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The equivalence of F_a-frames

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Abstract

Structured frames such as wavelet and Gabor frames in $L^2(\mathbb{R})$ have been extensively studied. But $L^2(\mathbb{R}_+)$ cannot admit wavelet and Gabor systems due to $\mathbb{R}_+$ being not a group under addition. In practice, $L^2(\mathbb{R}_+)$ models the causal signal space. The function-valued inner product-based F_a-frame for $L^2(\mathbb{R}_+)$ was first introduced by Hasankhani Fard and Dehghan, where an F_a-frame was called a function-valued frame. In this paper, we introduce the notions of F_a-equivalence and unitary F_a-equivalence between F_a-frames, and present a characterization of the F_a-equivalence and unitary F_a-equivalence. This characterization looks like that of equivalence and unitary equivalence between frames, but the proof is nontrivial due to the particularity of F_a-frames.

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1 Introduction

An at most countable sequence $\{e_i\}_{i\in I}$ in a separable Hilbert Space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} \bigl|\langle f, f_i\rangle\bigr|^2 \le B\|f\|^2$$

for $f \in \mathcal{H}$. It was first introduced by Duffin and Schaeffer in [5] to study nonharmonic Fourier series, but had not attracted much attention until Daubechies, Grossman and Meyer published their joint work [4] in 1986. Now the theory of frames has seen great achievements in abstract spaces as well as in function spaces ([3, 10, 11, 13, 14, 18, 25]). In particular, structured frames in $L^2(\mathbb{R})$ such as wavelet and Gabor frames have been extensively studied. However, structured frames in $L^2(\mathbb{R}_+)$ with $\mathbb{R}_+ = (0,\infty)$ have not. It is because $\mathbb{R}$ is a group under addition but $\mathbb{R}_+$ is not. This results in nonexistence of wavelet and Gabor systems in $L^2(\mathbb{R}_+)$. In practice, the time variable is nonnegative, and $L^2(\mathbb{R}_+)$ models the causal signal space. Motivated by this observation, some mathematicians studied Walsh series-based wavelet analysis in $L^2(\mathbb{R}_+)$ using Cantor group operation on $\mathbb{R}_+$ ([1, 6–9, 16, 17]). Recently, Hasankhani Fard and Dehghan in [12] introduced the notion of function-valued frame in $L^2(\mathbb{R}_+)$ which is referred to as "F_a-frame" in our papers. Let us first recall and extend some related notions.

Given $a > 1$, a measurable function f on $\mathbb{R}_+$ is said to be *a-dilation periodic* if $f(a\cdot) = f(\cdot)$ a.e. on $\mathbb{R}_+$, and a sequence $\{f_k\}_{k\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}_+$ is said to be *a-dilation periodic* if every f_k is a-dilation periodic. Let $L^2(\mathbb{Z} \times [1,a))$ denote the Hilbert space

$$L^2(\mathbb{Z} \times [1,a)) = \left\{f = \{f_k\}_{k\in\mathbb{Z}} : \int_1^a \sum_{k\in\mathbb{Z}} |f_k(x)|^2 \, dx < \infty, \{f_k\}_{k\in\mathbb{Z}} \text{ is } a\text{-dilation periodic}\right\}$$

equipped with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{Z}\times[1,a))} = \int_1^a \sum_{k\in\mathbb{Z}} f_k(x)\overline{g_k(x)} \, dx \quad \text{for } f, g \in L^2(\mathbb{Z} \times [1,a)).$$

The following definition is an extension of [12, Definition 2.1], and that in [23] which only dealt with functions in $L^2(\mathbb{R}_+)$. It is slightly different from [12, Definition 2.1], even for functions in $L^2(\mathbb{R}_+)$, but it is more convenient for our purpose. By [12, Theorem 2.2], the F_a-inner product herein has many properties similar to those of inner products.

Definition 1.1 Given $a > 1$, for $f, g \in L^2(\mathbb{R}_+)$ ($L^2(\mathbb{Z} \times [1,a))$), the F_a-inner product $\langle f, g\rangle_a$ of f and g is defined as the a-dilation periodic function on $\mathbb{R}_+$ given by

$$\langle f, g\rangle_a(\cdot) = \sum_{j\in\mathbb{Z}} a^j f(a^j\cdot)\overline{g(a^j\cdot)} \quad \left(\langle f, g\rangle_a(\cdot) = \sum_{k\in\mathbb{Z}} f_k(\cdot)\overline{g_k(\cdot)}\right) \tag{1.1}$$

a.e. on $[1,a)$. The F_a-norm $\|f\|_a$ of f is defined as $\|f\|_a(\cdot) = \sqrt{\langle f, f\rangle_a(\cdot)}$. And f and g are said to be *F_a-orthogonal* if $\langle f, g\rangle_a(\cdot) = 0$ a.e. on $[1,a)$. In symbols, $f \perp_{F_a} g$. It is to distinguish from the orthogonality "$\perp$" with respect to inner products.

Write

$$B_a = \{f \in L^\infty(\mathbb{R}_+) : f \text{ is } a\text{-dilation periodic}\},$$

and let $\{\Lambda_m\}_{m\in\mathbb{Z}}$ denote the a-dilation periodic function sequence on $\mathbb{R}_+$ satisfying

$$\Lambda_m(\cdot) = \frac{1}{\sqrt{a-1}} e^{2\pi i \frac{m\cdot}{a-1}} \quad \text{on } [1,a). \tag{1.2}$$

The following proposition is taken from [23, Lemma 2.3] which dealt with $L^2(\mathbb{R}_+)$. A similar argument shows that it is true for $L^2(\mathbb{Z} \times [1,a))$.

Proposition 1.1

(i) $\int_{[1,a)} |f(x)|^2 \, dx = \sum_{m\in\mathbb{Z}} |\langle f, \Lambda_m\rangle_{L^2[1,a)}|^2$ *for* $f \in L^1[1,a)$.

