Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On a reverse extended Hardy–Hilbert's inequality

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Abstract

By the use of the weight coefficients, the idea of introducing parameters and the Euler–Maclaurin summation formula, a reverse extended Hardy–Hilbert inequality and the equivalent forms are given. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are also considered.

MSC: 26D15

Keywords: Weight coefficient; Hardy–Hilbert's inequality; Reverse; Equivalent statement; Parameter

1 Introduction

Assuming that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following Hardy–Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

In 2006, by introducing the parameters $\lambda_i \in (0,2]$ $(i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,4]$, an extension of (1) was provided by [2] as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\lambda_1, \lambda_2)$ is the best possible ($B(u,v) = \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}}\,dt$ $(u, v > 0)$ is the beta function). For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (2) reduces to (1); for $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (2) reduces to Yang's work in [3]. Recently, applying (2), [4] gave a new inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ involving partial sums.

If $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)\,dx < \infty$, and $0 < \int_0^\infty g^q(y)\,dy < \infty$, then we still have the following Hardy–Hilbert integral inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(y)\,dy\right)^{1/q}, \tag{3}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is the best possible. Inequalities (1) and (3) with their extensions and reverses are important in analysis and its applications (cf. [5–15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ $(t > 0)$ is decreasing, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^\infty K(t)t^{s-1}\,dt < \infty$, $a_n \geq 0$, $0 < \sum_{n=1}^\infty a_n^p < \infty$, then we have

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

In the last ten years, some extensions of (4) with their applications and the reverses were provided by [16–20].

In 2016, by means of the techniques of real analysis, Hong et al. [21] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. Similar work about Hilbert-type integral inequalities is in [22–26].

In this paper, following the way of [2, 21], by the use of the weight coefficients, the idea of introduced parameters and Euler–Maclaurin summation formula, a reverse extended Hardy–Hilbert inequality as well as the equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remark 1–2.

2 Some lemmas

In what follows, we assume that $0 < p < 1$ $(q < 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathrm{N} = \{1, 2, \ldots\}$, $\lambda \in (0, 6]$, $\lambda_i \in (0, 2] \cap (0, \lambda)$ $(i = 1, 2)$,

$$O\left(\frac{1}{m^{\lambda_2}}\right) := \frac{(1+\theta_m)^{-\lambda}}{\lambda_2 B(\lambda_2, \lambda - \lambda_2)}\frac{1}{m^{\lambda_2}} \in (0,1) \quad \left(\theta_m \in \left(0, \frac{1}{m}\right), m \in \mathrm{N}\right).$$

We also assume that $a_m, b_n \geq 0$, such that

$$0 < \sum_{m=1}^\infty m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p < \infty, \quad \text{and} \quad 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q < \infty. \tag{5}$$

Lemma 1 *Define the following weight coefficient*:

$$\varpi(\lambda_2, m) := m^{\lambda-\lambda_2}\sum_{n=1}^\infty \frac{n^{\lambda_2-1}}{(m+n)^\lambda} \quad (m \in \mathrm{N}). \tag{6}$$

We have the following inequality:

$$B(\lambda_2, \lambda-\lambda_2)\left(1 - O\left(\frac{1}{m^{\lambda_2}}\right)\right) < \varpi(\lambda_2, m) < B(\lambda_2, \lambda-\lambda_2) \quad (m \in \mathrm{N}). \tag{7}$$

Proof For fixed $m \in \mathrm{N}$, we set function $g(m,t) := \frac{t^{\lambda_2-1}}{(m+t)^{\lambda}}$ $(t>0)$. Using the Euler–Maclaurin summation formula (cf. [2, 3]), for $\rho(t) := t-[t]-\frac{1}{2}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)\,dt + \frac{1}{2}g(m,1) + \int_1^{\infty}\rho(t)g'(m,t)\,dt \\ &= \int_0^{\infty} g(m,t)\,dt - h(m), \\ h(m) &:= \int_0^1 g(m,t)\,dt - \frac{1}{2}g(m,1) - \int_1^{\infty}\rho(t)g'(m,t)\,dt.\end{aligned}$$

We obtain $-\frac{1}{2}g(m,1) = \frac{-1}{2(m+1)^{\lambda}}$,

$$\begin{aligned}&\int_0^1 g(m,t)\,dt \\ &\quad = \int_0^1 \frac{t^{\lambda_2-1}}{(m+t)^{\lambda}}\,dt = \frac{1}{\lambda_2}\int_0^1 \frac{dt^{\lambda_2}}{(m+t)^{\lambda}} = \frac{1}{\lambda_2}\frac{t^{\lambda_2}}{(m+t)^{\lambda}}\bigg|_0^1 + \frac{\lambda}{\lambda_2}\int_0^1 \frac{t^{\lambda_2}\,dt}{(m+t)^{\lambda+1}} \\ &\quad = \frac{1}{\lambda_2}\frac{1}{(m+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\int_0^1 \frac{dt^{\lambda_2+1}}{(m+t)^{\lambda+1}} \\ &\quad > \frac{1}{\lambda_2}\frac{1}{(m+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\left[\frac{t^{\lambda_2+1}}{(m+t)^{\lambda+1}}\right]_0^1 + \frac{\lambda(\lambda+1)}{\lambda_2(\lambda_2+1)(m+1)^{\lambda+2}}\int_0^1 t^{\lambda_2+1}\,dt \\ &\quad = \frac{1}{\lambda_2}\frac{1}{(m+1)^{\lambda}} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\frac{1}{(m+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)}{\lambda_2(\lambda_2+1)(\lambda_2+2)}\frac{1}{(m+1)^{\lambda+2}}.\end{aligned}$$

