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New shrinking iterative methods for infinite families of monotone operators in a Banach space, computational experiments and applications

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Abstract

New shrinking iterative algorithms for approximating common zeros of two infinite families of maximal monotone operators in a real uniformly convex and uniformly smooth Banach space are designed. Two steps of multiple choices can be made in the new iterative algorithms, two groups of interactive containment sets C_n and Q_n are constructed and computational errors are considered, which are different from the previous ones. Strong convergence theorems are proved under mild assumptions and some new proof techniques can be found. Computational experiments for some special cases are conducted to show the effectiveness of the iterative algorithms and meanwhile some inequalities are proved to guarantee the strong convergence. Moreover, the applications of the abstract results on convex minimization problems and variational inequalities are exemplified.

Keywords: Lyapunov functional; Monotone operators; Shrinking iterative methods; Convex minimization problems; Variational inequalities; Visual Basic Six

1 Introduction

Throughout this paper, suppose E is a real Banach space with E^* being its dual space. Let C be a non-empty closed and convex subset of E. The symbols "$\langle x,f\rangle$", "$\to$" and "$\rightharpoonup$" denote the values of $f \in E^*$ at $x \in E$, the strong convergence and the weak convergence either in E or E^*, respectively.

For a nonlinear mapping $S : D(S) \subset E \to 2^E$, we use $F(S)$ to denote the set of fixed points of S, that is, $F(S) = \{x \in D(S) : x \in Sx\}$. For a nonlinear mapping $S : D(S) \subset E \to 2^{E^*}$, we use $S^{-1}0$ to denote the set of zeros of S, that is, $S^{-1}0 = \{x \in D(S) : 0 \in Sx\}$.

The normalized duality mapping $J_E : E \to 2^{E^*}$ is defined as follows [1]:

$$J_E(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

An operator $A : E \to 2^{E^*}$ is said to be monotone [1] if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, $\forall y_i \in Ax_i$, $i = 1, 2$. The monotone operator A is called maximal monotone if $R(J_E + \lambda A) = E^*$, $\forall \lambda > 0$.

The Lyapunov functional $\varphi : E \times E \to R^+$ is defined as follows [2]:

$$\varphi(x,y) = \|x\|^2 - 2\langle x, j_E(y)\rangle + \|y\|^2, \quad \forall x,y \in E, j_E(y) \in J_E(y).$$

If E is a real reflexive and strictly convex Banach space, then for each $x \in E$ there exists a unique element $x_0 \in C$ such that $\|x - x_0\| = \inf\{\|x - y\| : y \in C\}$. Such an element x_0 is denoted by $P_C x$ and P_C is called the metric projection of E onto C (see [2]).

If E is a real reflexive, smooth and strictly convex Banach space, then, for $\forall x \in E$, there exists a unique element $x_0 \in C$ satisfying $\varphi(x_0, x) = \inf\{\varphi(z,x) : z \in C\}$. In this case, $\forall x \in E$, define $\Pi_C : E \to C$ by $\Pi_C x = x_0$, and then Π_C is called the generalized projection from E onto C (see [2]).

A mapping $B : C \to C$ is called generalized non-expansive [3] if $F(B) \neq \emptyset$ and $\varphi(Bx, y) \leq \varphi(x,y)$, $\forall x \in C$ and $y \in F(B)$. A point $p \in C$ is said to be a strong asymptotic fixed point of B [4] if there exists a sequence $\{x_n\} \subset C$ with $x_n - Bx_n \to 0$ such that $x_n \to p$, as $n \to \infty$. We use $\widetilde{F}(B)$ to denote the set of strong asymptotic fixed points of B. A mapping B is called weakly relatively non-expansive [4] if $\widetilde{F}(B) = F(B) \neq \emptyset$ and $\varphi(p, Bx) \leq \varphi(p,x)$ for $x \in C$ and $p \in F(B)$.

A mapping $S : E \to C$ is said to be sunny [3] if $S(S(x) + t(x - S(x))) = S(x)$, $\forall x \in E$ and $t \geq 0$. A mapping $S : E \to C$ is said to be a retraction [3] if $S(z) = z$ for $\forall z \in C$. If E is a real smooth and strictly convex Banach space, then there exists a unique sunny generalized non-expansive retraction of E onto C, which is denoted by R_C.

Maximal monotone operator is a kind of important nonlinear mappings which draws much attention of mathematicians since it has rich practical background [5–8]. Some problems in nonlinear equations, minimization problems, variational inequalities and split problems and some others can be reduced to the problems for finding zeros of maximal monotone operators. Designing iterative algorithms to approximate zeros of maximal monotone operators is a hot topic, which can be seen in [9–13] and the references therein.

It is a natural idea to extend the study on designing iterative algorithms to approximate zeros of a maximal monotone operator to that for approximating common zeros of finite or infinite families of maximal monotone operators for the purpose of describing a complicated system in practical problems. Some related work can be found in [14–18] and the references therein.

Recall that in 2014 Wei et al. [15] introduced two composite operators $U_n := J_E^{-1}[a_0 J_E + \sum_{i=1}^m a_i J_E (J_E + r_{n,i} A_i)^{-1} J_E]$ and $W_n := J_E^{-1}\{b_0 J_E + \sum_{j=1}^l b_j J_E [(J_E + s_{n,j} B_j)^{-1} J_E (J_E + s_{n,j-1} B_{j-1})^{-1} \times J_E \cdots (J_E + s_{n,1} B_1)^{-1} J_E]\}$, where $A_i : E \to E^*$ and $B_j : E \to E^*$ are maximal monotone operators, for $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,l\}$. And the following iterative algorithm is presented for approximating the common zeros of $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^l$:

$$\begin{cases} x_1 \in E, \\ u_n = J_E^{-1}[(1-\alpha_n) J_E x_n], \\ v_n = J_E^{-1}[(1-\beta_n) J_E x_n + \beta_n J_E U_n u_n], \\ x_{n+1} = J_E^{-1}[\gamma_n J_E x_n + (1-\gamma_n) J_E W_n v_n], \quad n \in N. \end{cases} \tag{1.1}$$

Under the strong assumptions that the normalized duality mappings J_E and J_E^{-1} are weakly sequentially continuous, the result that $x_n \rightharpoonup v_0 = \lim_{n\to\infty} \Pi_{(\bigcap_{i=1}^m A_i^{-1}0)\cap(\bigcap_{j=1}^l B_j^{-1}0)}(x_n)$

is proved, as $n \to \infty$. Though only weak convergence is obtained, the idea of constructing composite operators is quite interesting.

In 2015, Wei et al. [16] deleted the strong assumptions imposed on both J_E and J_E^{-1} and obtained the result of strong convergence instead of weak convergence by constructing a sequence of shrinking projection sets. The iterative algorithm is presented in a real smooth and uniformly convex Banach space E as follows:

$$\begin{cases} x_1 \in E, \qquad u \in E, \\ u_n = J_E^{-1}\{\alpha_n J_E x_n \\ \qquad + (1-\alpha_n)J_E[(J_E + r_{n,m}A_m)^{-1}J_E(J_E + r_{n,m-1}A_{m-1})^{-1}J_E \cdots (J_E + r_{n,1}A_1)^{-1}J_E x_n]\}, \\ v_n = J_E^{-1}[\beta_n J_E u + (1-\beta_n)\sum_{j=1}^{l} a_j J_E(J_E + s_{n,j}B_j)^{-1}J_E u_n], \\ C_1 = E, \\ C_{n+1} = \{p \in C_n : \varphi(p,u_n) \le \varphi(p,x_n), \varphi(p,v_n) \le \beta_n\varphi(p,u) + (1-\beta_n)\varphi(p,u_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_1), \quad n \in N. \end{cases} \tag{1.2}$$

Under mild conditions, the result that $x_n \to \Pi_{(\bigcap_{i=1}^{m} A_i^{-1}0)\cap(\bigcap_{j=1}^{l} B_j^{-1}0)}(u)$, as $n \to \infty$, is proved, where $A_i : E \to E^*$ and $B_j : E \to E^*$ are maximal monotone mappings for $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,l\}$. Moreover, the iterative algorithm is applied to one kind p-Laplacian-like equation.

In 2015, Wei et al. [17] studied the maximal operators $A_i : E^* \to E$ and $B_j : E^* \to E$, for $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,l\}$. Since the domain of the operators is E^* not E, the sunny generalized non-expansive retraction $R_{C_{n+1}}$ is employed in the iterative construction instead of the generalized projection $\Pi_{C_{n+1}}$. The iterative algorithm is presented as follows:

$$\begin{cases} x_1 \in E, \qquad u \in E, \\ y_n = \alpha_n x_n + (1-\alpha_n)(I + r_{n,m}A_mJ_E)^{-1}(I + r_{n,m-1}A_{m-1}J_E)^{-1} \cdots (I + r_{n,1}A_1J_E)^{-1}x_n, \\ z_n = \beta_n u + (1-\beta_n)\sum_{j=1}^{l} a_j(I + s_{n,j}B_jJ_E)^{-1}y_n, \\ C_1 = E, \\ C_{n+1} = \{p \in C_n : \varphi(y_n,p) \le \varphi(x_n,p), \varphi(z_n,p) \le \beta_n\varphi(u,v) + (1-\beta_n)\varphi(y_n,p)\}, \\ x_{n+1} = R_{C_{n+1}}(x_1), \quad n \in N. \end{cases} \tag{1.3}$$

Under the assumption that J_E is weakly sequentially continuous, the result that $x_n \to R_{(\bigcap_{i=1}^{m}(A_iJ_E)^{-1}0)\cap(\bigcap_{j=1}^{l}(B_jJ_E)^{-1}0)}(x_1)$ is proved, as $n \to \infty$. The application of the iterative algorithm is applied to a kind of curvature systems.

In 2018, Wei et al. [18] extended the topic to the case for infinite family of maximal monotone operators $A_i : E \to E^*$ and infinite family of weakly relatively non-expansive mappings $B_i : E \to E$, for $i \in N$. In each iterative step n, two groups of subsets of E are constructed and multi-choice of the iterative element can be made avoiding the calculation of the generalized projection, which is different but contains the traditional projection iterative algorithm. The iterative algorithm can be seen as follows:

$$\begin{cases} x_1 \in E, \qquad e_1 \in E, \\ v_{n,i} = (J_E + s_{n,i}A_i)^{-1}J_E(x_n + e_n), \\ w_{n,i} = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n)J_E B_i v_{n,i}], \\ C_1 = E = Q_1, \\ C_{n+1,i} = \{z \in E : \langle v_{n,i} - z, J_E(x_n + e_n) - J_E v_{n,i}\rangle \geq 0\}, \\ C_{n+1} = (\bigcap_{i=1}^{\infty} C_{n+1,i}) \cap C_n, \\ Q_{n+1,i} = \{z \in C_{n+1,i} : \varphi(z, w_{n,i}) \leq \alpha_n \varphi(z, u_n) + (1-\alpha_n)\varphi(z, v_{n,i})\}, \\ Q_{n+1} = (\bigcap_{i=1}^{\infty} Q_{n+1,i}) \cap Q_n, \\ U_{n+1} = \{z \in Q_{n+1} : \|x_1 - z\|^2 \leq \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \tau_{n+1}\}, \\ x_{n+1} \in U_{n+1}, \quad n \in N, \end{cases} \tag{1.4}$$

where $\{e_n\} \subset E$ is the error sequence and $P_{Q_{n+1}}$ is the metric projection from E onto Q_{n+1}. The result that $x_n \to P_{\bigcap_{n=1}^{\infty} Q_n}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} F(B_i))$ is proved, as $n \to \infty$.