(ii) *For* $f, g \in L^2(\mathbb{R}_+)$ ($L^2(\mathbb{Z} \times [1,a))$) *and* $\varphi \in B_a$, *we have*

$$\langle f, g\rangle_a \in L^1[1,a), \qquad \langle f, \varphi g\rangle_a = \overline{\varphi}\langle f, g\rangle_a, \tag{1.3}$$

$$\langle f, g\rangle_{L^2(\mathbb{R}_+)} = \int_1^a \langle f, g\rangle_a(x) \, dx \quad \textit{if } f, g \in L^2(\mathbb{R}_+), \tag{1.4}$$

$$\langle f, g\rangle_{L^2(\mathbb{Z}\times[1,a))} = \int_1^a \langle f, g\rangle_a(x) \, dx \quad \textit{if } f, g \in L^2(\mathbb{Z} \times [1,a)), \tag{1.5}$$

$$\|f + g\|_a^2(\cdot) = \|f\|_a^2(\cdot) + \|g\|_a^2(\cdot) \quad \textit{a.e. on } [1,a) \textit{ iff } f \perp_{F_a} g. \tag{1.6}$$

(iii) $\sum_{m\in\mathbb{Z}} |\langle f, \Lambda_m g\rangle_{L^2(\mathbb{R}_+)}|^2 = \int_1^a |\langle f,g\rangle_a(x)|^2\,dx$ *for* $f,g \in L^2(\mathbb{R}_+)$, *and* $\sum_{m\in\mathbb{Z}} |\langle f, \Lambda_m g\rangle_{L^2(\mathbb{Z}\times[1,a))}|^2 = \int_1^a |\langle f,g\rangle_a(x)|^2\,dx$ *for* $f,g \in L^2(\mathbb{Z}\times[1,a))$.
(iv) *For* $f,g \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$, $f \perp_{F_a} g$ *if and only if* $f \perp \Lambda_m g$ *for* $m\in\mathbb{Z}$.
(v) *For* $f,g \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$, *if* $f \perp_{F_a} g$, *then* $f \perp \varphi\Lambda_m g$ *for* $m\in\mathbb{Z}$ *and* $\varphi\in B_a$.

The following definition is taken from [12, Definition 4.5] or [23, Definition 1.5].

Definition 1.2 A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is called an F_a-frame for $L^2(\mathbb{R}_+)$ if there exist constants $0 < A \le B < \infty$ such that, for each $f \in L^2(\mathbb{R}_+)$,

$$A\|f\|_a^2(\cdot) \le \sum_{k\in\mathbb{Z}} \bigl|\langle f,f_k\rangle_a(\cdot)\bigr|_a^2 \le B\|f\|_a^2(\cdot) \quad \text{a.e. on } [1,a), \tag{1.7}$$

where A and B are called frame bounds. It is called a Parseval (tight) F_a-frame for $L^2(\mathbb{R}_+)$ if $A = B = 1$ $(A = B)$ in (1.7). And it is called an F_a-Bessel sequence in $L^2(\mathbb{R}_+)$ with Bessel bound B if the right-hand side inequality of (1.7) holds.

For a sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$, its F_a*-span* is defined by

$$F_a\text{-span}\{f_n\} = \Biggl\{\sum_{k,m\in\mathbb{Z}} c_{k,m}\Lambda_m f_k : c = \{c_{k,m}\}_{k,m\in\mathbb{Z}} \in l_0\bigl(\mathbb{Z}^2\bigr)\Biggr\}, \tag{1.8}$$

and $\overline{F_a\text{-span}}\{f_k\}$ denotes the closure of F_a-span$\{f_k\}$ in $L^2(\mathbb{R}_+)$, where $l_0(\mathbb{Z}^2)$ is the set of finitely supported sequences on $\mathbb{Z}^2$. We say $\{f_k\}_{k\in\mathbb{Z}}$ is F_a*-complete* in $L^2(\mathbb{R}_+)$ if $\overline{F_a\text{-span}}\{f_n\} = L^2(\mathbb{R}_+)$. By [23, Lemma 2.6], $\{f_k\}_{k\in\mathbb{Z}}$ is F_a*-complete* in $L^2(\mathbb{R}_+)$ if and only if $f = 0$ is a unique solution to

$$\langle f,f_k\rangle_a(\cdot) = 0 \quad \text{a.e. on } [1,a) \text{ for } k\in\mathbb{Z}$$

in $L^2(\mathbb{R}_+)$. And $\{f_k\}_{k\in\mathbb{Z}}$ is called an F_a*-orthonormal system* in $L^2(\mathbb{R}_+)$ if $\langle f_k,f_{k'}\rangle_a(\cdot) = \delta_{k,k'}$ a.e. on $[1,a)$ for $k,k'\in\mathbb{Z}$, and called an F_a*-orthonormal basis* if it is an F_a-orthonormal system and F_a-complete in $L^2(\mathbb{R}_+)$.

Recall from [23, Theorem 2.2] and [12, Theorem 4.8] that a sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is an F_a-Bessel sequence (F_a-frame sequence, F_a-frame) in $L^2(\mathbb{R}_+)$ if and only if $\{\Lambda_m f_k\}_{m,k\in\mathbb{Z}}$ is a Bessel sequence (frame sequence, frame) in $L^2(\mathbb{R}_+)$ with the same bounds. Also by a standard argument, a sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $L^2(\mathbb{R}_+)$ is an F_a-orthonormal system (F_a-orthonormal basis) in $L^2(\mathbb{R}_+)$ if and only if $\{\Lambda_m f_k\}_{m,k\in\mathbb{Z}}$ is an orthonormal system (orthonormal basis) in $L^2(\mathbb{R}_+)$. According to this, using "F_a"-language we can say that F_a-frames $\{f_k\}_{k\in\mathbb{Z}}$ of the form $f_k(\cdot) = a^{\frac{k}{2}}\psi(a^k\cdot)$ with $\psi\in L^2(\mathbb{R}_+)$ have been studied more. Li and Zhang in [22] characterized F_a-frames, F_a-dual frames and Parseval F_a-frames for $L^2(\mathbb{R}_+)$ of the form $\{a^{\frac{k}{2}}\psi(a^k\cdot)\}_{k\in\mathbb{Z}}$, and as a special case, Li and Wang studied F_a-frame sets in [21]. Its multi-window and vector-valued cases and another variation were studied in [20, 23, 24, 27]. By [22, Corollary 3.1], for $0 \ne \psi \in L^2(\mathbb{R}_+)$, the following are equivalent:

