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Infinitely many solutions for a class of sublinear fractional Schrödinger equations with indefinite potentials

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Abstract

In this paper, we consider the following sublinear fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

where $s, p \in (0, 1)$, $N > 2s$, $(-\Delta)^s$ is a fractional Laplacian operator, and K, V both change sign in $\mathbb{R}^N$. We prove that the problem has infinitely many solutions under appropriate assumptions on K, V. The tool used in this paper is the symmetric mountain pass theorem.

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Keywords: Fractional Schrödinger equation; Indefinite potential; Symmetric mountain pass theorem

1 Introduction and main result

In this paper, we consider the following sublinear fractional Schrödinger equation:

$$(-\Delta)^s u + V(x)u = K(x)|u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $s, p \in (0, 1)$, $N > 2s$, $(-\Delta)^s$ is a fractional Laplacian operator, K, V both change sign in $\mathbb{R}^N$ and satisfy some conditions specified below.

Problem (1.1) gives the following nonlinear field equation:

$$i\frac{\partial \Psi}{\partial t} = (-\Delta)^s \Psi + (1 + E)\Psi - K(x)|\Psi|^{p-1}\Psi, \quad x \in \mathbb{R}^N, t \in \mathbb{R}^+. \tag{1.2}$$

The nonlinear field Eq. (1.2) reflects the stable diffusion process of Lévy particles in random field. Later, people found that this stable diffusion of Lévy process has also a very important application in the mechanical system, flame propagation, chemical reactions in the liquid, and the anomalous diffusion of physics in the plasma. For more details, readers can refer to [5, 25, 26, 45] and the references therein.

Problem (1.1) involves the fractional Laplacian $(-\triangle)^s$, which is a nonlocal operator. After this question was raised, it immediately aroused the interest of mathematicians (see [1, 4, 6–14, 16–22, 24, 27–29, 31, 33–44, 46–55] and the references therein).

For fractional equations on the whole space $\mathbb{R}^N$, the main difficulty one may face is that the Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is not compact for $q \in [2, 2_s^*)$. To overcome this difficulty, some authors [8, 10, 24, 31, 38, 50] considered fractional equations with the potential V satisfying the following conditions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$ and, for each $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty$, where V_0 is a constant and meas denotes Lebesgue measure in $\mathbb{R}^N$.

Due to condition (V), the subspace of $H^s(\mathbb{R}^N)$ embeds compactly into $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$, which is crucial in their paper. In fact, condition (V) is certain coercive condition. In the case of coercive condition $\lim_{|x|\to+\infty} V(x) = +\infty$, some authors, for example [12, 33], considered fractional equations on the whole space $\mathbb{R}^N$.

To overcome the difficulties caused by the lack of compactness, on the other hand, some authors restricted the energy functional to a subspace for $H^s(\mathbb{R}^N)$ of radially symmetric functions, which embeds compactly into $L^s(\mathbb{R}^N)$, for example, [9, 21, 34, 44, 54].

However, in this paper, we do not need some conditions like (V) or radially symmetric. That is, our paper does not use any compact embedding on the whole space $\mathbb{R}^N$.

It is worth noting that, for fractional equations on the whole space $\mathbb{R}^N$, most results need condition $V(x) \geq 0$ (see [1, 8–10, 12, 13, 16, 18, 20–22, 24, 28, 33, 34, 36–38, 44, 50, 52–54], in which some results were obtained in case of $V(x) = 1$ [16, 18, 21, 28, 44]). To the best of our knowledge, there are few results on the existence of solutions for fractional equations with a sign-changing potential except [11, 51]. In fact, replaced $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$ with $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$, condition similar to (V) is needed in [11]. In [51], Xu, Wei, and Dong considered the following p-Laplacian equation with positive nonlinearity:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u - \lambda|u|^{p-2}u = f(x,u) + g(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N,$$

where $N, p \geq 2$, $s \in (0,1)$, λ is a parameter, $(-\Delta)_p^s$ is the fractional p-Laplacian, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. In the case of $\lambda = 0$, they obtained the existence of a nontrivial solution to this equation. Furthermore, they proved that this equation has infinitely many nontrivial solutions when $\lambda \leq 0$ or $\lambda > 0$ is small enough.