Later, in [14], the iterative algorithm (1.4) was simplified in the sense that the evaluation of the sets $C_{n+1,i}$ and $Q_{n+1,i}$ for $i \in N$ are replaced by that of C_{n+1} and Q_{n+1} directly. The iterative algorithm is stated as follows:

$$\begin{cases} x_1 \in E, \qquad e_1 \in E, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_{n,i} J_E (J_E + r_{n,i}A_i)^{-1} J_E(x_n + e_n)], \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n)J_E B_i y_n], \\ C_1 = E = Q_1, \\ C_{n+1} = \{v \in C_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n), \\ \varphi(v, z_n) \leq \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, y_n)\}, \\ Q_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \lambda_{n+1}\}, \\ x_{n+1} \in Q_{n+1}, \quad n \in N. \end{cases} \tag{1.5}$$

The result that $x_n \to P_{\bigcap_{n=1}^{\infty} Q_n}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} F(B_i))$ is proved, as $n \to \infty$. Computational experiments are conducted for some special cases.

In this paper, our purpose is to extend the topic from two finite families of maximal monotone operators (e.g. [17]) to that for the infinite case. Two steps of multiple choices can be made in the new iterative algorithms and two groups of interactive containment sets C_n and Q_n are constructed, which are different from the previous ones(e.g. [18]). Some new proof techniques can be found, especially the wide use of inequalities. Computational experiments are conducted and the applications on convex minimization problems and variational inequalities are exemplified.

2 Preliminaries

A Banach space E is said to be uniformly convex [19] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, one has $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let $\lambda_E : [0,+\infty) \to [0,+\infty)$ be a function. Then λ_E is called the modulus of smoothness of E if it is defined as follows [19]:

$$\lambda_E(t) = \sup\left\{\frac{1}{2}\bigl(\|x+y\| + \|x-y\|\bigr) - 1 : x,y \in E, \|x\| = 1, \|y\| \le t\right\}.$$

A Banach space E is said to be uniformly smooth [19] if $\frac{\lambda_E(t)}{t} \to 0$, as $t \to 0$.

A uniformly convex and uniformly smooth Banach space E has Property (H) in the sense that, if for every sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, one has $x_n \to x$, as $n \to \infty$.

Lemma 2.1 ([19, 20]) *If E is real uniformly convex and uniformly smooth Banach space, then* (1) *J_E is single-valued, surjective and for $x \in E$ and $k \in (0,+\infty)$, $J_E(kx) = kJ_E(x)$;* (2) *$J_E^{-1} = J_{E^*}$ is the normalized duality mapping from E^* to E;* (3) *both J_E and J_E^{-1} are uniformly continuous on each bounded subset of E or E^*, respectively.*

Lemma 2.2 ([1]) *Let $A : E \to 2^{E^*}$ be a maximal monotone operator, then*

(1) *$A^{-1}0$ is a closed and convex subset of E;*
(2) *if $x_n \to x$ and $y_n \in Ax_n$ with $y_n \rightharpoonup y$, or $x_n \rightharpoonup x$ and $y_n \in Ax_n$ with $y_n \to y$, then $x \in D(A)$ and $y \in Ax$.*

Definition 2.3 ([21]) Let $\{C_n\}$ be a sequence of non-empty closed and convex subsets of E, then

(1) s-$\liminf C_n$, which is called strong lower limit of $\{C_n\}$, is defined as the set of all $x \in E$ such that there exists $x_n \in C_n$ for almost all n and it tends to x as $n \to \infty$ in the norm.
(2) w-$\limsup C_n$, which is called weak upper limit of $\{C_n\}$, is defined as the set of all $x \in E$ such that there exists a subsequence $\{C_{n_m}\}$ of $\{C_n\}$ and $x_{n_m} \in C_{n_m}$ for every n_m and it tends to x as $n_m \to \infty$ in the weak topology.
(3) If s-$\liminf C_n$ = w-$\limsup C_n$, then the common value is denoted by $\lim C_n$.

Lemma 2.4 ([21]) *Let $\{C_n\}$ be a decreasing sequence of closed and convex subsets of E, i.e. $C_n \subset C_m$ if $n \ge m$. Then $\{C_n\}$ converges in E and $\lim C_n = \bigcap_{n=1}^{\infty} C_n$.*

Lemma 2.5 ([22]) *Suppose E is a real uniformly smooth and uniformly convex Banach space. If $\lim C_n$ exists and is not empty, then $\{P_{C_n}x\}$ converges strongly to $P_{\lim C_n}x$ for every $x \in E$.*

Lemma 2.6 ([23]) *Let E be a real uniformly smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.7 ([23]) *Suppose E is a real uniformly convex and uniformly smooth Banach space and $A : E \to 2^{E^*}$ is a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Then $\forall x \in E$, $y \in A^{-1}0$ and $r > 0$, one has $\varphi(y, (J_E + rA)^{-1}J_E x) + \varphi((J_E + rA)^{-1}J_E x, x) \le \varphi(y, x)$.*

Lemma 2.8 ([23]) *Let E be a real strictly convex and smooth Banach space and C is a non-empty closed and convex subset of E. Then $\forall x \in E$, $\forall y \in C$, one has $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \le \varphi(y, x)$.*

Lemma 2.9 ([24]) *Let E be a real uniformly convex Banach space and $r \in (0, +\infty)$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, +\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|),$$

for $\alpha \in [0,1]$, $x, y \in E$ with $\|x\| \le r$ and $\|y\| \le r$.

3 Iterative algorithms and computational experiments

3.1 Iterative algorithms

Theorem 3.1 *Suppose E is a real uniformly convex and uniformly smooth Banach space and $J_E : E \to E^*$ is the normalized duality mapping. Let $A_i, B_i : E \to 2^{E^*}$ be maximal monotone operators, for each $i \in N$. Denote $\overline{U_n} = J_E^{-1}[a_0 J_E + \sum_{i=1}^{\infty} a_i J_E Q_{r_{n,i}}^{A_i}]$ and $\overline{W_n} = J_E^{-1}[b_0 J_E + \sum_{j=1}^{\infty} b_j J_E Q_{s_{n,j}}^{B_j} Q_{s_{n,j-1}}^{B_{j-1}} \cdots Q_{s_{n,1}}^{B_1}]$, where $Q_{r_{n,i}}^{A_i} = (J_E + r_{n,i}A_i)^{-1} J_E$ and $Q_{s_{n,j}}^{B_j} = (J_E + s_{n,j}B_j)^{-1} J_E$, for $i, j, n \in N$. Let $\{e_n\}$ and $\{\varepsilon_n\}$ be two error sequences in E, $\{r_{n,i}\}$, $\{s_{n,j}\}$, $\{\delta_n\}$ and $\{\vartheta_n\}$ be real number sequences in $(0, +\infty)$, for $i, j, n \in N$. Suppose $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ are real number sequences in $(0,1)$ such that $\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} b_i = 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $[0,1)$, for $n \in N$. Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \text{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E = X_1, \qquad Q_1 = E = Y_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ X_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \le \|P_{C_{n+1}}(x_1) - x_1\|^2 + \delta_n\}, \\ w_n \in X_{n+1}, \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ Y_{n+1} = \{v \in Q_{n+1} : \|x_1 - v\|^2 \le \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \vartheta_n\}, \\ x_{n+1} \in Y_{n+1}, \quad n \in N. \end{cases} \tag{3.1}$$

Under the assumptions that (i) $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \neq \emptyset$; (ii) $\inf_n r_{n,i} > 0$, $\inf_n s_{n,i} > 0$ for $i \in N$; (iii) $0 \le \sup_n \alpha_n < 1$, $0 \le \sup_n \beta_n < 1$; (iv) $\delta_n \to 0$, $\vartheta_n \to 0$; (v) $e_n \to 0$ and $\varepsilon_n \to 0$, as $n \to \infty$, one has

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty. \end{cases}$$

Proof The proof is split into ten steps.

Step 1. $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset C_n \cap Q_n$, for $n \in N$.

For this purpose, we shall use the inductive method.

If $n = 1$, it is obvious that $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset C_1 \cap Q_1 = E$. Suppose the result is true for $n = k$, that is, $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset C_k \cap Q_k$. Then $\forall p \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap$

$(\bigcap_{i=1}^{\infty} B_i^{-1}0)$, it follows from the definition of the Lyapunov functional, the convexity of $\|\cdot\|^2$ and Lemma 2.7 that

$$\begin{aligned}
\varphi(p,y_k) &= \|p\|^2 - 2\left\langle p, \alpha_k J_E x_k + (1-\alpha_k)\left[a_0 J_E(x_k+e_k) + \sum_{i=1}^{\infty} a_i J_E Q_{r_{k,i}}^{A_i}(x_k+e_k)\right]\right\rangle \\
&\quad + \left\|\alpha_k J_E x_k + (1-\alpha_k)\left[a_0 J_E(x_k+e_k) + \sum_{i=1}^{\infty} a_i J_E Q_{r_{k,i}}^{A_i}(x_k+e_k)\right]\right\|^2 \\
&\leq \|p\|^2 - 2\alpha_k\langle p, J_E x_k\rangle + \alpha_k\|x_k\|^2 - 2(1-\alpha_k)a_0\big\langle p, J_E(x_k+e_k)\big\rangle \\
&\quad - 2(1-\alpha_k)\sum_{i=1}^{\infty} a_i\big\langle p, J_E Q_{r_{k,i}}^{A_i}(x_k+e_k)\big\rangle \\
&\quad + (1-\alpha_k)a_0\|x_k+e_k\|^2 + (1-\alpha_k)\sum_{i=1}^{\infty} a_i\big\|Q_{r_{k,i}}^{A_i}(x_k+e_k)\big\|^2 \\
&= \alpha_k\varphi(p,x_k) + (1-\alpha_k)a_0\varphi(p,x_k+e_k) + (1-\alpha_{k+1})\sum_{i=1}^{\infty} a_i\varphi\big(p, Q_{r_{k,i}}^{A_i}(x_k+e_k)\big) \\
&\leq \alpha_k\varphi(p,x_k) + (1-\alpha_k)\varphi(p,x_k+e_k).
\end{aligned}$$

Thus $p \in C_{k+1}$. By induction, $p \in C_n$ for $n \in N$.