(i) $\{a^{\frac{k}{2}}\psi(a^k\cdot)\}_{k\in\mathbb{Z}}$ is a Parseval F_a-frame for $L^2(\mathbb{R}_+)$.
(ii) $\{a^{\frac{k}{2}}\psi(a^k\cdot)\}_{k\in\mathbb{Z}}$ is an F_a-orthonormal basis.
(iii) $\{a^{\frac{k}{2}}\psi(a^k\cdot)\}_{k\in\mathbb{Z}}$ is an F_a-orthonormal system.

Obviously, we do not have a similar result for frames. On the other hand, recall from [3, Theorem 5.4.7] that removing one vector from a frame leaves either a frame or an incomplete set. Example 2.1 below in Sect. 2 tells us that a similar conclusion does not hold for F_a-frames. It shows that removing one vector from an F_a-frame possibly leaves an F_a-complete set which is not an F_a-frame.

From the above discussion, there exist essential differences between frames and F_a-frames. This paper focuses on general F_a-frames. Two frames $\{f_i\}_{i\in I}$ and $\{\widetilde{f}_i\}_{i\in I}$ for a separable Hilbert space $\mathcal{H}$ are said be equivalent (unitarily equivalent) if there exists a bounded and invertible linear operator (unitary operator) T on $\mathcal{H}$ such that $\widetilde{f}_i = Tf_i$ for $i \in I$. The following proposition is taken from [2, 11, 15].

Proposition 1.2 *Let $\{f_i\}_{i\in I}$ and $\{\widetilde{f}_i\}_{i\in I}$ be frames for a separable Hilbert space $\mathcal{H}$. Then*

(i) *$\{f_i\}_{i\in I}$ and $\{\widetilde{f}_i\}_{i\in I}$ are equivalent if and only if their analysis operators have the same range, i.e.,*

$$\left\{\left\{\langle f,f_i\rangle\right\}_{i\in I}:f\in\mathcal{H}\right\}=\left\{\left\{\langle f,\widetilde{f}_i\rangle\right\}_{i\in I}:f\in\mathcal{H}\right\}.$$

(ii) *$\{f_i\}_{i\in I}$ and $\{\widetilde{f}_i\}_{i\in I}$ are unitarily equivalent if and only if*

$$\left\|\sum_{i\in I}c_if_i\right\|=\left\|\sum_{i\in I}c_i\widetilde{f}_i\right\|\quad \textit{for } c\in l^2(I).$$

A natural question is whether Proposition 1.2 can be extended to "F_a-frame" setting. This paper gives an affirmative answer. For this purpose, we first need to introduce "(unitary) equivalence" between F_a-frames. It is different from that of frames due to the particularity of F_a-frames.

Definition 1.3 Let $\mathcal{H},\mathcal{K}=L^2(\mathbb{R}_+)$ or $L^2(\mathbb{Z}\times[1,a))$, a bounded linear operator $T:\mathcal{H}\to\mathcal{K}$ is said to be a-factorable if

$$T(\varphi f)=\varphi T(f)\quad \text{for all } f\in\mathcal{H} \text{ and } \varphi\in B_a.$$

Definition 1.4 Two F_a-frames $F=\{f_k\}_{k\in\mathbb{Z}}$ and $\widetilde{F}=\{\tilde{f}_k\}_{k\in\mathbb{Z}}$ for $L^2(\mathbb{R}_+)$ are said to be F_a-equivalent (unitarily F_a-equivalent) if there exists an a-factorable, bounded and invertible linear operator (a-factorable and unitary operator) T on $L^2(\mathbb{R}_+)$ such that

$$Tf_k=\tilde{f}_k\quad \text{for } k\in\mathbb{Z}.$$

Let $F=\{f_k\}_{k\in\mathbb{Z}}$ be an F_a-Bessel sequence in $L^2(\mathbb{R}_+)$. Define the *F_a-analysis operator* $D_F:L^2(\mathbb{R}_+)\to L^2(\mathbb{Z}\times[1,a))$ and the *F_a-synthesis operator* $R_F:L^2(\mathbb{Z}\times[1,a))\to L^2(\mathbb{R}_+)$ by

$$D_Ff=\left\{\langle f,f_k\rangle_a\right\}_{k\in\mathbb{Z}}\quad \text{for } f\in L^2(\mathbb{R}_+)\tag{1.9}$$

and

$$R_Fg=\sum_{k\in\mathbb{Z}}g_kf_k\quad \text{for } g\in L^2\bigl(\mathbb{Z}\times[1,a)\bigr),\tag{1.10}$$

respectively. By [23, Theorem 2.1] they are well defined and bounded, and $D_F^* = R_F$. The *F_a-frame operator* S_F of F is defined by $S_F = R_F D_F$. Obviously, these three operators are all a-factorable. The main result of this paper is as follows.