In this article, we are interested in the existence of infinitely many solutions for problem (1.1) with potential function $V(x)$ changing sign in $\mathbb{R}^N$. Moreover, nonlinearity can be allowed to change sign. To state our main result, we assume the following:

(V_1) $V \in L^\infty(\mathbb{R}^N)$ and there exist $\alpha, R_0 > 0$ such that

$$V(x) \geq \alpha, \quad \forall |x| \geq R_0.$$

(V_2) $\|V^-\|_{\frac{N}{2s}} < \frac{1}{S}$, where $V^\pm(x) = \max\{\pm V(x), 0\}$ and S is the constant of Sobolev:

$$\|u\|_{2_s^*}^2 \leq S\|u\|_{H_0^s(\mathbb{R}^N)}^2, \quad \forall u \in H^s(\mathbb{R}^N), \text{where } 2_s^* = \frac{2N}{N-2s}.$$

(K) $K \in L^\infty(\mathbb{R}^N)$ and there exist $\beta > 0$, $R_1 > R_2 > 0$, $y_0 = (y_1, \ldots, y_N) \in \mathbb{R}^N$ such that

$$K(x) \leq -\beta, \quad \forall |x| > R_1; \qquad K(x) > 0, \quad \forall x \in B(y_0, R_2) \subset B(0, R_1).$$

Our main result of this paper can be stated as follows.

Theorem 1.1 *Assume* (V_1)–(V_2) *and* (K) *hold. Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.1 The ideas in this article come from the paper [3], where Schrödinger equations were considered. However, our proof is nontrivial since we present a simplified proof for the *PS* condition by comparing to that in [3]. In fact, the *PS* condition was proved in [3] by concentration compactness principle. It is noticed that the *PS* condition plays important role in the proof of the main results in [3].

2 Notations and preliminaries

In this paper, we use the following notations. Let

$$\|u\|_q = \left(\int_{\mathbb{R}^N} |u|^q \, dx\right)^{\frac{1}{q}}, \quad 1 \le q < +\infty.$$

Let E be a Banach space and $\varphi : E \to \mathbb{R}$ be a functional of class C^1. The Fréchet derivative of φ at u, $\varphi'(u)$ is an element of the dual space E^*, and we denote $\varphi'(u)$ evaluated at $v \in E$ by $\langle \varphi'(u), v\rangle$.

Let $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

and endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy\right)^{\frac{1}{2}},$$

here

$$[u]_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy\right)^{\frac{1}{2}}$$

is the so-called Gagliardo (semi) norm of u.

Using Fourier transform, the space $H^s(\mathbb{R}^N)$ can also be defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\mathcal{F} u|^2 \, d\xi < +\infty \right\},$$

where $\mathcal{F} u$ denotes the Fourier transform of u.

Let ℓ be the Schwartz space of rapidly decreasing C^∞ function on $\mathbb{R}^N$, $u \in \ell$, one has

$$(-\triangle)^s u(x) = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, dy,$$

the symbol *P.V.* stands for the Cauchy value, and $C(N,s)$ is a constant dependent only on the space dimension N and the order s.

From the results of [15], we have

$$(-\triangle)^s u = \mathcal{F}^{-1}\big(|\xi|^{2s}(\mathcal{F}u)\big) \quad \text{for any } \xi \in \mathbb{R}^N.$$

Then, by Proposition 3.4 and Proposition 3.6 of [15], we have

$$[u]_{H^s}^2 = \frac{2}{C(N,s)} \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u|^2\, d\xi = \frac{2}{C(N,s)} \big\|(-\triangle)^{\frac{s}{2}} u\big\|_2^2.$$

From the above facts, the norms on $H^s(\mathbb{R}^N)$ defined as follows

$$u \mapsto \left(\|u\|_2^2 + \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u|^2\, d\xi\right)^{\frac{1}{2}},$$
$$u \mapsto \left(\|u\|_2^2 + \big\|(-\triangle)^{\frac{s}{2}} u\big\|_2^2\right)^{\frac{1}{2}},$$
$$u \mapsto \|u\|_{H^s(\mathbb{R}^N)}$$

are all equivalent.

Lemma 2.1 ([15, 30, 34]) *Let $0 < s < 1$ such that $2s < N$. Then there exists $C = C(n,s)$ such that*

$$\|u\|_{2_s^*} \le C\|u\|_{H^s(\mathbb{R}^N)}$$

for every $u \in H^s(\mathbb{R}^N)$. Moreover, the embedding $H^s(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2_s^*]$ and locally compact whenever $p \in [2, 2_s^*)$.

Let the homogeneous Sobolev space

$$H_0^s(\mathbb{R}^N) = \big\{u \in L^{2_s^*}(\mathbb{R}^N) : |\xi|^s \mathcal{F}u \in L^2(\mathbb{R}^N)\big\}.$$

This space can be equivalently defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_0^2 \triangleq \|u\|_{H_0^s(\mathbb{R}^N)}^2 \triangleq \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u|^2\, d\xi.$$

The Sobolev space $E = H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ is endowed with the norm

$$\|u\| = \|u\|_0 + \|u\|_{p+1}.$$

Obviously, E is a reflexive Banach space.