We find

$$\begin{aligned}-g'(m,t) &= -\frac{(\lambda_2-1)t^{\lambda_2-2}}{(m+t)^{\lambda}} + \frac{\lambda t^{\lambda_2-1}}{(m+t)^{\lambda+1}} = \frac{(1-\lambda_2)t^{\lambda_2-2}}{(m+t)^{\lambda}} + \frac{\lambda t^{\lambda_2-2}}{(m+t)^{\lambda}} - \frac{\lambda m t^{\lambda_2-2}}{(m+t)^{\lambda+1}} \\ &= \frac{(\lambda+1-\lambda_2)t^{\lambda_2-2}}{(m+t)^{\lambda}} - \frac{\lambda m t^{\lambda_2-2}}{(m+t)^{\lambda+1}},\end{aligned}$$

and for $0<\lambda_2\le 2$, $\lambda_2<\lambda\le 6$, it follows that

$$(-1)^i\frac{d^i}{dt^i}\left[\frac{t^{\lambda_2-2}}{(m+t)^{\lambda}}\right] > 0, \qquad (-1)^i\frac{d^i}{dt^i}\left[\frac{t^{\lambda_2-2}}{(m+t)^{\lambda+1}}\right] > 0 \quad (i=0,1,2,3).$$

Still by the Euler–Maclaurin summation formula (cf. [2, 3]), we obtain

$$(\lambda+1-\lambda_2)\int_1^{\infty}\rho(t)\frac{t^{\lambda_2-2}}{(m+t)^{\lambda}}\,dt > -\frac{\lambda+1-\lambda_2}{12(m+1)^{\lambda}},$$

and

$$\begin{aligned}&-m\lambda\int_1^{\infty}\rho(t)\frac{t^{\lambda_2-2}}{(m+t)^{\lambda+1}}\,dt \\ &\quad > \frac{m\lambda}{12(m+1)^{\lambda+1}} - \frac{m\lambda}{720}\left[\frac{t^{\lambda_2-2}}{(m+t)^{\lambda+1}}\right]''_{t=1} \\ &\quad > \frac{(m+1)\lambda-\lambda}{12(m+1)^{\lambda+1}} - \frac{(m+1)\lambda}{720}\left[\frac{(\lambda+1)(\lambda+2)}{(m+1)^{\lambda+3}} + \frac{2(\lambda+1)(2-\lambda_2)}{(m+1)^{\lambda+2}} + \frac{(2-\lambda_2)(3-\lambda_2)}{(m+1)^{\lambda+1}}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda}{12(m+1)^{\lambda}} - \frac{\lambda}{12(m+1)^{\lambda+1}} \\
&\quad - \frac{\lambda}{720}\left[\frac{(\lambda+1)(\lambda+2)}{(m+1)^{\lambda+2}} + \frac{2(\lambda+1)(2-\lambda_2)}{(m+1)^{\lambda+1}} + \frac{(2-\lambda_2)(3-\lambda_2)}{(m+1)^{\lambda}}\right].
\end{aligned}
$$

Hence, we have $h(m) > \frac{h_1}{(m+1)^{\lambda}} + \frac{\lambda h_2}{(m+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)h_3}{(m+1)^{\lambda+2}}$, where

$$
h_1 := \frac{1}{\lambda_2} - \frac{1}{2} - \frac{1-\lambda_2}{12} - \frac{\lambda(2-\lambda_2)(3-\lambda_2)}{720}, \qquad h_2 := \frac{1}{\lambda_2(\lambda_2+1)} - \frac{1}{12} - \frac{(\lambda+1)(2-\lambda_2)}{360},
$$

and $h_3 := \frac{1}{\lambda_2(\lambda_2+1)(\lambda_2+2)} - \frac{\lambda+2}{720}$.