Theorem 1.1 *Let $F = \{f_k\}_{k\in\mathbb{Z}}$ and $\widetilde{F} = \{\widetilde{f_k}\}_{k\in\mathbb{Z}}$ be F_a-frames for $L^2(\mathbb{R}_+)$. Then*
(i) *F and $\widetilde{F}$ are F_a-equivalent if and only if*

$$\operatorname{range}(D_F) = \operatorname{range}(D_{\widetilde{F}}). \tag{1.11}$$

(ii) *F and $\widetilde{F}$ are unitarily F_a-equivalent if and only if*

$$\|R_F g\|_{L^2(\mathbb{R}_+)} = \|R_{\widetilde{F}} g\|_{L^2(\mathbb{R}_+)} \quad \textit{for } g \in L^2\bigl(\mathbb{Z}\times[1,a)\bigr). \tag{1.12}$$

The rest of this paper is organized as follows. Section 2 makes preparation for Theorem 1.1. Section 3 is devoted to proving Theorem 1.1.

2 Some preliminaries

This section is an auxiliary one. On one hand, we give an example that is an F_a-frame, but when removing some element, it leaves an F_a-complete set which is not an F_a-frame for $L_2(\mathbb{R}_+)$. It is well known that removing an element from a frame leaves either a frame or an incomplete set. This demonstrates that F_a-frames are very different from frames. On the other hand, we give some lemmas for later use. For this purpose, we first introduce some notations which are frequently used through the paper.

For a set E, we denote by $\mathcal{X}_E$ the characteristic function of E. Given $f_0 \in L^2(\mathbb{R}_+)$ ($L^2(\mathbb{Z}\times[1,a))$), a nonempty subset V of $L^2(\mathbb{R}_+)$ ($L^2(\mathbb{Z}\times[1,a))$) and an a-dilation periodic measurable function φ on $\mathbb{R}_+$, $f_0 \perp_{F_a} V$ means that $f_0 \perp_{F_a} g$ for each $g \in V$, φV, $V(\varphi)$ and $V^{\perp_{F_a}}$ denote the sets

$$\varphi V = \{\varphi f : f \in V\}, \tag{2.1}$$

$$V(\varphi) = \bigl\{\varphi f : f \in V, \varphi f \in L^2(\mathbb{R}_+)\bigr\} \quad \text{if } V \subset L^2(\mathbb{R}_+), \tag{2.2}$$

$$V(\varphi) = \bigl\{\varphi f : f \in V, \varphi f \in L^2\bigl(\mathbb{Z}\times[1,a)\bigr)\bigr\} \quad \text{if } V \subset L^2\bigl(\mathbb{Z}\times[1,a)\bigr), \tag{2.3}$$

$$V^{\perp_{F_a}} = \{f : f \perp_{F_a} g \text{ for each } g \in V\}, \tag{2.4}$$

respectively. Observe that $\varphi V = V(\varphi)$ if $\varphi \in B_a$. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and V be a closed linear subspace of $\mathcal{H}$. We denote by $V^\perp$ and P_V the orthogonal complement of V in $\mathcal{H}$ and the orthogonal projection from $\mathcal{H}$ onto V, respectively. For a bounded linear operator T from $\mathcal{H}$ to $\mathcal{K}$, we denote by $T|_V$, T^*, $\operatorname{range}(T)$ and $\ker(T)$ its restriction onto V, its adjoint operator, its range and its kernel, respectively. If T is also of closed range, we denote by $T^\dagger$ the pseudo-inverse of T, i.e.,

$$T^\dagger(y+z) = (T|_{(\ker(T))^\perp})^{-1} y \quad \text{for } y \in \operatorname{range}(T) \text{ and } z \in \bigl(\operatorname{range}(T)\bigr)^\perp.$$

Example 2.1 Let $a = 2$. Define $\{f_k\}_{k\in\mathbb{Z}}$ by

$$f_k(x) = \begin{cases} 2^{\frac{k}{2}}\mathcal{X}_{[2^{-k},2^{-k+1})}(x) & \text{if } k \ge 0; \\ \mathcal{X}_{[\frac{2}{3},\frac{4}{3})}(x) + (2-x)^{\frac{1}{3}}\mathcal{X}_{[\frac{4}{3},2)}(x) & \text{if } k = -1; \\ 2^{\frac{k+1}{2}}\mathcal{X}_{[2^{-k-1},2^{-k})}(x) & \text{if } k \le -2. \end{cases}$$

Then

(i) $\{f_k\}_{k\in\mathbb{Z}}$ is an F_a-frame for $L^2(\mathbb{R}_+)$.

(ii) $\{f_k\}_{0\neq k\in\mathbb{Z}}$ is not an F_a-frame, but it is F_a-complete in $L^2(\mathbb{R}_+)$.

Proof Obviously, $\{f_k\}_{k\in\mathbb{Z}}$ is a sequence in $L^2(\mathbb{R}_+)$. By a standard computation, we have, for each $f\in L^2(\mathbb{R}_+)$,

$$\sum_{0\neq k\in\mathbb{Z}}\bigl|\langle f,f_k\rangle_a(\cdot)\bigr|^2=\begin{cases}\|f\|_a^2(\cdot) & \text{a.e. on } [1,\frac{4}{3});\\ \sum_{0\neq j\in\mathbb{Z}}2^j|f(2^j\cdot)|^2 & \\ \quad+|2^{-1}f(2^{-1}\cdot)+f(\cdot)(2-\cdot)^{\frac{1}{3}}|^2 & \text{a.e. on } [\frac{4}{3},2),\end{cases}\tag{2.5}$$

$$\sum_{k\in\mathbb{Z}}\bigl|\langle f,f_k\rangle_a(\cdot)\bigr|^2=\begin{cases}\|f\|_a^2(\cdot)+|f(\cdot)|^2 & \text{a.e. on } [1,\frac{4}{3});\\ \|f\|_a^2(\cdot)+|2^{-1}f(2^{-1}\cdot)+f(\cdot)(2-\cdot)^{\frac{1}{3}}|^2 & \text{a.e. on } [\frac{4}{3},2).\end{cases}\tag{2.6}$$