The energy functional $\varphi : E \to \mathbb{R}$ corresponding to problem (1.1) is defined by

$$\varphi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u|^2\, d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x)|u|^{p+1}\, dx.$$

Under our conditions, $\varphi \in C^1(E)$ and its critical points are solutions of problem (1.1).

Definition 2.1 ([32]) Let E be a Banach space and A be a subset of E. Set A is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set A which does not contain the origin, we define a genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such k, we define $\gamma(A) = \infty$. We set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geq k$.

The following result is a version of the classical symmetric mountain pass theorem [2, 32]. For the proof, please see [23].

Theorem 2.1 ([23]) *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy*:

(I_1) *I is even, bounded from below, $I(0) = 0$, and I satisfies the Palais–Smale condition.*

(I_2) *For each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that*

$$\sup_{u \in A_k} I(u) < 0.$$

Then either of the following two conditions holds:

(i) *there exists a sequence u_k such that $I'(u_k) = 0, I(u_k) < 0$ and u_k converges to zero*; *or*

(ii) *there exist two sequences u_k and v_k such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k \to +\infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k \to +\infty} I(v_k) = 0$ and v_k converges to a non-zero limit.*

3 Proof of Theorem 1.1

Lemma 3.1 *Suppose that (V_1)–(V_2) and (K) hold. Then any PS sequence of φ is bounded in E.*

Proof Let $\{u_n\} \subset E$ be such that

$$\varphi(u_n) \text{ is bounded} \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{as } n \to \infty.$$

That is, there exists $C > 0$ such that $\varphi(u_n) \leq C$. So, according to Hölder's inequality and Sobolev's inequality, one has that

$$\begin{aligned}
C \geq \varphi(u_n) &= \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u_n|^2 \, d\xi + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_n^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u_n|^2 \, d\xi - \frac{1}{2} \int_{\mathbb{R}^N} V^-(x) u_n^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} K^+(x) |u_n|^{p+1} \, dx \\
&\geq \frac{1}{2} \|u_n\|_0^2 - \frac{1}{2} \left(\int_{\mathbb{R}^N} |V^-|^{\frac{N}{2s}} \, dx \right)^{\frac{2s}{N}} \left(\int_{\mathbb{R}^N} (|u_n|^2)^{\frac{2_s^*}{2}} \, dx \right)^{\frac{2}{2_s^*}} \\
&\quad - \frac{1}{p+1} \int_{\mathbb{R}^N} K^+(x) |u_n|^{p+1} \, dx \\
&\geq \left(\frac{1}{2} - \frac{S}{2} \|V^-\|_{\frac{N}{2s}} \right) \|u_n\|_0^2 - \frac{S^{\frac{p+1}{2}}}{p+1} \|K^+\|_{\frac{2_s^*}{2_s^* - (p+1)}} \|u_n\|_0^{p+1}.
\end{aligned}$$

Since $0 < p < 1$, there exists $\eta > 0$ such that

$$\|u_n\|_0^2 \leq \eta, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

On the other hand, we have that

$$\begin{aligned} C + \frac{\|u_n\|}{2} &\geq \varphi(u_n) - \frac{1}{2}\langle \varphi'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} K(x)|u_n|^{p+1}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} K^+(x)|u_n|^{p+1}\,dx + \left(\frac{1}{p+1} - \frac{1}{2}\right) \int_{\mathbb{R}^N} K^-(x)|u_n|^{p+1}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} \big(K^+(x) + \chi_{B(0,R_1)}(x)\big)|u_n|^{p+1}\,dx \\ &\quad + \left(\frac{1}{p+1} - \frac{1}{2}\right) \int_{\mathbb{R}^N} \big(K^-(x) + \chi_{B(0,R_1)}(x)\big)|u_n|^{p+1}\,dx, \end{aligned}$$

where $\|\cdot\|$ denotes the norm in E.