For $\lambda \in (0,6]$, $\frac{\lambda}{720} < \frac{1}{24}$, $\lambda_2 \in (0,2]$, we find

$$
h_1 > \frac{1}{\lambda_2} - \frac{1}{2} - \frac{1-\lambda_2}{12} - \frac{(2-\lambda_2)(3-\lambda_2)}{24} = \frac{24-20\lambda_2+7\lambda_2^2-\lambda_2^3}{24\lambda_2} > 0.
$$

In fact, setting $g(\sigma) := 24 - 20\sigma + 7\sigma^2 - \sigma^3$ ($\sigma \in (0,2]$), we obtain

$$
g'(\sigma) = -20 + 14\sigma - 3\sigma^2 = -3\left(\sigma - \frac{7}{3}\right)^2 - \frac{11}{3} < 0,
$$

and then

$$
h_1 > \frac{g(\lambda_2)}{24\lambda_2} \geq \frac{g(2)}{24\lambda_2} = \frac{4}{24\lambda_2} > 0 \quad \bigl(\lambda_2 \in (0,2]\bigr).
$$

We obtain $h_2 > \frac{1}{6} - \frac{1}{12} - \frac{18}{360} = \frac{1}{30} > 0$, and $h_3 \geq \frac{1}{24} - \frac{10}{720} = \frac{1}{36} > 0$. Hence, we have $h(m) > 0$, and then setting $t = mu$, it follows that

$$
\begin{aligned}
\varpi(\lambda_2, m) &= m^{\lambda-\lambda_2}\sum_{n=1}^{\infty} g(m,n) < m^{\lambda-\lambda_2}\int_0^{\infty} g(m,t)\,dt \\
&= m^{\lambda-\lambda_2}\int_0^{\infty} \frac{t^{\lambda_2-1}}{(m+t)^{\lambda}}\,dt = \int_0^{\infty} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}}\,du = B(\lambda_2, \lambda-\lambda_2).
\end{aligned}
$$

On the other hand, we also have

$$
\begin{aligned}
\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)\,dt + \frac{1}{2}g(m,1) + \int_1^{\infty} \rho(t)g'(m,t)\,dt \\
&= \int_1^{\infty} g(m,t)\,dt + H(m), \\
H(m) &:= \frac{1}{2}g(m,1) + \int_1^{\infty} \rho(t)g'(m,t)\,dt.
\end{aligned}
$$

We have obtained $\frac{1}{2}g(m,1) = \frac{1}{2(m+1)^{\lambda}}$ and

$$
g'(m,t) = \frac{-(\lambda+1-\lambda_2)t^{\lambda_2-2}}{(m+t)^{\lambda}} + \frac{\lambda m t^{\lambda_2-2}}{(m+t)^{\lambda+1}}.
$$

For $\lambda_2 \in (0,2] \cap (0,\lambda)$, $0 < \lambda \leq 6$, by the Euler–Maclaurin summation formula, we obtain

$$\begin{aligned}
&-(\lambda+1-\lambda_2)\int_1^\infty \rho(t)\frac{t^{\lambda_2-2}}{(m+t)^\lambda}\,dt \\
&\quad > \frac{\lambda+1-\lambda_2}{12(m+1)^\lambda} - \frac{\lambda+1-\lambda_2}{720}\left[\frac{t^{\lambda_2-2}}{(m+t)^\lambda}\right]''_{t=1} \\
&\quad = \frac{\lambda+1-\lambda_2}{12(m+1)^\lambda} \\
&\qquad - \frac{\lambda+1-\lambda_2}{720}\left[\frac{(2-\lambda_2)(3-\lambda_2)}{(m+t)^\lambda}t^{\lambda_2-2} + \frac{2\lambda(2-\lambda_2)}{(m+t)^{\lambda+1}}t^{\lambda_2-3} + \frac{\lambda(\lambda+1)}{(m+t)^{\lambda+2}}t^{\lambda_2-2}\right]_{t=1} \\
&\quad = \frac{\lambda+1-\lambda_2}{12(m+1)^\lambda} - \frac{\lambda+1-\lambda_2}{720}\left[\frac{(2-\lambda_2)(3-\lambda_2)}{(m+1)^\lambda} + \frac{2\lambda(2-\lambda_2)}{(m+1)^{\lambda+1}} + \frac{\lambda(\lambda+1)}{(m+1)^{\lambda+2}}\right], \\
&m\lambda\int_1^\infty \rho(t)\frac{t^{\lambda_2-2}}{(m+t)^{\lambda+1}}\,dt \\
&\quad > -\frac{m\lambda}{12(m+1)^{\lambda+1}} = -\frac{(m+1)\lambda-\lambda}{12(m+1)^{\lambda+1}} = \frac{-\lambda}{12(m+1)^\lambda} + \frac{\lambda}{12(m+1)^{\lambda+1}}.
\end{aligned}$$

Hence, we have $H(m) > \frac{H_1}{(m+1)^\lambda} + \frac{\lambda H_2(m)}{(m+1)^{\lambda+1}}$, where

$$\begin{aligned}
H_1 &:= \frac{7-\lambda_2}{12} - \frac{(\lambda+1-\lambda_2)(2-\lambda_2)(3-\lambda_2)}{720}, \\
H_2(m) &:= \frac{1}{12} - \frac{(\lambda+1-\lambda_2)(2-\lambda_2)}{720} - \frac{(\lambda+1-\lambda_2)(\lambda+1)}{720(m+1)}.
\end{aligned}$$

For $\lambda_2 \in (0,2] \cap (0,\lambda)$, $0 < \lambda \leq 6$, we find $H_1 > \frac{5}{12} - \frac{42}{720} > 0$, and

$$H_2(m) > \frac{1}{12} - \frac{14}{360} - \frac{49}{1440} = \frac{15}{1440} > 0.$$