And, using Lemma 2.7 repeatedly, one has

$$\begin{aligned}
\varphi(p,z_k) &\leq \|p\|^2 - 2\beta_k\langle p, J_E x_k\rangle + \beta_k\|x_k\|^2 - 2(1-\beta_k)b_0\big\langle p, J_E(w_k+\varepsilon_k)\big\rangle \\
&\quad - 2(1-\beta_k)\sum_{j=1}^{\infty} b_j\big\langle p, J_E Q_{s_{k,j}}^{B_j}\cdots Q_{s_{k,1}}^{B_1}(w_k+\varepsilon_k)\big\rangle + (1-\beta_k)b_0\|w_k+\varepsilon_k\|^2 \\
&\quad + (1-\beta_k)\sum_{j=1}^{\infty} b_j\big\|Q_{s_{k,j}}^{B_j}\cdots Q_{s_{k,1}}^{B_1}(w_k+\varepsilon_k)\big\|^2 \\
&= \beta_k\varphi(p,x_k) + (1-\beta_k)b_0\varphi(p,w_k+\varepsilon_k) \\
&\quad + (1-\beta_k)\sum_{j=1}^{\infty} b_j\varphi\big(p, Q_{s_{k,j}}^{B_j}\cdots Q_{s_{k,1}}^{B_1}(w_k+\varepsilon_k)\big) \\
&\leq \beta_k\varphi(p,x_k) + (1-\beta_k)\varphi(p,w_k+\varepsilon_k).
\end{aligned}$$

Thus $p \in Q_{k+1}$. By induction $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset Q_n$, for $n \in N$, which implies that $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset C_n \cap Q_n$, for $n \in N$.

Step 2. C_n and Q_n are non-empty closed and convex subsets of E, for each $n \in N$.

It follows from Step 1 that both C_n and Q_n are non-empty subsets of E for $n \in N$.

It is obvious that both C_1 and Q_1 are closed and convex subsets of E. Suppose that both C_k and Q_k are closed and convex subsets of E, then noticing the fact that

$$\begin{aligned}
&\varphi(v,y_k) \leq \alpha_k\varphi(v,x_k) + (1-\alpha_k)\varphi(v,x_k+e_k) \\
&\quad\Leftrightarrow\quad \big\langle v, \alpha_k J_E x_k + (1-\alpha_k)J_E(x_k+e_k) - J_E y_k\big\rangle \leq \frac{(1-\alpha_k)\|x_k+e_k\|^2 + \alpha_k\|x_k\|^2 - \|y_k\|^2}{2},
\end{aligned}$$

one sees that C_{k+1} is closed and convex. Therefore, by induction, C_n is closed and convex for each $n \in N$.

Notice that

$$\begin{aligned} &\varphi(v, z_k) \leq \beta_k \varphi(v, x_k) + (1-\beta_k)\varphi(v, w_k + \varepsilon_k) \\ &\quad \Leftrightarrow \quad \langle p, \beta_k J_E x_k + (1-\beta_k) J_E(w_k + \varepsilon_k) - J_E z_k \rangle \\ &\qquad \leq \frac{(1-\beta_k)\|w_k + \varepsilon_k\|^2 + \beta_k \|x_k\|^2 - \|z_k\|^2}{2}. \end{aligned}$$

Combining with the fact that C_n is closed and convex for $n \in N$, one sees that Q_{k+1} is closed and convex. By induction, Q_n is closed and convex, for each $n \in N$.

Step 3. $P_{C_n}(x_1) \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, $P_{Q_n}(x_1) \to P_{\bigcap_{n=1}^{\infty} Q_n}(x_1)$, as $n \to \infty$.

Since $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \neq \emptyset$, from Steps 1 and 2 and (3.1), we know that C_n is a non-empty closed, convex and decreasing subset of E. Using Lemmas 2.4 and 2.5, we know that $P_{C_n}(x_1) \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Similarly, we have $P_{Q_n}(x_1) \to P_{\bigcap_{n=1}^{\infty} Q_n}(x_1)$, as $n \to \infty$.

Step 4. $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) = P_{\bigcap_{n=1}^{\infty} Q_n}(x_1)$.

It suffices to show that $\bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} Q_n$.

In fact, from (3.1), $Q_n \subset C_n$, then $\bigcap_{n=1}^{\infty} Q_n \subset \bigcap_{n=1}^{\infty} C_n$. On the other hand, since $C_1 = E$ and $C_{n+1} \subset Q_n$, then $\bigcap_{n=1}^{\infty} C_{n+1} = \bigcap_{n=1}^{\infty} C_n \subset \bigcap_{n=1}^{\infty} Q_n$, which ensures that $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) = P_{\bigcap_{n=1}^{\infty} Q_n}(x_1)$.

Step 5. $\{w_n\}$ and $\{x_n\}$ are well-defined.

In fact, we only need to show that $X_n \neq \emptyset$ and $Y_n \neq \emptyset$, for each $n \in N$.

Since $\|P_{C_{n+1}}(x_1) - x_1\| = \inf_{q \in C_{n+1}} \|q - x_1\|$, for δ_n there exists $k_n \in C_{n+1}$ such that

$$\|x_1 - k_n\|^2 \leq \Bigl(\inf_{q \in C_{n+1}} \|q - x_1\|\Bigr)^2 + \delta_n = \bigl\|P_{C_{n+1}}(x_1) - x_1\bigr\|^2 + \delta_n.$$

Then $X_n \neq \emptyset$, which implies that $\{w_n\}$ is well-defined.

Similarly, $Y_n \neq \emptyset$, which implies that $\{x_n\}$ is well-defined.

Step 6. Both $\{w_n\}$ and $\{x_n\}$ are bounded.

Since $w_n \in X_{n+1}$,

$$\|x_1 - w_n\|^2 \leq \bigl\|P_{C_{n+1}}(x_1) - x_1\bigr\|^2 + \delta_n.$$

Since $\{P_{C_n}(x_1)\}$ is convergent from Step 3 and $\delta_n \to 0$, $\{w_n\}$ is bounded.

Similarly, $\{x_n\}$ is bounded.

Step 7. $w_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1) = P_{\bigcap_{n=1}^{\infty} Q_n}(x_1)$ and $x_n \to P_{\bigcap_{n=1}^{\infty} Q_n}(x_1) = P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $w_n \in X_{n+1} \subset C_{n+1}$ and C_n is convex, for $\forall t \in (0,1)$, $tP_{C_{n+1}}(x_1) + (1-t)w_n \in C_{n+1}$. Thus $\|P_{C_{n+1}}(x_1) - x_1\| \leq \|tP_{C_{n+1}}(x_1) + (1-t)w_n - x_1\|$. Using Lemma 2.9, we have

$$\begin{aligned} &\bigl\|P_{C_{n+1}}(x_1) - x_1\bigr\|^2 \\ &\quad \leq \bigl\|tP_{C_{n+1}}(x_1) + (1-t)w_n - x_1\bigr\|^2 \\ &\quad \leq t\bigl\|P_{C_{n+1}}(x_1) - x_1\bigr\|^2 + (1-t)\|w_n - x_1\|^2 - t(1-t)g\bigl(\bigl\|P_{C_{n+1}}(x_1) - w_n\bigr\|\bigr). \end{aligned}$$

Therefore, $tg(\|P_{C_{n+1}}(x_1) - w_n\|) \le \|w_n - x_1\|^2 - \|P_{C_{n+1}}(x_1) - x_1\|^2 \le \delta_n \to 0$, as $n \to \infty$. Then $w_n - P_{C_{n+1}}(x_1) \to 0$, as $n \to \infty$. Combining with Steps 3 and 4, we have $w_n \to P_{\bigcap_{n=1}^\infty C_n}(x_1) = P_{\bigcap_{n=1}^\infty Q_n}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in Y_{n+1} \subset Q_{n+1}$ and Q_n is convex, for $\forall t \in (0,1)$, $tP_{Q_{n+1}}(x_1) + (1-t)x_{n+1} \in Q_{n+1}$. Thus $\|P_{Q_{n+1}}(x_1) - x_1\| \le \|tP_{Q_{n+1}}(x_1) + (1-t)x_{n+1} - x_1\|$. Using Lemma 2.9 again, we have

$$\begin{aligned}&\bigl\|P_{Q_{n+1}}(x_1) - x_1\bigr\|^2 \\ &\quad\le \bigl\|tP_{Q_{n+1}}(x_1) + (1-t)x_{n+1} - x_1\bigr\|^2 \\ &\quad\le t\bigl\|P_{Q_{n+1}}(x_1) - x_1\bigr\|^2 + (1-t)\|x_{n+1} - x_1\|^2 - t(1-t)g\bigl(\bigl\|P_{Q_{n+1}}(x_1) - x_{n+1}\bigr\|\bigr).\end{aligned}$$

Therefore, $tg(\|P_{Q_{n+1}}(x_1) - x_{n+1}\|) \le \|x_{n+1} - x_1\|^2 - \|P_{Q_{n+1}}(x_1) - x_1\|^2 \le \vartheta_n \to 0$, as $n \to \infty$. Combining with Steps 3 and 4, we have $x_n \to P_{\bigcap_{n=1}^\infty Q_n}(x_1) = P_{\bigcap_{n=1}^\infty C_n}(x_1)$, as $n \to \infty$.

Step 8. $y_n \to P_{\bigcap_{n=1}^\infty C_n}(x_1) = P_{\bigcap_{n=1}^\infty Q_n}(x_1)$ and $z_n \to P_{\bigcap_{n=1}^\infty C_n}(x_1) = P_{\bigcap_{n=1}^\infty Q_n}(x_1)$, as $n \to \infty$.

Since $w_n \in X_{n+1} \subset C_{n+1} \subset Q_n$, for $n \ge 2$,

$$\varphi(w_n, y_n) \le \alpha_n \varphi(w_n, x_n) + (1-\alpha_n)\varphi(w_n, x_n + e_n)$$

and

$$\varphi(w_n, z_{n-1}) \le \beta_{n-1}\varphi(w_n, x_{n-1}) + (1-\beta_{n-1})\varphi(w_n, w_{n-1} + \varepsilon_{n-1}).$$

Since $e_n \to 0$ and $\varepsilon_n \to 0$, from Lemma 2.6 and Steps 6 and 7, we have $w_n - y_n \to 0$ and $w_n - z_{n-1} \to 0$, as $n \to \infty$. Therefore, $y_n \to P_{\bigcap_{n=1}^\infty C_n}(x_1) = P_{\bigcap_{n=1}^\infty Q_n}(x_1)$ and $z_n \to P_{\bigcap_{n=1}^\infty C_n}(x_1) = P_{\bigcap_{n=1}^\infty Q_n}(x_1)$, as $n \to \infty$.

Step 9. $P_{\bigcap_{n=1}^\infty C_n}(x_1) = P_{\bigcap_{n=1}^\infty Q_n}(x_1) \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty B_i^{-1}0)$.