Thanks to (K), we have that

$$K^+(x) = 0 \quad \text{for all } |x| > R_1.$$

Then, by $K \in L^\infty(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} \big|K^+(x) + \chi_{B(0,R_1)}(x)\big|^{\frac{2_s^*}{2_s^*-(p+1)}}\,dx = \int_{B(0,R_1)} \big|K^+(x) + \chi_{B(0,R_1)}(x)\big|^{\frac{2_s^*}{2_s^*-(p+1)}}\,dx < \infty.$$

Hence, by Hölder's inequality and Sobolev's inequality, we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big(K^+(x) + \chi_{B(0,R_1)}(x)\big)|u_n|^{p+1}\,dx \\ &\leq \left(\int_{\mathbb{R}^N} \big(K^+(x) + \chi_{B(0,R_1)}(x)\big)^{\frac{2_s^*}{2_s^*-(p+1)}}\,dx\right)^{\frac{2_s^*-(p+1)}{2_s^*}} \times \left(\int_{\mathbb{R}^N} \big(|u_n|^{p+1}\big)^{\frac{2_s^*}{p+1}}\,dx\right)^{\frac{p+1}{2_s^*}} \\ &\leq S^{\frac{p+1}{2}} \big\|K^+ + \chi_{B(0,R_1)}\big\|_{\frac{2_s^*}{2_s^*-(p+1)}} \|u_n\|_0^{p+1}. \end{aligned} \tag{3.2}$$

Using (K) again, we know that $K^-(x) \geq \beta$ for all $|x| > R_1$. Then we have that

$$\int_{\mathbb{R}^N} \big(K^-(x) + \chi_{B(0,R_1)}(x)\big)|u_n|^{p+1}\,dx \geq \min(\beta, 1)\|u_n\|_{p+1}^{p+1}. \tag{3.3}$$

According to (3.1), (3.2), and (3.3), there exists a constant $C_1 > 0$ such that

$$\|u_n\|_{p+1}^{p+1} \leq C_1 + C_1\|u_n\|_{p+1} \quad \text{for all } n \in \mathbb{N}.$$

Since $0 < p < 1$, there exists a constant $C_2 > 0$ such that

$$\|u_n\|_{p+1} \leq C_2, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Hence, it follows from (3.1) and (3.4) that $\{u_n\}$ is bounded in E. □

Lemma 3.2 *Suppose that* (V_1)–(V_2) *and* (K) *hold. Then* φ *satisfies the PS condition on* E.

Proof Let $\{u_n\} \subset E$ be such that

$$\varphi(u_n) \text{ is bounded} \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 3.1, $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, from Lemma 2.1 we can assume that

$$u_n \rightharpoonup u \text{ in } E; \qquad u_n \to u \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N), \quad 2 \le q < 2^*_s; \qquad u_n \to u \text{ a.e in } \mathbb{R}^N. \tag{3.5}$$

So, $\forall \psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |\xi|^{2s} \mathcal{F} u_n \mathcal{F} \psi \, d\xi + \int_{\mathbb{R}^N} V(x) u_n \psi \, dx \to \int_{\mathbb{R}^N} |\xi|^{2s} \mathcal{F} u \mathcal{F} \psi \, d\xi + \int_{\mathbb{R}^N} V(x) u \psi \, dx.$$

By $u_n \to u$ in $L^{p+1}(\mathrm{supp}(\psi))$ [15, 30] and Lebesgue's dominated convergence theorem, one has that

$$\int_{\mathbb{R}^N} K(x) |u_n|^{p-1} u_n \psi \, dx \to \int_{\mathbb{R}^N} K(x) |u|^{p-1} u \psi \, dx.$$

Hence, we have

$$0 = \lim_{n \to +\infty} \langle \varphi'(u_n), \psi \rangle = \langle \varphi'(u), \psi \rangle, \quad \forall \psi \in C_0^\infty(\mathbb{R}^N).$$

Then

$$\langle \varphi'(u), u \rangle = 0.$$

Let $v_n = u_n - u$, then $u_n = v_n + u$, we have that

$$\begin{aligned}
\langle \varphi'(u_n), u_n \rangle &= \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} u_n|^2 \, d\xi + \int_{\mathbb{R}^N} V(x) u_n^2 \, dx - \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx \\
&= \int_{\mathbb{R}^N} |\xi|^{2s} \big(|\mathcal{F} v_n|^2 + |\mathcal{F} u|^2 + 2 \mathcal{F} v_n \mathcal{F} u \big) \, d\xi \\
&\quad + \int_{\mathbb{R}^N} \big(V(x) v_n^2 + V(x) u^2 + 2V(x) v_n u \big) \, dx \\
&\quad - \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx + \int_{\mathbb{R}^N} K(x) |u|^{p+1} \, dx - \int_{\mathbb{R}^N} K(x) |u|^{p+1} \, dx \\
&= \langle \varphi'(u), u \rangle + \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} v_n|^2 \, d\xi + \int_{\mathbb{R}^N} V(x) v_n^2 \, dx \\
&\quad - \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} \, dx + \int_{\mathbb{R}^N} K(x) |u|^{p+1} \, dx + o_n(1) \\
&\ge \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F} v_n|^2 \, d\xi - \int_{\mathbb{R}^N} V^-(x) v_n^2 \, dx \\
&\quad - \int_{\mathbb{R}^N} K(x) \big(|u_n|^{p+1} - |u|^{p+1} \big) \, dx + o_n(1).
\end{aligned}$$