It follows that $H(m) > 0$, and then

$$\begin{aligned}
m^{\lambda-\lambda_2}\sum_{n=1}^\infty g(m,n) &> m^{\lambda-\lambda_2}\int_1^\infty g(m,t)\,dt \\
&= m^{\lambda-\lambda_2}\int_0^\infty g(m,t)\,dt - m^{\lambda-\lambda_2}\int_0^1 g(m,t)\,dt \\
&= B(\lambda_2,\lambda-\lambda_2)\left[1 - \frac{1}{B(\lambda_2,\lambda-\lambda_2)}\int_0^{\frac{1}{m}} \frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du\right] > 0.
\end{aligned}$$

By the integral mid-value theorem, we find

$$\int_0^{\frac{1}{m}} \frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du = \frac{1}{(1+\theta_m)^\lambda}\int_0^{\frac{1}{m}} u^{\lambda_2-1}\,du = \frac{1}{(1+\theta_m)^\lambda}\frac{1}{\lambda_2 m^{\lambda_2}} \quad \left(\theta_m \in \left(0,\frac{1}{m}\right)\right),$$

namely, (7) follows. □

Lemma 2 *We have the following reverse extended Hardy–Hilbert inequality*:

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \\ &> B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\ &\quad\times\left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right]m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{8}$$

Proof In the same way as obtaining (7), for $n\in\mathbf{N}$, we obtain the following inequality of the weight coefficient:

$$\omega(\lambda_1,n) := n^{\lambda-\lambda_1}\sum_{m=1}^{\infty}\frac{n^{\lambda_1-1}}{(m+n)^{\lambda}} < B(\lambda_1,\lambda-\lambda_1). \tag{9}$$

By the reverse Hölder inequality (cf. [27]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\left[\frac{n^{(\lambda_2-1)/p}}{m^{(\lambda_1-1)/q}}a_m\right]\left[\frac{m^{(\lambda_1-1)/q}}{n^{(\lambda_2-1)/p}}b_n\right] \\ &\geq \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\frac{n^{\lambda_2-1}}{m^{(\lambda_1-1)(p-1)}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}\frac{m^{\lambda_1-1}}{n^{(\lambda_2-1)(q-1)}}b_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\sum_{m=1}^{\infty}\varpi(\lambda_2,m)m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega(\lambda_1,n)n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by (7) and (9), in view of $0<p<1$, $q<0$, we have (8). □

Remark 1 By (8), for $\lambda_1+\lambda_2=\lambda\in(0,4]$, $0<\lambda_i\leq 2$ $(i=1,2)$, we find

$$\begin{aligned} &\omega(\lambda_1,n) < B(\lambda_1,\lambda_2), \\ &B(\lambda_1,\lambda_2)\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right) < \varpi(\lambda_2,m) < B(\lambda_1,\lambda_2) \quad (m,n\in\mathrm{N}), \\ &O\left(\frac{1}{m^{\lambda_2}}\right) = \frac{(1+\theta_m)^{-\lambda}}{\lambda_2 B(\lambda_1,\lambda_2)}\frac{1}{m^{\lambda_2}}\in(0,1) \quad \left(\theta_m\in\left(0,\frac{1}{m}\right)\right), \\ &0<\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p<\infty, \qquad 0<\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q<\infty. \end{aligned}$$

and the following reverse inequality:

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \\ &\quad > B(\lambda_1,\lambda_2)\left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right]m^{p(1-\lambda_1)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{10}$$

Lemma 3 *For any* $\varepsilon > 0$, *we have*

$$L := \sum_{m=1}^{\infty} O\left(\frac{1}{m^{\lambda_2+\varepsilon+1}}\right) = O(1). \tag{11}$$

Proof There exists a constant $M > 0$, such that

$$|L| \leq M \sum_{m=1}^{\infty} \frac{1}{m^{\lambda_2+\varepsilon+1}} = M\left(1 + \sum_{m=2}^{\infty} \frac{1}{m^{\lambda_2+\varepsilon+1}}\right).$$

By the decreasing property of the series, it follows that

$$|L| \leq M\left(1 + \int_1^{\infty} \frac{1}{x^{\lambda_2+\varepsilon+1}}\,dx\right) < M\left(1 + \frac{1}{\lambda_2}\right) < \infty.$$

Hence, Eq. (11) follows. □

Lemma 4 *For* $\lambda_1 + \lambda_2 = \lambda \in (0,4]$, *the constant factor* $B(\lambda_1, \lambda_2)$ *in* (10) *is the best possible.*

Proof For any $0 < \varepsilon < p\lambda_1$, we set

$$\tilde{a}_m := m^{\lambda_1 - \frac{\varepsilon}{p} - 1}, \qquad \tilde{b}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (m, n \in \mathrm{N}).$$

If there exists a constant $M \geq B(\lambda_1, \lambda_2)$, such that (10) is valid when replacing $B(\lambda_1, \lambda_2)$ by M, then in particular, substitution of $a_m = \tilde{a}_m$ and $b_n = \tilde{b}_n$ in (10), we have

$$\begin{aligned}
\tilde{I} &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\lambda}} \tilde{a}_m \tilde{b}_n \\
&> M\left\{\sum_{m=1}^{\infty}\left[1 - O\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p(1-\lambda_1)-1} \tilde{a}_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} \tilde{b}_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