For $\forall q \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty B_i^{-1}0)$, using Lemma 2.7 and (3.1), we have

$$\begin{aligned}\varphi(q, y_n) &\le \alpha_n\varphi(q, x_n) + (1-\alpha_n)\varphi\bigl(q, \overline{U_n}(x_n + e_n)\bigr) \\ &\le \alpha_n\varphi(q, x_n) + (1-\alpha_n)\Biggl[a_0\varphi(q, x_n + e_n) + \sum_{i=1}^\infty a_i\varphi\bigl(q, Q_{r_{n,i}}^{A_i}(x_n + e_n)\bigr)\Biggr] \\ &= \alpha_n\varphi(q, x_n) + (1-\alpha_n)\Biggl[a_0\varphi(q, x_n + e_n) + \sum_{i=1, i\ne i_0}^\infty a_i\varphi\bigl(q, Q_{r_{n,i}}^{A_i}(x_n + e_n)\bigr) \\ &\quad + a_{i_0}\varphi\bigl(q, Q_{r_{n,i_0}}^{A_{i_0}}(x_n + e_n)\bigr)\Biggr] \\ &\le \alpha_n\varphi(q, x_n) + (1-\alpha_n)\bigl[(1-a_{i_0})\varphi(q, x_n + e_n) + a_{i_0}\varphi\bigl(q, Q_{r_{n,i_0}}^{A_{i_0}}(x_n + e_n)\bigr)\bigr] \\ &\le \alpha_n\varphi(q, x_n) + (1-\alpha_n)\bigl\{(1-a_{i_0})\varphi(q, x_n + e_n) + a_{i_0}\bigl[\varphi(q, x_n + e_n) \\ &\quad - \varphi\bigl(Q_{r_{n,i_0}}^{A_{i_0}}(x_n + e_n), x_n + e_n\bigr)\bigr]\bigr\} \\ &= \alpha_n\varphi(q, x_n) + (1-\alpha_n)\varphi(q, x_n + e_n) - (1-\alpha_n)a_{i_0}\varphi\bigl(Q_{r_{n,i_0}}^{A_{i_0}}(x_n + e_n), x_n + e_n\bigr).\end{aligned}$$

Thus

$$(1-\alpha_n)a_{i_0}\varphi\bigl(Q_{r_{n,i_0}}^{A_{i_0}}(x_n + e_n), x_n + e_n\bigr) \le \alpha_n\varphi(q, x_n) + (1-\alpha_n)\varphi(q, x_n + e_n) - \varphi(q, y_n),$$

which ensures that $x_n + e_n - Q_{r_{n,i_0}}^{A_{i_0}}(x_n + e_n) \to 0$, as $n \to \infty$, since $0 \le \sup_n \alpha_n < 1$.

Repeating the above process, $x_n + e_n - Q^{A_i}_{r_{n,i}}(x_n + e_n) \to 0$, for each $\forall i \in N$, as $n \to \infty$. Thus, $Q^{A_i}_{r_{n,i}}(x_n + e_n) \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, for $\forall i \in N$, as $n \to \infty$.

Let $u_{n,i} = Q^{A_i}_{r_{n,i}}(x_n + e_n)$, then $J_E u_{n,i} + r_{n,i}A_i u_{n,i} = J_E(x_n + e_n)$. Note that $u_{n,i} \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, $e_n \to 0$ and $\inf_n r_{n,i} > 0$, then $A_i u_{n,i} \to 0$, as $n \to \infty$. In view of Lemmas 2.1 and 2.2, $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$.

For $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)$, using Lemma 2.7 again, we have

$$\begin{aligned}
\varphi(q,z_n) &\le \beta_n\varphi(q,x_n) + (1-\beta_n)\varphi\bigl(q,\overline{W_n}(w_n+\varepsilon_n)\bigr) \\
&\le \beta_n\varphi(q,x_n) + (1-\beta_n)\Biggl[b_0\varphi(q,w_n+\varepsilon_n) \\
&\quad + \sum_{j=1}^{\infty} b_j\varphi\bigl(q, Q^{B_j}_{s_{n,j}}Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)\bigr)\Biggr] \\
&\le \beta_n\varphi(q,x_n) + (1-\beta_n)b_0\varphi(q,w_n+\varepsilon_n) \\
&\quad + (1-\beta_n)\sum_{j=1}^{\infty} b_j\varphi\bigl(q, Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)\bigr) \\
&\quad - (1-\beta_n)\sum_{j=1}^{\infty} b_j\varphi\bigl(Q^{B_j}_{s_{n,j}}Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n), Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)\bigr).
\end{aligned}$$

Then using Lemma 2.7 repeatedly and noticing the results of Steps 7 and 8, one has

$$\begin{aligned}
&(1-\beta_n)\sum_{j=1}^{\infty} b_j\varphi\bigl(Q^{B_j}_{s_{n,j}}Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n), Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)\bigr) \\
&\quad\le \beta_n\varphi(q,x_n) + (1-\beta_n)b_0\varphi(q,w_n+\varepsilon_n) + (1-\beta_n)\sum_{j=1}^{\infty} b_j\varphi\bigl(q, Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)\bigr) \\
&\qquad - \varphi(q,z_n) \\
&\quad\le \beta_n\varphi(q,x_n) + (1-\beta_n)\varphi(q,w_n+\varepsilon_n) - \varphi(q,z_n) \to 0, \quad \textit{as } n \to \infty,
\end{aligned}$$

which implies that $\varphi(Q^{B_j}_{s_{n,j}}Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n), Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)) \to 0$, and then Lemma 2.6 implies that $Q^{B_j}_{s_{n,j}}Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) - Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) \to 0$, as $n \to \infty$.

Repeating the above process, by induction, we have

$$\begin{cases}
Q^{B_{j-1}}_{s_{n,j-1}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) - Q^{B_{j-2}}_{s_{n,j-2}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) \to 0, \\
Q^{B_{j-2}}_{s_{n,j-2}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) - Q^{B_{j-3}}_{s_{n,j-3}}\cdots Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) \to 0, \\
\vdots \\
Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) - (w_n+\varepsilon_n) \to 0,
\end{cases} \tag{3.2}$$

as $n \to \infty$.

Therefore, $Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n) \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$. Imitating the proof of $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) \in \bigcap_{i=1}^{\infty} A_i^{-1}0$, we know that $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) \in B_1^{-1}0$.

Now, set $v_{n,1} = Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)$ and $v_{n,2} = Q^{B_2}_{s_{n,2}}Q^{B_1}_{s_{n,1}}(w_n+\varepsilon_n)$, then $J_E v_{n,2} + s_{n,2}B_2 v_{n,2} = J_E v_{n,1}$. Since $v_{n,1} \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, from (3.2) we have $v_{n,2} \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$. Since

$\inf_n s_{n,2} > 0$, by using Lemma 2.1, $B_2 J_E v_{n,2} \to 0$, as $n \to \infty$. Lemma 2.2 implies that $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) \in B_2^{-1}0$.

By induction, we easily show that $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) \in B_j^{-1}0$, for each $j \in N$. Therefore, $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) \in \bigcap_{j=1}^{\infty} B_j^{-1}0$, which implies that $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) = P_{\bigcap_{n=1}^{\infty} Q_n}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)$.

Step 10. $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) = P_{\bigcap_{n=1}^{\infty} Q_n}(x_1) = P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1)$.

From Step 9, we see that

$$\left\|P_{\bigcap_{n=1}^{\infty} C_n}(x_1) - x_1\right\| \geq \left\|P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) - x_1\right\|.$$

From Step 1, we see that

$$\left\|P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) - x_1\right\| \geq \left\|P_{\bigcap_{n=1}^{\infty} C_n}(x_1) - x_1\right\|.$$

Therefore,

$$\left\|P_{\bigcap_{n=1}^{\infty} C_n}(x_1) - x_1\right\| = \left\|P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) - x_1\right\|.$$

Since $P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ is unique, $P_{\bigcap_{n=1}^{\infty} C_n}(x_1) = P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1)$.

This completes the proof. □

Corollary 3.2 *If we choose* $w_n = P_{C_{n+1}}(x_1)$, *then* (3.1) *reduces to the following one*:

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \textit{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E, \qquad Q_1 = E = Y_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ w_n = P_{C_{n+1}}(x_1), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \leq \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ Y_{n+1} = \{v \in Q_{n+1} : \|x_1 - v\|^2 \leq \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \vartheta_n\}, \\ x_{n+1} \in Y_{n+1}, \quad n \in N. \end{cases} \tag{3.3}$$

Under the assumptions that (*i*), (*ii*), (*iii*) *and* (*v*) *in Theorem* 3.1 *and* (*iv*)′ $\vartheta_n \to 0$, *as* $n \to \infty$, *one has*

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\cap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty. \end{cases}$$

Proof The proof can also be split into ten steps. Copy the proof of Steps 1–5 and Steps 8–10 in Theorem 3.1 and modify Steps 6 and 7 as follows, we can still get the result.

Step 6. Both $\{w_n\}$ and $\{x_n\}$ are bounded.

Since $w_n = P_{C_{n+1}}(x_1)$, we have $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset C_{n+1}$, $\|w_n - x_1\| \leq \|q - x_1\|$, which implies that $\{w_n\}$ is bounded.

Since $x_{n+1} \in Y_{n+1}$,

$$\|x_1 - x_{n+1}\|^2 \leq \left\|P_{Q_{n+1}}(x_1) - x_1\right\|^2 + \delta_n.$$

Since $P_{Q_n}(x_1) \to P_{\bigcap_{n=1}^{\infty} Q_n}(x_1)$ and $\delta_n \to 0$, $\{x_n\}$ is bounded.

Step 7. $w_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ and $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

It follows from Lemmas 2.4 and 2.5 that $w_n = P_{C_{n+1}}(x_1) \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$. Copy Step 7 in Theorem 3.1, $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

This completes the proof. □

Similar to Corollary 3.2, we have the following two results:

Corollary 3.3 *If we choose $x_{n+1} = P_{Q_{n+1}}(x_1)$, then (3.1) reduces to the following one:*

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \text{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E = X_1, \qquad Q_1 = E, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ X_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \delta_n\}, \\ w_n \in X_{n+1}, \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \leq \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ x_{n+1} = P_{Q_{n+1}}(x_1), \quad n \in N. \end{cases} \tag{3.4}$$

Under the assumptions of (i), (ii), (iii) and (v) in Theorem 3.1 and (iv)″ $\delta_n \to 0$, one has

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \text{as } n \to \infty. \end{cases}$$

Corollary 3.4 *If we choose $w_n = P_{C_{n+1}}(x_1)$ and $x_{n+1} = P_{Q_{n+1}}(x_1)$, then (3.1) reduces to the following one:*

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \text{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E = Q_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ w_n = P_{C_{n+1}}(x_1), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \leq \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ x_{n+1} = P_{Q_{n+1}}(x_1), \quad n \in N. \end{cases} \tag{3.5}$$

Under the assumptions of (*i*), (*ii*), (*iii*) *and* (*v*), *one has*

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty. \end{cases}$$

Corollary 3.5 *If we choose* $w_n = \Pi_{C_{n+1}}(x_n)$, *then* (3.1) *reduces to the following one*:

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \text{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E, \qquad Q_1 = E = Y_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ w_n = \Pi_{C_{n+1}}(x_n), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ Y_{n+1} = \{v \in Q_{n+1} : \|x_1 - v\|^2 \le \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \vartheta_n\}, \\ x_{n+1} \in Y_{n+1}, \quad n \in N. \end{cases} \tag{3.6}$$

Under the assumptions that (*i*), (*ii*), (*iii*) *and* (*v*) *in Theorem* 3.1 *and* (*iv*)′ *in Corollary* 3.2, *one has*

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0)\cap(\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), & \text{as } n \to \infty. \end{cases}$$

Proof Copy Steps 1–5 and 9 and 10 in Theorem 3.1, we are left to show the results of Steps 6, 7 and 8 are still true.

Step 6. Both $\{w_n\}$ and $\{x_n\}$ are bounded.