From (2.6), it follows that, for each $f\in L^2(\mathbb{R}_+)$,

$$\|f\|_a^2(\cdot)\le\sum_{k\in\mathbb{Z}}\bigl|\langle f,f_k\rangle_a(\cdot)\bigr|^2\le 3\|f\|_a^2(\cdot)\quad\text{a.e. on } [1,2).$$

Thus (i) holds. Next we prove (ii). By (2.5) it follows that, for $f\in L^2(\mathbb{R}_+)$,

$$\sum_{0\neq k\in\mathbb{Z}}\bigl|\langle f,f_k\rangle_a(\cdot)\bigr|^2=0\quad\text{a.e. on } [1,a)$$

implies that $f=0$. This shows that $\{f_k\}_{0\neq k\in\mathbb{Z}}$ is F_a-complete in $L^2(\mathbb{R}_+)$. Take $f\in L^2(\mathbb{R}_+)$ by

$$f(x)=\mathcal{X}_{[\frac{2}{3},\frac{4}{3})}(x)-2^{-1}(2-x)^{-\frac{1}{3}}\mathcal{X}_{[\frac{4}{3},2)}(x).$$

Then

$$\|f\|_a^2(x)=2^{-1}+2^{-2}(2-x)^{-\frac{2}{3}}\quad\text{for } x\in\Bigl[\frac{4}{3},2\Bigr).$$

But $\sum_{0\neq k\in\mathbb{Z}}|\langle f,f_k\rangle_a(x)|^2=2^{-1}$ for $x\in[\frac{4}{3},2)$ by (2.5). Observe that $\lim_{x\to 2}\|f\|_a^2(x)=\infty$. It follows that there exists no positive constant A such that

$$A\|f\|_a^2(\cdot)\le\sum_{0\neq k\in\mathbb{Z}}\bigl|\langle f,f_k\rangle_a(\cdot)\bigr|^2\quad\text{a.e. on } [1,2).$$

Therefore, $\{f_k\}_{0\neq k\in\mathbb{Z}}$ is not an F_a-frame for $L^2(\mathbb{R}_+)$. □

By a standard argument, we have the following.

Lemma 2.1 *Let $\mathcal{A}$ be a bounded linear surjection from a Hilbert space $\mathcal{H}$ onto another Hilbert space $\mathcal{K}$. Then*

$$\mathcal{A}^\dagger=\mathcal{A}^*\bigl(\mathcal{A}\mathcal{A}^*\bigr)^{-1}.$$

By a standard argument similar to the case of frame, we have the following lemma, which is also a special case of [19, Lemma 2.5].

Lemma 2.2 *Let $\{f_k\}_{k\in\mathbb{Z}}$ be an F_a-frame for $L^2(\mathbb{R}_+)$ with frame bounds A and B, and S_F be its frame operator. Then S_F is a bounded and invertible linear operator on $L^2(\mathbb{R}_+)$, $\{S_F^{-1}f_k\}_{k\in\mathbb{Z}}$ is an F_a-frame for $L^2(\mathbb{R}_+)$ with frame bounds B^{-1} and A^{-1}, and*

$$f = \sum_{k\in\mathbb{Z}} \langle f, S_F^{-1}f_k\rangle_a f_k \quad \textit{for } f \in L^2(\mathbb{R}_+).$$

The following lemma demonstrates that the orthogonal complement operation preserves unimodular factor product invariant property of initial sets.

Lemma 2.3 *Given $\varphi \in B_a$ with $|\varphi| = 1$ and a nonempty subset V of $L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$, let $\varphi V = V$. Then $\varphi V^\perp = V^\perp$.*

Proof Observe that, for $f \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$, $f\perp\varphi V$ if and only if $\overline{\varphi}f\perp V$. It follows that $(\varphi V)^\perp = \varphi V^\perp$. On the other hand, $(\varphi V)^\perp = V^\perp$ if $\varphi V = V$. Therefore, $\varphi V^\perp = V^\perp$. □

The following lemma is an extension of [19, Lemma 2.3] which dealt with the subspaces of $L^2(\mathbb{R}_+)$. The proof herein is simpler than that of [19, Lemma 2.3].

Lemma 2.4 *Let V be a closed linear subspace of $L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$. Then the following are equivalent*:

(i) $\Lambda_m V = V$ *for* $m \in \mathbb{Z}$.
(ii) $V^{\perp_{F_a}} = V^\perp$.
(iii) $V(\varphi) \subset V$ *for an arbitrary a-dilation periodic measurable function on* $\mathbb{R}_+$.

Proof By Proposition 1.1(iv), for $f \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$, $f\perp_{F_a} V$ if and only if $f\perp\Lambda_m V$ for each $m\in\mathbb{Z}$. On the other hand, (i) is equivalent to $V^\perp = (\Lambda_m V)^\perp$ for each $m\in\mathbb{Z}$. It follows that (i) is equivalent to (ii). Since (i) is equivalent to $\Lambda_m V \subset V$ for each $m \in \mathbb{Z}$, (iii) implies (i). Next we prove (ii) implies (iii) to finish the proof. Suppose (ii) holds. Observe that $V^\perp$ is a closed subspace, and $V^\perp = \Lambda_m V^\perp$ for each $m\in\mathbb{Z}$ by Lemma 2.3. Applying the equivalence between (i) and (ii) to $V^\perp$, we obtain

$$V = \left(V^\perp\right)^\perp = \left(V^\perp\right)^{\perp_{F_a}}.$$

It follows that

$$V = \left(V^{\perp_{F_a}}\right)^{\perp_{F_a}} \tag{2.7}$$

by (ii). On the other hand,

$$\langle f, \varphi g\rangle_a = \varphi\langle f, g\rangle_a$$

for $g \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$ and a-dilation periodic measurable functions φ on $\mathbb{R}_+$ satisfying $\varphi g \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$. It follows that $V^{\perp_{F_a}} \subset (V(\varphi))^{\perp_{F_a}}$, and thus

$$\left[\left(V(\varphi)\right)^{\perp_{F_a}}\right]^{\perp_{F_a}} \subset V$$

by (2.7). This leads to (iii) by the fact that $V(\varphi) \subset [(V(\varphi))^{\perp_{F_a}}]^{\perp_{F_a}}$. The proof is completed. □

Lemma 2.5 *Let V be a closed linear subspace of $L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$ satisfying $\Lambda_m V = V$ for each $m \in \mathbb{Z}$. Then*

$$P_V(\varphi f) = \varphi P_V f$$

for $f \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$ and $\varphi \in B_a$.