By (11) and the decreasing property of series, we obtain

$$\begin{aligned}
\tilde{I} &> M\left\{\sum_{m=1}^{\infty}\left[1 - O\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p(1-\lambda_1)-1} m^{p\lambda_1-\varepsilon-p}\right\}^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} n^{q\lambda_2-\varepsilon-q}\right]^{\frac{1}{q}} \\
&= M\left(\sum_{m=1}^{\infty} m^{-\varepsilon-1} - \sum_{m=1}^{\infty} O\left(\frac{1}{m^{\lambda_2+\varepsilon+1}}\right)\right)^{\frac{1}{p}} \left(1 + \sum_{n=2}^{\infty} n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\
&> M\left(\int_1^{\infty} x^{-\varepsilon-1}\,dx - O(1)\right)^{\frac{1}{p}} \left(1 + \int_1^{\infty} y^{-\varepsilon-1}\,dy\right)^{\frac{1}{q}} \\
&= \frac{M}{\varepsilon}\big(1 - \varepsilon O(1)\big)^{\frac{1}{p}} (\varepsilon + 1)^{\frac{1}{q}}.
\end{aligned}$$

By (9), setting $\hat{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0,2) \cap (0,\lambda)$ $(0 < \hat{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} < \lambda)$, we find

$$\begin{aligned}
\tilde{I} &= \sum_{n=1}^{\infty}\left[n^{(\lambda_2+\frac{\varepsilon}{p})}\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}m^{(\lambda_1-\frac{\varepsilon}{p})-1}\right]n^{-\varepsilon-1} \\
&= \sum_{n=1}^{\infty}\omega(\hat{\lambda}_1,n)n^{-\varepsilon-1} < B(\hat{\lambda}_1,\hat{\lambda}_2)\left(1+\sum_{n=2}^{\infty}n^{-\varepsilon-1}\right) \\
&< B(\hat{\lambda}_1,\hat{\lambda}_2)\left(1+\int_1^{\infty}y^{-\varepsilon-1}\,dy\right) = \frac{\varepsilon+1}{\varepsilon}B\left(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right).
\end{aligned}$$

Then we have

$$(\varepsilon+1)B\left(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right) > \varepsilon\tilde{I} > M\bigl(1-\varepsilon O(1)\bigr)^{\frac{1}{p}}(\varepsilon+1)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, we find $B(\lambda_1,\lambda_2) \geq M$. Hence, $M = B(\lambda_1,\lambda_2)$ is the best possible constant factor of (10). □

Setting $\tilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\tilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda,$$

and we can reduce (8) to the following:

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} \\
&> B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\
&\quad\times\left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right]m^{p(1-\tilde{\lambda}_1)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\tilde{\lambda}_2)-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{12}$$

Lemma 5 *If $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, the constant factor $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ in (12) is the best possible, then we have $\lambda-\lambda_1-\lambda_2 = 0$, namely, $\lambda = \lambda_1+\lambda_2$.*

Proof For $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, we obtain

$$\begin{aligned}
&\tilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} > \frac{(1-p)\lambda_1}{p} + \frac{\lambda_1}{q} = 0, \qquad \tilde{\lambda}_1 < \frac{\lambda_1+p(\lambda-\lambda_1)}{p} + \frac{\lambda_1}{q} = \lambda, \\
&0 < \tilde{\lambda}_2 = \lambda - \tilde{\lambda}_1 < \lambda.
\end{aligned}$$

Hence, we have $B(\tilde{\lambda}_1,\tilde{\lambda}_2) \in \mathrm{R}_+ = (0,\infty)$.

If the constant factor $B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$ in (12) is the best possible, then in view of (10), the unique best possible constant factor must be $B(\tilde{\lambda}_1,\tilde{\lambda}_2)$ $(\in \mathrm{R}_+)$, namely,

$$B(\tilde{\lambda}_1,\tilde{\lambda}_2) = B^{\frac{1}{p}}(\lambda-\lambda_2,\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1).$$

By the reverse Hölder inequality, we find

$$\begin{aligned}
B(\tilde{\lambda}_1,\tilde{\lambda}_2) &= B\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q},\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right) \\
&= \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du = \int_0^\infty \frac{1}{(1+u)^\lambda}\bigl(u^{\frac{\lambda-\lambda_2-1}{p}}\bigr)\bigl(u^{\frac{\lambda_1-1}{q}}\bigr)\,du \\
&\geq \left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda-\lambda_2-1}\,du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda_1-1}\,du\right]^{\frac{1}{q}} \\
&= B^{\frac{1}{p}}(\lambda-\lambda_2,\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1).
\end{aligned} \tag{13}$$

We observe that (13) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero and (cf. [27])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \quad \text{a.e. in } \mathrm{R}_+.$$