Copy Theorem 3.1, $\{x_n\}$ is bounded. Since $w_n = \Pi_{C_{n+1}}(x_n)$, $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset C_{n+1}$, using Lemma 2.8, $\varphi(q, w_n) + \varphi(w_n, x_n) \le \varphi(q, x_n)$. Thus $\{\varphi(q, w_n)\}$ is bounded. Since $\varphi(q, w_n) \ge (\|w_n\| - \|q\|)^2$, $\{w_n\}$ is bounded.

Step 7. $w_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ and $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Copy Theorem 3.1, $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in Y_{n+1} \subset Q_{n+1} \subset C_{n+1}$, using Lemma 2.8, $\varphi(x_{n+1}, w_n) + \varphi(w_n, x_n) \le \varphi(x_{n+1}, x_n) \to 0$, as $n \to \infty$. Thus $\varphi(w_n, x_n) \to 0$, which implies from Lemma 2.6 that $w_n - x_n \to 0$ and then $w_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ as $n \to \infty$.

Step 8. $y_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ and $z_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in Y_{n+1} \subset Q_{n+1} \subset C_{n+1}$, $\varphi(x_{n+1}, y_n) \le \alpha_n \varphi(x_{n+1}, x_n) + (1-\alpha_n)\varphi(x_{n+1}, x_n + e_n) \to 0$, which implies from Lemma 2.6 that $x_{n+1} - y_n \to 0$ as $n \to \infty$. Thus $y_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $w_n = \Pi_{C_{n+1}}(x_n) \in C_{n+1} \subset Q_n$, we have $\varphi(w_n, z_{n-1}) \leq \beta_{n-1}\varphi(w_n, x_{n-1}) + (1-\beta_{n-1}) \times \varphi(w_n, w_{n-1} + \varepsilon_{n-1}) \to 0$, as $n \to \infty$.

Thus $w_n - z_{n-1} \to 0$ which implies that $z_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

This completes the proof. □

Corollary 3.6 *If we choose* $x_{n+1} = \Pi_{Q_{n+1}}(w_n)$, *then* (3.1) *reduces to the following one*:

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \textit{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E = X_1, \qquad Q_1 = E, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \leq \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ X_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \delta_n\}, \\ w_n \in X_{n+1}, \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \leq \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ x_{n+1} = \Pi_{Q_{n+1}}(w_n), \quad n \in N. \end{cases} \tag{3.7}$$

Under the assumptions that (*i*), (*ii*), (*iii*) *and* (*v*) *in Theorem* 3.1 *and* (*iv*)″ *in Corollary* 3.3, *one has*

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0), \quad \textit{as } n \to \infty. \end{cases}$$

Proof Copy Steps 1–5 and 9 and 10 in Theorem 3.1, we are left to show the results of Steps 6, 7 and 8 are still true.

Step 6. Both $\{w_n\}$ and $\{x_n\}$ are bounded.

Copy Theorem 3.1, $\{w_n\}$ is bounded. Since $x_{n+1} = \Pi_{Q_{n+1}}(w_n)$, $\forall q \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) \subset Q_{n+1}$, Lemma 2.8 implies that $\varphi(q, x_{n+1}) + \varphi(x_{n+1}, w_n) \leq \varphi(q, w_n)$. Thus $\{x_n\}$ is bounded.

Step 7. $w_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ and $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Copy Theorem 3.1, $w_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $w_{n+1} \in X_{n+2} \subset C_{n+2} \subset Q_{n+1}$, using Lemma 2.8, we have $\varphi(w_{n+1}, x_{n+1}) + \varphi(x_{n+1}, w_n) \leq \varphi(w_{n+1}, w_n) \to 0$, as $n \to \infty$. Thus $w_{n+1} - x_{n+1} \to 0$ and thus $x_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ as $n \to \infty$.

Step 8. $y_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$ and $z_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in Q_{n+1} \subset C_{n+1}$, $\varphi(x_{n+1}, y_n) \leq \alpha_n \varphi(x_{n+1}, x_n) + (1-\alpha_n)\varphi(x_{n+1}, x_n + e_n) \to 0$, which implies from Lemma 2.6 that $x_{n+1} - y_n \to 0$ as $n \to \infty$. Thus $y_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

Since $x_{n+1} \in Q_{n+1}$, we have $\varphi(x_{n+1}, z_n) \leq \beta_n \varphi(x_{n+1}, x_n) + (1-\beta_n)\varphi(x_{n+1}, w_n + \varepsilon_n) \to 0$, as $n \to \infty$. Thus $z_n \to P_{\bigcap_{n=1}^{\infty} C_n}(x_1)$, as $n \to \infty$.

This completes the proof. □

Corollary 3.7 *If we choose $w_n = \Pi_{C_{n+1}}(x_n)$ and $x_{n+1} = P_{Q_{n+1}}(x_1)$, then (3.1) becomes to (3.8). If we choose $w_n = P_{C_{n+1}}(x_1)$ and $x_{n+1} = \Pi_{Q_{n+1}}(w_n)$, then (3.1) becomes to (3.9).*

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \text{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E = Q_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ w_n = \Pi_{C_{n+1}}(x_n), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ x_{n+1} = P_{Q_{n+1}}(x_1), \quad n \in N, \end{cases} \tag{3.8}$$

and

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E \quad \text{chosen arbitrarily}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) J_E \overline{U_n}(x_n + e_n)], \\ C_1 = E = Q_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ w_n = P_{C_{n+1}}(x_1), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) J_E \overline{W_n}(w_n + \varepsilon_n)], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ x_{n+1} = \Pi_{Q_{n+1}}(w_n), \quad n \in N. \end{cases} \tag{3.9}$$

Under the assumptions that (i), (ii), (iii) and (v) in Theorem 3.1, one has

$$\begin{cases} x_n \to P_{(\bigcap_{i=1}^\infty A_i^{-1}0)\cap(\bigcap_{i=1}^\infty B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty B_i^{-1}0), \quad \text{as } n \to \infty, \\ w_n \to P_{(\bigcap_{i=1}^\infty A_i^{-1}0)\cap(\bigcap_{i=1}^\infty B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty B_i^{-1}0), \quad \text{as } n \to \infty, \\ y_n \to P_{(\bigcap_{i=1}^\infty A_i^{-1}0)\cap(\bigcap_{i=1}^\infty B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty B_i^{-1}0), \quad \text{as } n \to \infty, \\ z_n \to P_{(\bigcap_{i=1}^\infty A_i^{-1}0)\cap(\bigcap_{i=1}^\infty B_i^{-1}0)}(x_1) \in (\bigcap_{i=1}^\infty A_i^{-1}0) \cap (\bigcap_{i=1}^\infty B_i^{-1}0), \quad \text{as } n \to \infty. \end{cases}$$

Remark 3.8 Compared to [15–17], we have the following differences: (1) infinite maximal monotone operators are studied instead of finite cases; (2) the limit of the iterative sequences, $P_{(\bigcap_{i=1}^\infty A_i^{-1}0)\cap(\bigcap_{i=1}^\infty B_i^{-1}0)}(x_1)$, is easier for computation theoretically since metric projection only involves $\|\cdot\|$ while generalized projection involves Lyapunov functional φ; (3) computational errors are considered in each step; (4) for each given iterative step n, multi-choice can be made on both $\{w_n\}$ and $\{x_n\}$ in (3.1); (5) the normalized duality mappings J_E or J_E^{-1} are no longer needed to be weakly sequentially continuous.

Remark 3.9 Compared to [14] and [18], we have the following differences: (1) for each iterative step n, multi-choice can be made on both $\{w_n\}$ and $\{x_n\}$ in (3.1); (2) four key sets $\{C_n\}$, $\{Q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are defined which permits more choices for the iterative sequences; (3) both $\{C_n\}$ and $\{Q_n\}$ are decreasing sets in (3.1) and satisfy the following inter-relationship: $C_{n+1} \subset Q_n \subset C_n \subset Q_{n-1}$ for $n \ge 2$; (4) $\bigcap_{n=1}^\infty C_n = \bigcap_{n=1}^\infty Q_n$ can be proved which guarantees the limit of the iterative sequences is unique.

Remark 3.10 Corollaries 3.2–3.4 can be seen as a group of results and Corollaries 3.5 and 3.6 can be seen as another. In Corollaries 3.2–3.4, we want to say that if we take w_n or x_n or both as the value of metric projections, the results are still true. In Corollaries 3.5 and 3.6, we want to say that if we take w_n or x_n as the value of generalized projections, the results are still true. In this sense, Theorem 3.1 is a new and general result.

3.2 Computational experiments

Remark 3.11 If E reduces to a Hilbert space H, then the Lyapunov functional is reduced to

$$\varphi(x,y) = \|x-y\|^2, \quad \forall x,y \in H.$$

Remark 3.12 Take $E = (-\infty,+\infty)$. Suppose $A_i, B_i : (-\infty,+\infty) \to (-\infty,+\infty)$ are defined as follows: $A_i x = \frac{x}{2^i}$ and $B_i x = 2^i x$ for $x \in (-\infty,+\infty)$ and $i \in N$. Then A_i and B_i are maximal monotone for $i \in N$ and $(\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0) = \{0\}$. Let $a_i = \frac{1}{2^{i+1}} = b_i$ for $i \in \{0\} \cup N$, $\beta_n = \delta_n = \vartheta_n = e_n = \varepsilon_n = \frac{1}{n}$ and $\alpha_n = \frac{1}{2^n}$ for $n \in N$. Let $r_{n,i} = (2^{n+i-1}-1)2^i$ and $s_{n,i} = \frac{2^n-1}{2^i}$ for $i,n \in N$. It is easy to check that all of the assumptions of Theorem 3.1 are satisfied for this special case.