Proof Fix $\varphi \in B_a$. Then $V(\varphi) = \varphi V$. By Lemma 2.3, $V^\perp$ is also a closed linear subspace of $L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$ satisfying $\Lambda_m V^\perp = V^\perp$ for each $m \in \mathbb{Z}$. Applying Lemma 2.4 to $V^\perp$ leads to $\varphi V^\perp \subset V^\perp$. It follows that $\varphi P_{V^\perp} f \in V^\perp$ for $f \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$, and thus

$$P_V(\varphi P_{V^\perp} f) = 0 \quad \text{for } f \in L^2(\mathbb{R}_+)\ \bigl(L^2\bigl(\mathbb{Z}\times[1,a)\bigr)\bigr). \tag{2.8}$$

By Lemma 2.4, we have $\varphi P_V f \in V$ which implies that

$$P_V(\varphi P_V f) = \varphi P_V f$$

for $f \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$. This together with (2.8) leads to

$$\begin{aligned} P_V(\varphi f) &= P_V(\varphi P_V f + \varphi P_{V^\perp} f) \\ &= \varphi P_V f \end{aligned}$$

for $f \in L^2(\mathbb{R}_+)$ $(L^2(\mathbb{Z}\times[1,a)))$. The proof is completed. □

Lemma 2.6 *Let V and W be closed subspaces of $L^2(\mathbb{R}_+)$ or $L^2(\mathbb{Z}\times[1,a))$ satisfying $\Lambda_m V = V$ and $\Lambda_m W = W$ for each $m \in \mathbb{Z}$, and $T: V \to W$ be an a-factorable bounded linear operator from V to W. Then*

(i) *T^*, T^*T and TT^* are a-factorable, and $\langle Tf, g\rangle_a = \langle f, T^*g\rangle_a$ for $f \in V$, $g \in W$.*
(ii) *T^{-1} is a-factorable if T is invertible.*

Proof For simplicity, for $f \in V$ and $g \in W$, we use $\langle Tf, g\rangle$ and $\langle f, T^*g\rangle$ to denote the inner products of Tf and g, and f and T^*g in the corresponding spaces, i.e.,

$$\langle Tf, g\rangle = \begin{cases} \langle Tf, g\rangle_{L^2(\mathbb{R}_+)} & \text{if } W \subset L^2(\mathbb{R}_+); \\ \langle Tf, g\rangle_{L^2(\mathbb{Z}\times[1,a))} & \text{if } W \subset L^2(\mathbb{Z}\times[1,a)) \end{cases}$$

and

$$\langle f, T^*g\rangle = \begin{cases} \langle f, T^*g\rangle_{L^2(\mathbb{R}_+)} & \text{if } V \subset L^2(\mathbb{R}_+); \\ \langle f, T^*g\rangle_{L^2(\mathbb{Z}\times[1,a))} & \text{if } V \subset L^2(\mathbb{Z}\times[1,a)). \end{cases}$$

(i) If T^* is a-factorable, so are T^*T and TT^* since T is a-factorable. Arbitrarily fix $f \in V$, $g \in W$ and $\varphi \in B_a$. Then $\varphi V \subset V$ and $\varphi W \subset W$ by Lemma 2.4. Since T is a-factorable,

$$\bigl\langle T(\varphi f), g\bigr\rangle = \langle \varphi Tf, g\rangle = \langle Tf, \overline{\varphi} g\rangle = \bigl\langle f, T^*(\overline{\varphi} g)\bigr\rangle$$

and

$$\langle T(\varphi f), g\rangle = \langle \varphi f, T^* g\rangle = \langle f, \overline{\varphi} T^* g\rangle.$$

It follows that

$$\langle f, T^*(\overline{\varphi} g)\rangle = \langle f, \overline{\varphi} T^* g\rangle,$$

and thus $T^*(\overline{\varphi} g) = \overline{\varphi} T^* g$ by the arbitrariness of f. And again by the arbitrariness of φ and g, T^* is a-factorable. Next we prove that

$$\langle Tf, g\rangle_a = \langle f, T^* g\rangle_a \quad \text{for } f \in V \text{ and } g \in W. \tag{2.9}$$

Observe that, for $f \in V$ and $g \in W$, $\langle Tf, g\rangle = \langle f, T^* g\rangle$. By Proposition 1.1(ii), it may be rewritten as

$$\int_1^a \langle Tf, g\rangle_a(x)\,dx = \int_1^a \langle f, T^* g\rangle_a(x)\,dx \quad \text{for } f \in V \text{ and } g \in W. \tag{2.10}$$

Given an arbitrary $E \subset [1, a)$ with $|E| > 0$, replace f by $\mathcal{X}_{\bigcup_{j\in\mathbb{Z}} a^j E} f$ in (2.10) (this can be done by Lemma 2.4). Then we have

$$\int_E \langle Tf, g\rangle_a(x)\,dx = \int_E \langle f, T^* g\rangle_a(x)\,dx$$

due to the fact that T is a-factorable. It leads to (2.9) by the arbitrariness of E and [26, Theorem 1.40].