Assuming that $A \neq 0$, it follows that

$$u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A} \quad \text{a.e. in } \mathrm{R}_+,$$

and then $\lambda-\lambda_2-\lambda_1 = 0$, namely, $\lambda = \lambda_1+\lambda_2$. □

3 Main results and some particular cases

Theorem 1 *Inequality* (8) *is equivalent to the following inequalities*:

$$\begin{aligned}
J &:= \left\{\sum_{n=1}^\infty n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}\left[\sum_{m=1}^\infty \frac{1}{(m+n)^\lambda}a_m\right]^p\right\}^{\frac{1}{p}} \\
&> B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left\{\sum_{m=1}^\infty\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right]m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}},
\end{aligned} \tag{14}$$

$$\begin{aligned}
J_1 &:= \left\{\sum_{m=1}^\infty \frac{m^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{[1-O(\frac{1}{m^{\lambda_2}})]^{q-1}}\left[\sum_{n=1}^\infty \frac{1}{(m+n)^\lambda}b_n\right]^q\right\}^{\frac{1}{q}} \\
&> B^{\frac{1}{p}}(\lambda_2,\lambda-\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{15}$$

If the constant factor in (8) *is the best possible, then so is the constant factor in* (14) *and* (15).

Proof Suppose that (14) is valid. By the Hölder inequality, we have

$$\begin{aligned}
I &= \sum_{n=1}^\infty\left[n^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}\sum_{m=1}^\infty \frac{1}{(m+n)^\lambda}a_m\right]\left[n^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}b_n\right] \\
&\geq J\left\{\sum_{n=1}^\infty n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{16}$$

Then, by (14), we obtain (8). On the other hand, assuming that (8) is valid, we set

$$b_n := n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}\left[\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}a_m\right]^{p-1}, \quad n \in \mathbf{N}.$$

If $J = \infty$, then (14) is naturally valid; if $J = 0$, then it is impossible to make (14) valid, namely, $J > 0$. Suppose that $0 < J < \infty$. By (8), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q \\
&\quad = J^p = I \\
&\quad > B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \\
&\qquad \times \left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}, \\
&J = \left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{p}} \\
&\quad > B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}},
\end{aligned}$$

namely, (14) follows. Hence, inequality (8) is equivalent to (14).

Suppose that (15) is valid. By the Hölder inequality, we have

$$\begin{aligned}
I &= \sum_{m=1}^{\infty}\left[\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right)^{\frac{1}{p}} m^{\frac{1}{q}-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})} a_m\right]\left[\frac{m^{\frac{-1}{q}+(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}}{(1-O(\frac{1}{m^{\lambda_2}}))^{1/p}}\sum_{n=1}^{\infty}\frac{1}{(m+n)^{\lambda}} b_n\right] \\
&\geq \left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} J_1.
\end{aligned} \tag{17}$$

Then, by (15), we obtain (8). On the other hand, assuming that (8) is valid, we set

$$a_m := \frac{m^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{(1-O(\frac{1}{m^{\lambda_2}}))^{q-1}}\left[\sum_{n=1}^{\infty}\frac{1}{(m+n)^{\lambda}} b_n\right]^{q-1}, \quad m \in \mathrm{N}.$$

If $J_1 = \infty$, then (15) is naturally valid; if $J_1 = 0$, then it is impossible to make (15) valid, namely, $J_1 > 0$. Suppose that $0 < J_1 < \infty$. By (8), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p \\
&\quad = J_1^q = I \\
&\quad > B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \left\{\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}}
\end{aligned}$$

$$\times \left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}},$$

$$J_1 = \left\{\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right) m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{q}}$$
$$> B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)\left\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}},$$

namely, (15) follows. Hence, inequality (8) is equivalent to (15) and then inequalities (8), (14) and (15) are equivalent.

If the constant factor in (8) is the best possible, then so is the constant factor in (14) and (15). Otherwise, by (16) (or (17)), we would reach a contradiction that the constant factor in (8) is not the best possible. □

Theorem 2 *The following statements* (i), (ii), (iii) *and* (iv) *are equivalent*:

(i) $B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)$ *is independent of* p, q;

(ii) $B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)$ *is expressible as a single integral*;

(iii) $B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)$ *in* (8) *is the best possible constant factor*;

(iv) *if* $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, *then* $\lambda = \lambda_1+\lambda_2$.

If the statement (*iv*) *follows, namely,* $\lambda = \lambda_1+\lambda_2$, *then we have* (10) *and the following equivalent inequalities with the best possible constant factor* $B(\lambda_1, \lambda_2)$:

$$\left\{\sum_{n=1}^{\infty} n^{p\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}} a_m\right]^p\right\}^{\frac{1}{p}}$$
$$> B(\lambda_1, \lambda_2)\left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^{\lambda_2}}\right)\right] m^{p(1-\lambda_1)-1} a_m^p\right\}^{\frac{1}{p}}, \tag{18}$$

$$\left\{\sum_{m=1}^{\infty}\frac{m^{q\lambda_1-1}}{[1-O(\frac{1}{m^{\lambda_2}})]^{q-1}}\left[\sum_{n=1}^{\infty}\frac{1}{(m+n)^{\lambda}} b_n\right]^q\right\}^{\frac{1}{q}} > B(\lambda_1, \lambda_2)\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \tag{19}$$