Corollary 3.13 *Taking the example in Remark 3.12, we can choose the following iterative sequences among infinite choices generated by iterative algorithm (3.1) in Theorem 3.1:*

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1-\frac{1}{2^n})(\frac{1}{2}+\frac{1}{3\times 2^n})(x_n+\frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n}+(1-\frac{1}{2^n})(x_n+\frac{1}{n})^2-y_n^2}{2[\frac{x_n}{2^n}+(1-\frac{1}{2^n})(x_n+\frac{1}{n})-y_n]}, \quad n \in N, \\ \overline{a_n} = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ w_n = x_1 - \sqrt{(x_1-\overline{a_n})^2+\frac{1}{n}}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n+\frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n}+(1-\frac{1}{n})(w_n+\frac{1}{n})^2-z_n^2}{2[\frac{x_n}{n}+(1-\frac{1}{n})(w_n+\frac{1}{n})-z_n]}, \quad n \in N\setminus\{1\}, \\ \overline{b_n} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \\ x_{n+1} = x_1 - \sqrt{(x_1-\overline{b_n})^2+\frac{1}{n}}, \quad n \in N, \end{cases} \tag{3.10}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1-\frac{1}{2^n})(\frac{1}{2}+\frac{1}{3\times 2^n})(x_n+\frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n}+(1-\frac{1}{2^n})(x_n+\frac{1}{n})^2-y_n^2}{2[\frac{x_n}{2^n}+(1-\frac{1}{2^n})(x_n+\frac{1}{n})-y_n]}, \quad n \in N, \\ w_n = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n+\frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n}+(1-\frac{1}{n})(w_n+\frac{1}{n})^2-z_n^2}{2[\frac{x_n}{n}+(1-\frac{1}{n})(w_n+\frac{1}{n})-z_n]}, \quad n \in N\setminus\{1\}, \\ \overline{b_n} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \\ x_{n+1} = x_1 - \sqrt{(x_1-\overline{b_n})^2+\frac{1}{n}}, \quad n \in N, \end{cases} \tag{3.11}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ w_n = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ x_{n+1} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \end{cases} \tag{3.12}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ \overline{a_n} = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ w_n = x_1 - \sqrt{(x_1 - \overline{a_n})^2 + \frac{1}{n}}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ x_{n+1} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \end{cases} \tag{3.13}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ \overline{a_n} = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ w_n = \frac{x_1 - \sqrt{(x_1-\overline{a_n})^2 + \frac{1}{n}} + \overline{a_n}}{2}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ \overline{b_n} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \\ x_{n+1} = x_1 - \sqrt{(x_1 - \overline{b_n})^2 + \frac{1}{n}}, \quad n \in N, \end{cases} \tag{3.14}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ \overline{a_n} = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ w_n = \frac{x_1 - \sqrt{(x_1-\overline{a_n})^2 + \frac{1}{n}} + \overline{a_n}}{2}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ x_{n+1} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \end{cases} \tag{3.15}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ \overline{a_n} = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ w_n = \frac{x_1 - \sqrt{(x_1-\overline{a_n})^2 + \frac{1}{n}} + \overline{a_n}}{2}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ \overline{b_n} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \\ x_{n+1} = \frac{x_1 - \sqrt{(x_1-\overline{b_n})^2 + \frac{1}{n}} + \overline{b_n}}{2}, \quad n \in N, \end{cases} \tag{3.16}$$

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ \overline{a_n} = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ w_n = x_1 - \sqrt{(x_1 - \overline{a_n})^2 + \frac{1}{n}}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ \overline{b_n} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \\ x_{n+1} = \frac{x_1 - \sqrt{(x_1-\overline{b_n})^2 + \frac{1}{n}} + \overline{b_n}}{2}, \quad n \in N, \end{cases} \tag{3.17}$$

and

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ y_n = \frac{x_n}{2^n} + (1 - \frac{1}{2^n})(\frac{1}{2} + \frac{1}{3\times 2^n})(x_n + \frac{1}{n}), \quad n \in N, \\ v_n = \frac{\frac{x_n^2}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n})^2 - y_n^2}{2[\frac{x_n}{2^n} + (1-\frac{1}{2^n})(x_n+\frac{1}{n}) - y_n]}, \quad n \in N, \\ w_n = \min_{m\le n, m\in N}\{v_m, t_{m-1}\}, \quad n \in N, \\ z_n = \frac{1}{n}x_n + (1-\frac{1}{n})\frac{2^n}{2^{n+1}-1}(w_n + \frac{1}{n}), \quad n \in N, \\ t_n = \frac{\frac{x_n^2}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n})^2 - z_n^2}{2[\frac{x_n}{n} + (1-\frac{1}{n})(w_n+\frac{1}{n}) - z_n]}, \quad n \in N \setminus \{1\}, \\ \overline{b_n} = \min_{m\le n, m\in N}\{v_m, t_m\}, \quad n \in N, \\ x_{n+1} = \frac{x_1 - \sqrt{(x_1-\overline{b_n})^2 + \frac{1}{n}} + \overline{b_n}}{2}, \quad n \in N. \end{cases} \tag{3.18}$$

Then $\{x_n\}$ *generated by* (3.10)–(3.18) *converges strongly to* $0 \in (\bigcap_{i=1}^{\infty} A_i^{-1}0) \cap (\bigcap_{i=1}^{\infty} B_i^{-1}0)$, *as* $n \to \infty$.

Proof We shall only show that $\{x_n\}$ in (3.10) can be obtained by iterative algorithm (3.1) and the result of strong convergence is true. Similarly, (3.11)–(3.18) are available.

Compute y_n and z_n in (3.1) for the example, where $n \in N$.

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)\overline{U_n}(x_n + e_n) \\ &= \alpha_n x_n + (1-\alpha_n)a_0(x_n+e_n) + (1-\alpha_n)\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(x_n+e_n) \\ &= \alpha_n x_n + (1-\alpha_n)\frac{x_n+e_n}{2} + (1-\alpha_n)\sum_{i=1}^{\infty}\frac{1}{4^i}\frac{x_n+e_n}{2^n} \\ &= \frac{x_n}{2^n} + \left(1-\frac{1}{2^n}\right)\left(\frac{1}{2}+\frac{1}{3\times 2^n}\right)\left(x_n+\frac{1}{n}\right), \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} z_n &= \beta_n x_n + (1-\beta_n)\overline{W_n}(w_n+\varepsilon_n) \\ &= \beta_n x_n + (1-\beta_n)b_0(w_n+\varepsilon_n) \\ &\quad + (1-\beta_n)\sum_{j=1}^{\infty} b_j(I+s_{n,j}B_j)^{-1}(I+s_{n,j-1}B_{j-1})^{-1}\cdots(I+s_{n,1}B_1)^{-1}(w_n+\varepsilon_n) \\ &= \beta_n x_n + (1-\beta_n)\frac{w_n+\varepsilon_n}{2} + (1-\beta_n)\sum_{j=1}^{\infty}\frac{w_n+\varepsilon_n}{2^{(n+1)j+1}} \\ &= \frac{x_n}{n} + \left(1-\frac{1}{n}\right)\frac{2^n}{2^{n+1}-1}\left(w_n+\frac{1}{n}\right). \end{aligned} \tag{3.20}$$

Compute C_{n+1} and Q_{n+1} in (3.1) for the example, where $n \in N$:

$$\begin{aligned} C_{n+1} &= Q_n \cap \bigl\{v \in R : 2\bigl[\alpha_n x_n + (1-\alpha_n)(x_n+e_n) - y_n\bigr]v \\ &\quad \le \alpha_n x_n^2 + (1-\alpha_n)(x_n+e_n)^2 - y_n^2\bigr\}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} Q_{n+1} &= C_{n+1} \cap \bigl\{v \in R : 2\bigl[\beta_n x_n + (1-\beta_n)(w_n+\varepsilon_n) - z_n\bigr]v \\ &\quad \le \beta_n x_n^2 + (1-\beta_n)(w_n+\varepsilon_n)^2 - z_n^2\bigr\}. \end{aligned} \tag{3.22}$$

Next, we shall use inductive method to show that the following is true:

$$\begin{cases} x_1 = 1, \qquad t_0 = t_1 = 1, \\ C_1 = Q_1 = X_1 = Y_1 = (-\infty, +\infty), \\ C_2 = (-\infty, \frac{41}{24}] = Q_2, \qquad X_2 = [0, \frac{41}{24}] = Y_2, \\ C_{n+1} = (-\infty, \overline{a_n}], \quad n \in N\setminus\{1\}, \\ X_{n+1} = [x_1 - \sqrt{(x_1-\overline{a_n})^2+\frac{1}{n}}, \overline{a_n}], \quad n \in N\setminus\{1\}, \\ \textit{we may choose} \quad w_n = x_1 - \sqrt{(x_1-\overline{a_n})^2+\frac{1}{n}}, \quad n \in N, \\ Q_{n+1} = (-\infty, \overline{b_n}], \quad n \in N\setminus\{1\}, \\ Y_{n+1} = [x_1 - \sqrt{(x_1-\overline{b_n})^2+\frac{1}{n}}, \overline{b_n}], \quad n \in N\setminus\{1\}, \\ \textit{we may choose} \quad x_{n+1} = x_1 - \sqrt{(x_1-\overline{b_n})^2+\frac{1}{n}}, \quad n \in N, \\ 0 < \overline{b_n} \le \overline{a_n} \le 1, \quad n \in N. \end{cases} \tag{3.23}$$

In fact, if $n = 1$, using (3.19) and (3.10), $y_1 = \frac{7}{6}$, $v_1 = \frac{41}{24}$ and $\overline{a_1} = \min\{v_1, t_0\} = 1$. Then from (3.1), $C_2 = (-\infty, \frac{41}{24}]$, $P_{C_2}(x_1) = x_1$, and then $X_2 = C_2 \cap [0,2] = [0, \frac{41}{24}]$. Thus we may choose $w_1 = x_1 - \sqrt{(x_1 - \overline{a_1})^2 + \frac{1}{1}} = 0$. Then using (3.20), $z_1 = 1$. Since $\beta_1 x_1 + (1-\beta_1)(w_1 + \varepsilon_1) - z_1 = \beta_1 x_1^2 + (1-\beta_1)(w_1+\varepsilon_1)^2 - z_1^2 = 0$, we have $Q_2 = C_2 \cap (-\infty, +\infty) = C_2$ and then $Y_2 = X_2$. And $\overline{b_1} = \min\{v_1, t_1\} = 1$, thus we may choose $x_2 = 1 - \sqrt{(1-1)^2 + 1} = 0$. Therefore, (3.23) is true for $n = 1$.

If $n = 2$, it is easy to calculate that $y_2 = \frac{7}{32}$, $v_2 = \frac{143}{320}$ and $0 < \overline{a_2} = \min\{v_1, t_0, v_2, t_1\} = v_2 = \frac{143}{320} < 1$. Then from (3.21), $C_3 = Q_2 \cap (-\infty, v_2] = (-\infty, v_1] \cap (-\infty, v_2] = (-\infty, v_2] = (-\infty, \overline{a_2}]$, $P_{C_3}(x_1) = \overline{a_2}$, and then $X_3 = [x_1 - \sqrt{(x_1 - \overline{a_2})^2 + \frac{1}{2}}, \overline{a_2}]$. Thus we may choose $w_2 = x_1 - \sqrt{(x_1 - \overline{a_2})^2 + \frac{1}{2}} = 0.1022543$. Thus $z_2 = 0.1720727$ and $t_2 = 0.587915$. And then from (3.22), $Q_3 = C_3 \cap (-\infty, t_2] = (-\infty, \overline{a_2}] \cap (-\infty, t_2] = (-\infty, \overline{b_2}]$, $Y_3 = Q_3 \cap [x_1 - \sqrt{(x_1 - \overline{b_2})^2 + \frac{1}{2}}, x_1 + \sqrt{(x_1 - \overline{b_2})^2 + \frac{1}{2}}] = [x_1 - \sqrt{(x_1 - \overline{b_2})^2 + \frac{1}{2}}, \overline{b_2}]$. Thus we may choose $x_3 = x_1 - \sqrt{(x_1 - \overline{b_2})^2 + \frac{1}{2}}$. It is easy to check that $0 < \overline{b_2} \le \overline{a_2} \le 1$. Therefore, (3.23) is true for $n = 2$.

