $l=\delta=1$, fractional integral inequalities for fractional operators defined by Rahman et al. in [15] can be deduced, by setting $p=0$ and $l=\delta=1$, fractional integral inequalities for fractional operators defined by Shukla and Prajapati in [20] (see also [21]) can be deduced, by setting $p=0$ and $l=\delta=k=1$, fractional integral inequalities for fractional operators defined by Prabhakar in [14] can be deduced, by setting $p=\omega=0$ fractional integral inequalities for Riemann–Liouville fractional integrals can be deduced. Also all the results of this paper hold for s-convex, m-convex, exponentially convex, exponentially s-convex, and convex functions. In particular results for (s,m)-convex functions, which are proved in [7], can be obtained.

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