(ii) Suppose T is invertible. For $g \in W$ and $\varphi \in B_a$, we have

$$T(\varphi T^{-1} g) = \varphi T T^{-1} g = \varphi g$$

Since T is a-factorable. It follows that $\varphi T^{-1} g = T^{-1}(\varphi g)$ for $g \in W$. The proof is completed. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1 (i) Necessity. Suppose F and $\widetilde{F}$ are F_a-equivalent. Then there exists an a-factorable, bounded and invertible linear operator T on $L^2(\mathbb{R}_+)$ such that

$$Tf_k = \widetilde{f}_k \quad \text{for } k \in \mathbb{Z}.$$

By Lemma 2.6, it follows that, for each $f \in L^2(\mathbb{R}_+)$,

$$\langle f, \widetilde{f}_k\rangle_a = \langle T^* f, f_k\rangle_a \quad \text{for } k \in \mathbb{Z}. \tag{3.1}$$

Since T is bounded and invertible, so is T^*. This implies that $\operatorname{range}(T^*) = L^2(\mathbb{R}_+)$. Therefore, (3.1) implies that $\operatorname{range}(D_F) = \operatorname{range}(D_{\widetilde{F}})$.

Sufficiency. Suppose $\mathrm{range}(D_F) = \mathrm{range}(D_{\widetilde{F}}) = V$. Obviously, $\Lambda_m V = V$ for $m \in \mathbb{Z}$. By Lemma 2.2 and [23, Theorem 2.1], $\mathrm{range}(R_F) = L^2(\mathbb{R}_+)$. This implies that V is closed due to the fact that $D_F = R_F^*$. Let $R_F|_V$ be the restriction of R_F on V. We first claim that $(R_F|_V)^*(R_F|_V)$ is bounded and invertible, and its inverse $[(R_F|_V)^*(R_F|_V)]^{-1}$ is a-factorable. Let us check it. For $g \in V$ and $f \in L^2(\mathbb{R}_+)$, we have

$$\begin{aligned}\langle g, (R_F|_V)^* f\rangle_{L^2(\mathbb{Z}\times[1,a))} &= \langle (R_F|_V) g, f\rangle_{L^2(\mathbb{R}_+)} \\ &= \langle R_F g, f\rangle_{L^2(\mathbb{R}_+)} \\ &= \langle g, R_F^* f\rangle_{L^2(\mathbb{Z}\times[1,a))}.\end{aligned}$$

This implies that

$$(R_F|_V)^* f = R_F^* f \quad \text{for } f \in L^2(\mathbb{R}_+) \tag{3.2}$$

by the arbitrariness of g. Since $\mathrm{range}(R_F) = L^2(\mathbb{R}_+)$ and $R_F^* = D_F$,

$$V = \mathrm{range}\bigl(R_F^*\bigr) = \mathrm{range}\bigl(R_F^* R_F\bigr). \tag{3.3}$$

Also observe that

$$\mathrm{range}(R_F) = R_F\bigl[\bigl(\ker(R_F)\bigr)^{\perp}\bigr] = R_F(V) = \mathrm{range}(R_F|_V) \tag{3.4}$$

due to $(\ker(R_F))^{\perp} = V$. Collecting (3.2)–(3.4) gives

$$V = \mathrm{range}\bigl[(R_F|_V)^*(R_F|_V)\bigr]. \tag{3.5}$$

Since $R_F|_V$ is injective, so is $(R_F|_V)^*(R_F|_V)$. This together with (3.5) leads to $(R_F|_V)^*(R_F|_V)$ being a bounded bijection on V. On the other hand, $R_F|_V$ is a-factorable since R_F is a-factorable and $\Lambda_m V = V$ for $m \in \mathbb{Z}$. By Lemma 2.6, $(R_F|_V)^*(R_F|_V)$ and $[(R_F|_V)^*(R_F|_V)]^{-1}$ are both a-factorable. We have proved the claim. Now we define $T : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by

$$T = R_{\widetilde{F}}\bigl[(R_F|_V)^*(R_F|_V)\bigr]^{-1} D_F.$$

Then it is well defined and bounded. Next we prove that T is an a-factorable bijection satisfying $\widetilde{f_k} = T f_k$ for $k \in \mathbb{Z}$ to finish the proof of sufficiency. By Lemma 2.2 and [23, Theorem 2.1] and the fact that $(\ker(R_{\widetilde{F}}))^{\perp} = V$, we have

$$L^2(\mathbb{R}_+) = \mathrm{range}(R_{\widetilde{F}}) = R_{\widetilde{F}}(V). \tag{3.6}$$

Also observing $(R_F|_V)^*(R_F|_V)$ being a bijection on V leads to

$$V = \mathrm{range}\bigl(\bigl[(R_F|_V)^*(R_F|_V)\bigr]^{-1} D_F\bigr).$$

It follows that

$$L^2(\mathbb{R}_+) = \mathrm{range}(T) \tag{3.7}$$

by (3.6). Since F is an F_a-frame for $L^2(\mathbb{R}_+)$, D_F is injective, and thus

$$\big[(R_F|_V)^*(R_F|_V)\big]^{-1}D_F$$

is injective. Also $R_{\widetilde{F}}$ is injective when restricted on V, and

$$\operatorname{range}\big(\big[(R_F|_V)^*(R_F|_V)\big]^{-1}D_F\big) \subset V.$$

It follows that T is injective. Therefore, T is bijective. Since $R_{\widetilde{F}}$, D_F and $[(R_F|_V)^*(R_F|_V)]^{-1}$ are all a-factorable by Lemma 2.6, so is T. Finally, we prove that

$$\widetilde{f}_k = Tf_k \quad \text{for } k \in \mathbb{Z}. \tag{3.8}$$

For $k \in \mathbb{Z}$, define $e^{(k)} = \{e_l^{(k)}(\cdot)\}_{l\in\mathbb{Z}} \in L^2(\mathbb{Z}\times[1,a))$ by

$$e_l^{(k)}(\cdot) = \begin{cases} 0 & \text{if } l \neq k; \\ 1 & \text{if } l = k \end{cases}$$

on $[1,a)$. Then, for $k \in \mathbb{Z}$,

$$f_k = R_F e^{(k)} = R_F\big(P_V e^{(k)} + P_{V^\perp} e^{(k)}\big) = R_F P_V e^{(k)}$$

due to $V^\perp = \ker(R_F)$, and thus

$$Tf_k = R_{\widetilde{F}}\big[(R_F|_V)^*(R_F|_V)\big]^{-1}R_F^* R_F P_V e^{(k)}.$$