Suppose (3.23) is true for $n = k$ ($k \ge 2$), that is,

$$\begin{cases} C_{k+1} = (-\infty, \overline{a_k}], \\ X_{k+1} = [x_1 - \sqrt{(x_1 - \overline{a_k})^2 + \frac{1}{k}}, \overline{a_k}], \\ \textit{we may choose} \quad w_k = x_1 - \sqrt{(x_1 - \overline{a_k})^2 + \frac{1}{k}}, \\ Q_{k+1} = (-\infty, \overline{b_k}], \\ Y_{k+1} = [x_1 - \sqrt{(x_1 - \overline{b_k})^2 + \frac{1}{k}}, \overline{b_k}], \\ \textit{we may choose} \quad x_{k+1} = x_1 - \sqrt{(x_1 - \overline{b_k})^2 + \frac{1}{k}}, \\ 0 < \overline{b_k} \le \overline{a_k} \le 1. \end{cases}$$

Then, if $n = k+1$, we can easily see from definitions of $\overline{a_n}$ and $\overline{b_n}$ that $\overline{b_{k+1}} \le \overline{a_{k+1}} \le t_0 = 1$. Since $0 < \overline{b_k} \le 1$, we have $1 + \frac{1}{k+1} > \sqrt{(x_1 - \overline{b_k})^2 + \frac{1}{k}}$, which implies that $x_{k+1} + e_{k+1} = x_{k+1} + \frac{1}{k+1} > 0$. Therefore,

$$\begin{aligned} &\alpha_{k+1} x_{k+1} + (1-\alpha_{k+1})(x_{k+1} + e_{k+1}) - y_{k+1} \\ &\quad = \left(1 - \frac{1}{2^{k+1}}\right)\left(\frac{1}{2} - \frac{1}{3 \times 2^{k+1}}\right)(x_{k+1} + e_{k+1}) > 0. \end{aligned} \tag{3.24}$$

Note that

$$\begin{aligned} 2(1-\alpha_{k+1})\left(\frac{1}{2} + \frac{1}{3 \times 2^{k+1}}\right)^2 &= \left(1 - \frac{1}{2^{k+1}}\right)\left(1 + \frac{1}{3 \times 2^k}\right)\left(\frac{1}{2} + \frac{1}{3 \times 2^{k+1}}\right) \\ &= \left(1 + \frac{1}{3 \times 2^k} - \frac{1}{2^{k+1}} - \frac{1}{6 \times 4^k}\right)\left(\frac{1}{2} + \frac{1}{3 \times 2^{k+1}}\right) \\ &= \left(1 - \frac{1}{6 \times 2^k} - \frac{1}{6 \times 4^k}\right)\left(\frac{1}{2} + \frac{1}{3 \times 2^{k+1}}\right) < 1, \end{aligned}$$

then

$$
\begin{aligned}
y_{k+1}^2 &= \alpha_{k+1}^2 x_{k+1}^2 + 2\alpha_{k+1}(1-\alpha_{k+1})\left(\frac{1}{2}+\frac{1}{3\times 2^{k+1}}\right)x_{k+1}(x_{k+1}+e_{k+1}) \\
&\quad + (1-\alpha_{k+1})^2\left(\frac{1}{2}+\frac{1}{3\times 2^{k+1}}\right)^2(x_{k+1}+e_{k+1})^2 \\
&\le 2\alpha_{k+1}^2 x_{k+1}^2 + 2(1-\alpha_{k+1})^2\left(\frac{1}{2}+\frac{1}{3\times 2^{k+1}}\right)^2(x_{k+1}+e_{k+1})^2 \\
&\le \alpha_{k+1}x_{k+1}^2 + (1-\alpha_{k+1})(x_{k+1}+e_{k+1})^2.
\end{aligned}
\tag{3.25}
$$

Therefore, (3.24) and (3.25) imply that $v_{k+1} > 0$. Since $\overline{b_k} > 0$ and $v_{k+1} > 0$, we have $\overline{a_{k+1}} > 0$. That is, $0 < \overline{a_{k+1}} \le 1$.

Using (3.21), $C_{k+2} = Q_{k+1} \cap (-\infty, v_{k+1}] = (-\infty, \overline{b_k}] \cap (-\infty, v_{k+1}] = (-\infty, \overline{a_{k+1}}]$. Then $X_{k+2} = C_{k+2} \cap [x_1 - \sqrt{(x_1-\overline{a_{k+1}})^2 + \frac{1}{k+1}}, x_1 + \sqrt{(x_1-\overline{a_{k+1}})^2 + \frac{1}{k+1}}] = [x_1 - \sqrt{(x_1-\overline{a_{k+1}})^2 + \frac{1}{k+1}}, \overline{a_{k+1}}]$. Thus we may choose $w_{k+1} = x_1 - \sqrt{(x_1-\overline{a_{k+1}})^2 + \frac{1}{k+1}}$.

Since $(1+\frac{1}{k+1})^2 > (1-\overline{a_{k+1}})^2 + \frac{1}{k+1}$, we have $w_{k+1} + \varepsilon_{k+1} = 1 - \sqrt{(1-\overline{a_{k+1}})^2 + \frac{1}{k+1}} + \frac{1}{k+1} > 0$, which ensures that

$$
\begin{aligned}
&\beta_{k+1}x_{k+1} + (1-\beta_{k+1})(w_{k+1}+\varepsilon_{k+1}) - z_{k+1} \\
&\quad = \left(1-\frac{1}{k+1}\right)\frac{2^{k+2}-2^{k+1}-1}{2^{k+2}-1}(w_{k+1}+\varepsilon_{k+1}) > 0.
\end{aligned}
\tag{3.26}
$$

Note that

$$
\begin{aligned}
&2(1-\beta_{k+1})^2\left(\frac{2^{k+1}}{2^{k+2}-1}\right)^2 \le 1-\beta_{k+1} \\
&\quad \Longleftrightarrow \quad \left(1-\frac{1}{k+1}\right)\frac{2^{k+2}}{2^{k+2}-1}\frac{2^{k+1}}{2^{k+2}-1} \le 1 \\
&\quad \Longleftrightarrow \quad (k+1)\times 8\times 2^k \le (k+1) + 8(k+1)\times 4^k + 8\times 4^k.
\end{aligned}
$$

This last inequality above is obviously true for $k \in N$. Thus

$$
\begin{aligned}
z_{k+1}^2 &= \beta_{k+1}^2 x_{k+1}^2 + 2\beta_{k+1}(1-\beta_{k+1})\frac{2^{k+1}}{2^{k+2}-1}x_{k+1}(w_{k+1}+\varepsilon_{k+1}) \\
&\quad + (1-\beta_{k+1})^2\left(\frac{2^{k+1}}{2^{k+2}-1}\right)^2(w_{k+1}+\varepsilon_{k+1})^2 \\
&\le 2\beta_{k+1}^2 x_{k+1}^2 + 2(1-\beta_{k+1})^2\left(\frac{2^{k+1}}{2^{k+2}-1}\right)^2(w_{k+1}+\varepsilon_{k+1})^2 \\
&\le \beta_{k+1}x_{k+1}^2 + (1-\beta_{k+1})(w_{k+1}+\varepsilon_{k+1})^2.
\end{aligned}
\tag{3.27}
$$

Equation (3.26) and (3.27) imply that $t_{k+1} > 0$, which ensures that $\overline{b_{k+1}} > 0$ since $\overline{a_{k+1}} > 0$. Using (3.22), $Q_{k+2} = C_{k+2} \cap (-\infty, t_{k+1}] = (-\infty, \overline{a_{k+1}}] \cap (-\infty, t_{k+1}] = (-\infty, \overline{b_{k+1}}]$, and $Y_{k+2} = Q_{k+2} \cap [x_1 - \sqrt{(x_1-\overline{b_{k+1}})^2 + \frac{1}{k+1}}, x_1 + \sqrt{(x_1-\overline{b_{k+1}})^2 + \frac{1}{k+1}}] = [x_1 - \sqrt{(x_1-\overline{b_{k+1}})^2 + \frac{1}{k+1}}, \overline{b_{k+1}}]$.

Thus we may choose $x_{k+2} = x_1 - \sqrt{(x_1 - \overline{b_{k+1}})^2 + \frac{1}{k+1}}$. By now, we have proved that (3.23) is true for $n \in N$.

Therefore, $\{x_n\}$ defined in (3.10) is valid.

Finally, we shall show that $x_n \to 0$, as $n \to \infty$.

From (3.10) or (3.23), we can easily see that $\{x_n\}$ is bounded. Let $\{x_{n_j}\}$ be any subsequence of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = \xi$. Then using (3.10), we may see that $\lim_{j\to\infty} y_{n_j} = \frac{\xi}{2}$ and $\lim_{j\to\infty} v_{n_j} = \frac{3\xi}{4}$. Since $x_{n_j+1} = x_1 - \sqrt{(x_1 - \overline{b_{n_j}})^2 + \frac{1}{n_j}}$, we have $\lim_{j\to\infty} \overline{b_{n_j}} = \xi$. Note that $0 < \overline{b_{n_j}} \le v_{n_j}$, then $0 \le \xi \le \frac{3}{4}\xi$, which implies that $\xi = 0$. This means that each strongly convergent subsequence of $\{x_n\}$ converges strongly to 0. Therefore, $x_n \to 0$, as $n \to \infty$. And, it is not difficult to see that $y_n \to 0$, $v_n \to 0$, $w_n \to 0$, and $z_n \to 0$, as $n \to \infty$.

This completes the proof. □

Remark 3.14 Do computational experiments on (3.11) in Corollary 3.13. By using codes of Visual Basic Six, we get Table 1 and Fig. 1.

Table 1 Numerical Results of $\{x_n\}$ and $\{w_n\}$ with initial $x_1 = 1.0$ based on (3.11)

n	x_n	w_n
1	1.000000000000000	1.000000000000000
2	0.000000000000000	0.446875
3	0.102254342463858	0.341360682151952
4	0.124135236267222	0.285370643452427
5	0.127821625331379	0.247725551310041
6	0.12483324665994	0.219411312688766
7	0.119099684740128	0.196813440416536
8	0.112325627072255	0.178149458121737
9	0.105327818032999	0.162399735334363
10	0.0985093930167346	0.148914357463224
11	0.0920645557462596	0.137245126226504
12	0.0860771021967925	0.127064722634339
13	0.080571192640345	0.118123908962632
14	0.0755387964514549	0.110227022688371
15	0.0709550481367587	0.103216987101004
16	0.0667870345463115	0.0969656052456123
17	0.0629988697395166	0.0913669545415246
18	0.0595546051413147	0.086332693840198
19	0.0564198487140243	0.0817886054753062
20	0.05356260429844	0.077671969713881

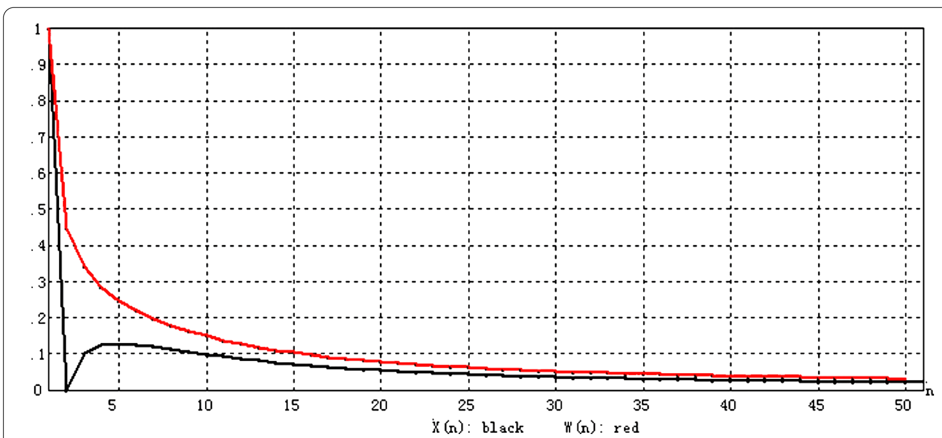

Figure 1 Convergence of $\{x_n\}$ and $\{w_n\}$ corresponding to Table 1

4 Applications

4.1 Application to convex minimization problems

Suppose $f : E \to (-\infty, +\infty]$ is a proper convex and lower-semicontinuous function. Then the subdifferential of f, ∂f, is defined as follows: $\forall x \in E$,

$$\partial f(x) = \{y \in E^* : f(x) + \langle z - x, y\rangle \le f(y), \forall z \in E\}.$$