Proof (i) ⇒ (ii). By (i), we have

$$B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)$$
$$= \lim_{p\to 1^-}\lim_{q\to -\infty} B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) = B(\lambda_2, \lambda-\lambda_2).$$

namely, $B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)$ is expressible as a single integral

$$B(\lambda_2, \lambda-\lambda_2) = \int_0^{\infty}\frac{1}{(1+u)^{\lambda}} u^{\lambda_2-1}\, du.$$

(ii) ⇒ (iv). If $B^{\frac{1}{p}}(\lambda_2, \lambda-\lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)$ is expressible as a convergent single integral

$$B\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}\right),$$

then (13) keeps the form of equality. In view of the proof of Lemma 5, it follows that $\lambda = \lambda_1 + \lambda_2$.

(iv) $\Rightarrow$ (i). If $\lambda = \lambda_1 + \lambda_2$, then

$$B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) = B(\lambda_1, \lambda_2),$$

which is independent of p, q. Hence, it follows that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (iv). By Lemma 5, we have $\lambda = \lambda_1 + \lambda_2$.

(iv) $\Rightarrow$ (iii). By Lemma 4, for $\lambda = \lambda_1 + \lambda_2$,

$$B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \quad \bigl(= B(\lambda_1, \lambda_2)\bigr)$$

is the best possible constant factor of (8). Therefore, we have (iii) $\Leftrightarrow$ (iv).

Hence, the statements (i), (ii), (iii) and (iv) are equivalent. □

Remark 2 For $\lambda_1 = \lambda_2 = \frac{\lambda}{2} \in (0, 2]$ $(0 < \lambda \le 4)$ in (10), (18) and (19), we have the following equivalent inequalities with the best possible constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$:

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} \\ &\quad > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{m=1}^{\infty} \left[1 - O\left(\frac{1}{m^{\lambda/2}}\right)\right] m^{p(1-\frac{\lambda}{2})-1} a_m^p \right\}^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right]^{\frac{1}{q}}, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left\{ \sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{1}{(m+n)^{\lambda}} a_m \right]^p \right\}^{\frac{1}{p}} \\ &\quad > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{m=1}^{\infty} \left[1 - O\left(\frac{1}{m^{\lambda/2}}\right)\right] m^{p(1-\frac{\lambda}{2})-1} a_m^p \right\}^{\frac{1}{p}}, \end{aligned} \tag{21}$$

$$\left\{ \sum_{m=1}^{\infty} \frac{m^{\frac{q\lambda}{2}-1}}{[1 - O(\frac{1}{m^{\lambda/2}})]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^{\lambda}} b_n \right]^q \right\}^{\frac{1}{q}} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left[\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right]^{\frac{1}{q}}. \tag{22}$$

In particular, (i) for $\lambda = 2$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^2} > \left\{ \sum_{m=1}^{\infty} \left[1 - O\left(\frac{1}{m}\right)\right] \frac{a_m^p}{m} \right\}^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{n} \right)^{\frac{1}{q}}, \tag{23}$$

$$\left\{ \sum_{n=1}^{\infty} n^{p-1} \left[\sum_{m=1}^{\infty} \frac{1}{(m+n)^2} a_m \right]^p \right\}^{\frac{1}{p}} > \left\{ \sum_{m=1}^{\infty} \left[1 - O\left(\frac{1}{m}\right)\right] \frac{a_m^p}{m} \right\}^{\frac{1}{p}}, \tag{24}$$

$$\left\{ \sum_{m=1}^{\infty} \frac{m^{q-1}}{[1 - O(\frac{1}{m})]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^2} b_n \right]^q \right\}^{\frac{1}{q}} > \left(\sum_{n=1}^{\infty} \frac{b_n^q}{n} \right)^{\frac{1}{q}}; \tag{25}$$

(ii) for $\lambda = 4$, we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^4} > \frac{1}{6} \left\{ \sum_{m=1}^{\infty} \left[1 - O\left(\frac{1}{m^2}\right)\right] \frac{a_m^p}{m^{p+1}} \right\}^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{n^{q+1}} \right)^{\frac{1}{q}}, \tag{26}$$

$$\left\{\sum_{n=1}^{\infty} n^{2p-1}\left[\sum_{m=1}^{\infty} \frac{1}{(m+n)^4} a_m\right]^p\right\}^{\frac{1}{p}} > \frac{1}{6}\left\{\sum_{m=1}^{\infty}\left[1-O\left(\frac{1}{m^2}\right)\right]\frac{a_m^p}{m^{p+1}}\right\}^{\frac{1}{p}}, \tag{27}$$

$$\left\{\sum_{m=1}^{\infty} \frac{m^{2q-1}}{[1-O(\frac{1}{m^2})]^{q-1}}\left[\sum_{n=1}^{\infty} \frac{1}{(m+n)^4} b_n\right]^q\right\}^{\frac{1}{q}} > \frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{n^{q+1}}\right)^{\frac{1}{q}}. \tag{28}$$

4 Conclusions

In this paper, by the use of the weight coefficients, the idea of introducing parameters and the Euler–Maclaurin summation formula, a reverse extended Hardy–Hilbert inequality as well as the equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remark 1, 2. The lemmas and theorems provide an extensive account of this type of inequalities.