It follows that

$$\begin{aligned} Tf_k &= R_{\widetilde{F}}\big[(R_F|_V)^*(R_F|_V)\big]^{-1}(R_F|_V)^*(R_F|_V)P_V e^{(k)} \\ &= R_{\widetilde{F}} P_V e^{(k)} \end{aligned}$$

by (3.2). Also observing that

$$\widetilde{f}_k = R_{\widetilde{F}} e^{(k)} = R_{\widetilde{F}} P_V e^{(k)}$$

leads to (3.8).

(ii) Necessity. Suppose F and $\widetilde{F}$ are unitarily F_a-equivalent. Then there exists an a-factorable and unitary operator T on $L^2(\mathbb{R}_+)$ such that

$$Tf_k = \widetilde{f}_k \quad \text{for } k \in \mathbb{Z}.$$

It follows that

$$R_{\widetilde{F}} g = \sum_{k\in\mathbb{Z}} g_k Tf_k = TR_F g$$

for $g \in L^2(\mathbb{Z}\times[1,a))$, and thus (1.12) holds by the unitarity of T.

Sufficiency. Suppose (1.12) holds. Then $\ker(R_F) = \ker(R_{\widetilde{F}})$. On the other hand, $\mathrm{range}(D_F)$ and $\mathrm{range}(D_{\widetilde{F}})$ are closed by the arguments in (i). It follows that

$$\mathrm{range}(D_F) = \mathrm{range}(D_{\widetilde{F}})$$

due to the fact that $R_F^* = D_F$ and $R_{\widetilde{F}}^* = D_{\widetilde{F}}$. Therefore, F and $\widetilde{F}$ are F_a-equivalent, and

$$T = R_{\widetilde{F}}\big[(R_F|_V)^*(R_F|_V)\big]^{-1}D_F$$

is an a-factorable, bounded bijection on $L^2(\mathbb{R}_+)$ satisfying $\widetilde{f_k} = Tf_k$ for $k \in \mathbb{Z}$ by (i) and its proof. Next we prove that T is unitary to finish the proof. Write $V = \mathrm{range}(D_F) = \mathrm{range}(D_{\widetilde{F}})$, and define $\widetilde{D}_F : L^2(\mathbb{R}_+) \to V$ by

$$\widetilde{D}_F f = D_F f \quad \text{for } f \in L^2(\mathbb{R}_+).$$

Observe that $\widetilde{D}_F$ is different from D_F since V need not be equal to $L^2(\mathbb{Z} \times [1,a))$ and D_F is from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{Z} \times [1,a))$. Obviously, it is well defined. For $f \in L^2(\mathbb{R}_+)$ and $g \in V$, we have

$$\begin{aligned}
\langle \widetilde{D}_F f, g\rangle_{L^2(\mathbb{Z}\times[1,a))} &= \langle D_F f, g\rangle_{L^2(\mathbb{Z}\times[1,a))} \\
&= \big\langle f, D_F^* g\big\rangle_{L^2(\mathbb{R}_+)} \\
&= \big\langle f, (R_F|_V) g\big\rangle_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

It follows that

$$(\widetilde{D}_F)^* = R_F|_V. \tag{3.9}$$

Since $\widetilde{D}_F$ is surjective, we have

$$(\widetilde{D}_F)^\dagger = (\widetilde{D}_F)^*\big[\widetilde{D}_F(\widetilde{D}_F)^*\big]^{-1}$$

by Lemma 2.1. Thus

$$\begin{aligned}
(\widetilde{D}_F)^\dagger &= (R_F|_V)\big[(R_F|_V)^*(R_F|_V)\big]^{-1} \\
&= R_F\big[(R_F|_V)^*(R_F|_V)\big]^{-1}
\end{aligned}$$

by (3.9). Also observe that

$$(\widetilde{D}_F)^\dagger D_F f = (\widetilde{D}_F)^\dagger \widetilde{D}_F f = P_{(\ker(\widetilde{D}_F))^\perp} f = P_{(\ker(D_F))^\perp} f$$

for $f \in L^2(\mathbb{R}_+)$. It follows that

$$R_F\big[(R_F|_V)^*(R_F|_V)\big]^{-1} D_F f = P_{(\ker(D_F))^\perp} f = f$$

for $f \in L^2(\mathbb{R}_+)$ due to D_F being injective. Again substituting g for

$$\left[(R_F|_V)^*(R_F|_V)\right]^{-1} D_F f$$

in (1.12), we obtain

$$\begin{aligned}\|Tf\|_{L^2(\mathbb{R}_+)} &= \left\|R_F\left[(R_F|_V)^*(R_F|_V)\right]^{-1} D_F f\right\|_{L^2(\mathbb{R}_+)} \\ &= \|f\|_{L^2(\mathbb{R}_+)}.\end{aligned}$$

This shows that T is norm-preserving and thus is unitary. The proof is completed. □

4 Conclusions

The space $L^2(\mathbb{R}_+)$ does not admit wavelet and Gabor systems due to $\mathbb{R}_+$ being not a group under addition. This paper addresses the F_a-frame for $L^2(\mathbb{R}_+)$. We introduce the notions of F_a-equivalence and unitary F_a-equivalence between F_a-frames, and characterize the F_a-equivalence and unitary F_a-equivalence. This characterization looks like that of equivalence and unitary equivalence between frames, but the proof is nontrivial due to the particularity of F_a-frames.

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