Theorem 4.1 *Let E, α_n, β_n, e_n, ε_n, δ_n and ϑ_n be the same as those in Theorem 3.1. Let $f, g : E \to (-\infty, +\infty]$ be two proper convex and lower-semicontinuous functions. Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E, \\ \overline{u_n} = \operatorname{argmin}_{z\in E}\{f(z) + \frac{\|z\|^2}{2r_n} - \frac{1}{r_n}\langle z, J_E(x_n + e_n)\rangle\}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n)a_0 J_E(x_n + e_n) + (1-\alpha_n)(1-a_0)J_E\overline{u_n}], \\ C_1 = E = X_1, \qquad Q_1 = E = Y_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n\varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ X_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \le \|P_{C_{n+1}}(x_1) - x_1\|^2 + \delta_n\}, \\ w_n \in X_{n+1}, \\ \overline{\overline{u_n}} = \operatorname{argmin}_{z\in E}\{g(z) + \frac{\|z\|^2}{2s_n} - \frac{1}{s_n}\langle z, J_E(w_n + \varepsilon_n)\rangle\}, \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n)b_0 J_E(w_n + \varepsilon_n) + (1-\beta_n)(1-b_0)J_E\overline{\overline{u_n}}], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n\varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ Y_{n+1} = \{v \in Q_{n+1} : \|x_1 - v\|^2 \le \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \vartheta_n\}, \\ x_{n+1} \in Y_{n+1}, \quad n \in N. \end{cases} \tag{4.1}$$

Under the assumptions that $(\partial f)^{-1}0 \cap (\partial g)^{-1}0 \ne \emptyset$, $\inf_n r_n > 0$ and $\inf_n s_n > 0$, we have $x_n \to P_{(\partial f)^{-1}0\cap(\partial g)^{-1}0}(x_1)$, as $n \to \infty$.

Proof Similar to [25], $\overline{u_n} = \operatorname{argmin}_{z\in E}\{f(z) + \frac{\|z\|^2}{2r_n} - \frac{1}{r_n}\langle z, J_E(x_n + e_n)\rangle\}$ is equivalent to $0 \in \partial f(\overline{u_n}) + \frac{1}{r_n}J_E\overline{u_n} - \frac{1}{r_n}J_E(x_n + e_n)$. Then $\overline{u_n} = (J_E + r_n\partial f)^{-1}J_E(x_n + e_n)$. And, $\overline{\overline{u_n}} = \operatorname{argmin}_{z\in E}\{g(z) + \frac{\|z\|^2}{2s_n} - \frac{1}{s_n}\langle z, J_E(w_n + \varepsilon_n)\rangle\}$ is equivalent to $0 \in \partial g(\overline{\overline{u_n}}) + \frac{1}{s_n}J_E\overline{\overline{u_n}} - \frac{1}{s_n}J_E(w_n + \varepsilon_n)$. Then $\overline{\overline{u_n}} = (J_E + s_n\partial g)^{-1}J_E(w_n + \varepsilon_n)$. Using Theorem 3.1, the result is available.

This completes the proof. □

Remark 4.2 Similarly, we can modify (4.1) and get the corresponding convergence theorems with respect to Corollaries 3.2–3.7.

4.2 Application to variational inequalities

Let C be the non-empty closed and convex subset of E. Let $T : C \to E^*$ be a single-valued, monotone and hemi-continuous mapping. The variational inequality problem is to find $u \in C$ such that

$$\langle y - u, Tu\rangle \ge 0, \quad \forall y \in C. \tag{4.2}$$

The symbol $\mathrm{VI}(C, T)$ denotes the set of solution of the variational inequality problem (4.2).

It follows from [26] that $A : E \to 2^{E^*}$ defined by

$$Ax = \begin{cases} Tx + N_C x, & x \in C, \\ \emptyset, & x \overline{\in} C, \end{cases}$$

is maximal monotone and $A^{-1}0 = \mathrm{VI}(C,T)$, where $N_C(x) = \{z \in E^* : \langle y - x, z\rangle \le 0, \forall y \in C\}$.

Theorem 4.3 *Let E, α_n, β_n, e_n, ε_n, δ_n and ϑ_n be the same as those in Theorem 3.1. Let C be the non-empty closed and convex subset of E. Let $T_1, T_2 : C \to E^*$ be two single-valued, monotone and hemi-continuous mappings. Let $A, B : E \to 2^{E^*}$ be defined as follows:*

$$Ax = \begin{cases} T_1 x + N_C x, & x \in C, \\ \emptyset, & x \overline{\in} C, \end{cases}$$

and

$$Bx = \begin{cases} T_2 x + N_C x, & x \in C, \\ \emptyset, & x \overline{\in} C. \end{cases}$$

Let $\{x_n\}$ be generated by the following iterative algorithm:

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E, \\ \overline{u_n} = \mathrm{VI}(C, T_1 + \frac{1}{r_n} J_E - \frac{1}{r_n} J_E(x_n + e_n)), \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n) a_0 J_E(x_n + e_n) + (1-\alpha_n)(1-a_0) J_E \overline{u_n}], \\ C_1 = E = X_1, \qquad Q_1 = E = Y_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ X_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \le \|P_{C_{n+1}}(x_1) - x_1\|^2 + \delta_n\}, \\ w_n \in X_{n+1}, \\ \overline{\overline{u_n}} = \mathrm{VI}(C, T_2 + \frac{1}{s_n} J_E - \frac{1}{s_n} J_E(w_n + \varepsilon_n)), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n) b_0 J_E(w_n + \varepsilon_n) + (1-\beta_n)(1-b_0) J_E \overline{\overline{u_n}}], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ Y_{n+1} = \{v \in Q_{n+1} : \|x_1 - v\|^2 \le \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \vartheta_n\}, \\ x_{n+1} \in Y_{n+1}, \quad n \in N. \end{cases} \tag{4.3}$$

Under the assumptions that $\mathrm{VI}(C,T_1) \cap \mathrm{VI}(C,T_2) \neq \emptyset$, $\inf_n r_n > 0$ *and* $\inf_n s_n > 0$, *we have* $x_n \to P_{\mathrm{VI}(C,T_1)\cap \mathrm{VI}(C,T_2)}(x_1)$, *as* $n \to \infty$.

Proof

$$\begin{aligned} \overline{u_n} &= \mathrm{VI}\left(C, T_1 + \frac{1}{r_n} J_E - \frac{1}{r_n} J_E(x_n + e_n)\right) \\ &\Leftrightarrow \quad \left\langle y - \overline{u_n}, T_1 \overline{u_n} + \frac{1}{r_n} J_E \overline{u_n} - \frac{1}{r_n} J_E(x_n + e_n)\right\rangle \ge 0, \quad \forall y \in C \\ &\Leftrightarrow \quad J_E(x_n + e_n) \in r_n A \overline{u_n} + J_E \overline{u_n} \Longleftrightarrow \overline{u_n} = (J_E + r_n A)^{-1} J_E(x_n + e_n). \end{aligned}$$

Similarly, we have $\overline{\overline{u_n}} = (J_E + s_n B)^{-1} J_E(w_n + \varepsilon_n)$. Using Theorem 3.1, the result is available.

This completes the proof. □

Remark 4.4 Similarly, we can modify (4.3) and get the corresponding convergence theorems with respect to Corollaries 3.2–3.7.

4.3 Approximating to common solution of both minimization problems and variational inequalities

Theorem 4.5 *Let E, α_n, β_n, e_n, ε_n, δ_n, ϑ_n and f be the same as those in Theorem 4.1. Let C be the non-empty closed and convex subset of E. Suppose $T : C \to E^*$ is a single-valued, monotone and hemi-continuous mapping and $A : E \to 2^{E^*}$ is defined by*

$$Ax = \begin{cases} Tx + N_C x, & x \in C, \\ \emptyset, & x \overline{\in} C. \end{cases}$$

Let $\{x_n\}$ be generated by the following iterative algorithm:

$$\begin{cases} x_1, e_1, \varepsilon_1 \in E, \\ \overline{u_n} = \operatorname{argmin}_{z\in E}\{f(z) + \frac{\|z\|^2}{2r_n} - \frac{1}{r_n}\langle z, J_E(x_n + e_n)\rangle\}, \\ y_n = J_E^{-1}[\alpha_n J_E x_n + (1-\alpha_n)a_0 J_E(x_n + e_n) + (1-\alpha_n)(1-a_0)J_E\overline{u_n}], \\ C_1 = E = X_1, \qquad Q_1 = E = Y_1, \\ C_{n+1} = \{v \in Q_n : \varphi(v, y_n) \le \alpha_n \varphi(v, x_n) + (1-\alpha_n)\varphi(v, x_n + e_n)\}, \\ X_{n+1} = \{v \in C_{n+1} : \|x_1 - v\|^2 \le \|P_{C_{n+1}}(x_1) - x_1\|^2 + \delta_n\}, \\ w_n \in X_{n+1}, \\ \overline{\overline{u_n}} = \mathrm{VI}(C, T + \frac{1}{s_n}J_E - \frac{1}{s_n}J_E(w_n + \varepsilon_n)), \\ z_n = J_E^{-1}[\beta_n J_E x_n + (1-\beta_n)b_0 J_E(w_n + \varepsilon_n) + (1-\beta_n)(1-b_0)J_E\overline{\overline{u_n}}], \\ Q_{n+1} = \{v \in C_{n+1} : \varphi(v, z_n) \le \beta_n \varphi(v, x_n) + (1-\beta_n)\varphi(v, w_n + \varepsilon_n)\}, \\ Y_{n+1} = \{v \in Q_{n+1} : \|x_1 - v\|^2 \le \|P_{Q_{n+1}}(x_1) - x_1\|^2 + \vartheta_n\}, \\ x_{n+1} \in Y_{n+1}, \quad n \in N. \end{cases} \tag{4.4}$$

Under the assumptions that $(\partial f)^{-1}0 \cap \mathrm{VI}(C, T) \neq \emptyset$, $\inf_n r_n > 0$ and $\inf_n s_n > 0$, we have $x_n \to P_{(\partial f)^{-1}0\cap \mathrm{VI}(C,T)}(x_1)$, as $n \to \infty$.

Proof Similar to Theorems 4.1 and 4.3, the result can be easily obtained. This completes the proof. □

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Authors' contributions
The first and the fourth author are responsible for abstract results and paper writing. The second author is responsible for the numerical experiment and the third author is responsible for applications of the abstract results. All authors read and approved the final manuscript.

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