Acknowledgements
The authors thank the referee for useful proposals to reform the paper.

Funding
This work is supported by the National Natural Science Foundation (No. 61772140), and Science and Technology Planning Project Item of Guangzhou City (No. 201707010229).

Availability of data and materials
The data and material in this paper are original.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. ZH and YS participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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Publisher's Note

Received: 2 September 2019 Accepted: 4 March 2020 Published online: 12 March 2020

References

1. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities. Cambridge University Press, Cambridge (1934)
2. Krnic, M., Pecaric, J.: Extension of Hilbert's inequality. J. Math. Anal. Appl. **324**(1), 150–160 (2006)
3. Yang, B.C.: On a generalization of Hilbert double series theorem. J. Nanjing Univ. Math. Biq. **18**(1), 145–152 (2001)
4. Adiyasuren, V., Batbold, T., Azar, L.E.: A new discrete Hilbert-type inequality involving partial sums. J. Inequal. Appl. **2019**, 127 (2019)
5. Yang, B.C.: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing (2009)
6. Krnic, M., Pecaric, J.: General Hilbert's and Hardy's inequalities. Math. Inequal. Appl. **8**(1), 29–51 (2005)
7. Perić, I., Vuković, P.: Multiple Hilbert's type inequalities with a homogeneous kernel. Banach J. Math. Anal. **5**(2), 33–43 (2011)
8. Huang, Q.L.: A new extension of Hardy–Hilbert-type inequality. J. Inequal. Appl. **2015**, 397 (2015)
9. He, B.: A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor. J. Math. Anal. Appl. **431**, 889–902 (2015)
10. Xu, J.S.: Hardy–Hilbert's inequalities with two parameters. Adv. Math. **36**(2), 63–76 (2007)
11. Xie, Z.T., Zeng, Z., Sun, Y.F.: A new Hilbert-type inequality with the homogeneous kernel of degree −2. Adv. Appl. Math. Sci. **12**(7), 391–401 (2013)
12. Zeng, Z., Raja Rama Gandhi, K., Xie, Z.T.: A new Hilbert-type inequality with the homogeneous kernel of degree −2 and with the integral. Bull. Math. Sci. Appl. **3**(1), 11–20 (2014)
13. Xin, D.M.: A Hilbert-type integral inequality with the homogeneous kernel of zero degree. Math. Theory Appl. **30**(2), 70–74 (2010)
14. Azar, L.E.: The connection between Hilbert and Hardy inequalities. J. Inequal. Appl. **2013**, 452 (2013)

15. Adiyasuren, V., Batbold, T., Krni´c, M.: Hilbert-type inequalities involving differential operators, the best constants and applications. Math. Inequal. Appl. **18**, 111–124 (2015)
16. Rassias, M.T., Yang, B.C.: On half-discrete Hilbert's inequality. Appl. Math. Comput. **220**, 75–93 (2013)
17. Yang, B.C., Krnić, M.: A half-discrete Hilbert-type inequality with a general homogeneous kernel of degree 0. J. Math. Inequal. **6**(3), 401–417 (2012)
18. Rassias, M.T., Yang, B.C.: A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function. Appl. Math. Comput. **225**, 263–277 (2013)
19. Rassias, M.T., Yang, B.C.: On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function. Appl. Math. Comput. **242**, 800–813 (2013)
20. Yang, B.C., Debnath, L.: Half-Discrete Hilbert-Type Inequalities. World Scientific Publishing, Singapore (2014)
21. Hong, Y., Wen, Y.M.: A necessary and sufficient condition of that Hilbert type series inequality with homogeneous kernel has the best constant factor. Ann. Math. **37A**(3), 329–336 (2016)
22. Hong, Y.: On the structure character of Hilbert's type integral inequality with homogeneous kernel and applications. J. Jilin Univ. Sci. Ed. **55**(2), 189–194 (2017)
23. Hong, Y., Huang, Q.L., Yang, B.C., Liao, J.Q.: The necessary and sufficient conditions for the existence of a kind of Hilbert-type multiple integral inequality with the non-homogeneous kernel and its applications. J. Inequal. Appl. **2017**, 316 (2017)
24. Xin, D.M., Yang, B.C., Wang, A.Z.: Equivalent property of a Hilbert-type integral inequality related to the beta function in the whole plane. J. Funct. Spaces **2018**, Article ID 2691816 (2018)
25. Hong, Y., He, B., Yang, B.C.: Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory. J. Math. Inequal. **12**(3), 777–788 (2018)
26. Yang, B.C., Chen, Q.: On a Hardy–Hilbert-type inequality with parameters. J. Inequal. Appl. **2015**, 339 (2015)
27. Kuang, J.C.: Applied Inequalities. Shangdong Science and Technology Press, Jinan (2004)