Thanks to (3.5) and Lemma 4.2 in [3], we have that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)\big[|u_n|^{p+1}-|u|^{p+1}\big]\,dx=\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)|v_n|^{p+1}\,dx.$$

So, we have that

$$\begin{aligned}\langle\varphi'(u_n),u_n\rangle\geq{}&\int_{\mathbb{R}^N}|\xi|^{2s}|\mathcal{F}v_n|^2\,d\xi-\int_{\mathbb{R}^N}V^-(x)v_n^2\,dx\\&-\int_{\mathbb{R}^N}K(x)|v_n|^{p+1}\,dx+o_n(1)\\={}&\int_{\mathbb{R}^N}|\xi|^{2s}|\mathcal{F}v_n|^2\,d\xi-\int_{\mathbb{R}^N}V^-(x)v_n^2\,dx\\&-\int_{\mathbb{R}^N}\big(K^+(x)+\chi_{B(0,R_1)}(x)\big)|v_n|^{p+1}\,dx\\&+\int_{\mathbb{R}^N}\big(K^-(x)+\chi_{B(0,R_1)}(x)\big)|v_n|^{p+1}\,dx+o_n(1).\end{aligned}\tag{3.6}$$

Claim 1 $\int_{\mathbb{R}^N}V^-(x)v_n^2\,dx\to 0$ *as* $n\to+\infty$.

In fact, by (V_1), we have that $V^-(x)=0$ for all $|x|\geq R_0$. So, from $v_n\to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $2\leq q<2_s^*$, and $V\in L^\infty(\mathbb{R}^N)$, we obtain $\int_{\mathbb{R}^N}V^-(x)v_n^2\,dx\to 0$ as $n\to+\infty$.

Claim 2 $\int_{\mathbb{R}^N}(K^+(x)+\chi_{B(0,R_1)}(x))|v_n|^{p+1}\,dx\to 0$ *as* $n\to+\infty$.

In fact, thanks to (K), we have that $K^+(x)=0$ for all $|x|>R_1$. So, by $K\in L^\infty(\mathbb{R}^N)$ and $v_n\to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $2\leq q<2_s^*$, we get

$$\int_{\mathbb{R}^N}\big(K^+(x)+\chi_{B(0,R_1)}(x)\big)|v_n|^{p+1}\,dx\to 0$$

as $n\to+\infty$.

From Claim 1, Claim 2, (3.3), and (3.6), we obtain that

$$0=\lim_{n\to+\infty}\big(\|v_n\|_0^2+\min(\beta,1)\|v_n\|_{p+1}^{p+1}\big).$$

That is, $v_n\to 0$ in E. The proof is complete. □

Lemma 3.3 *Assume that* (V_1)–(V_2) *and* (K) *hold. Then, for each* $k\in\mathbb{N}$, *there exists* $A_k\in\Gamma_k$ *such that*

$$\sup_{u\in A_k}\varphi(u)<0.$$

Proof The proof is based on some ideas of Kajikiya [23] and is very similar to the one contained in [3]. For readers' convenience, we give the proof. Let R_2 and y_0 be fixed as in (K) and denote

$$D(R_2)=\big\{(x_1,\dots,x_n)\in\mathbb{R}^N:|x_i-y_i|<R_2,1\leq i\leq N\big\}.$$

Let $k \in \mathbb{N}$ be an arbitrary number and define $n = \min\{n \in \mathbb{N} : n^N \geq k\}$. By planes parallel to each face of $D(R_2)$, let $D(R_2)$ be equally divided into n^N small parts D_i with $1 \leq i \leq n^N$. In fact, the length a of the edge D_i is $\frac{R_2}{n}$. Let $F_i \subset D_i$ be new cubes such that F_i has the same center as that of D_i. The faces of F_i and D_i are parallel, and the length of the edge of F_i is $\frac{a}{2}$. Let ϕ_i, $1 \leq i \leq k$, satisfy: $\operatorname{supp}(\phi_i) \subset D_i$; $\operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) = \emptyset$ $(i \neq j)$; $\phi_i(x) = 1$ for $x \in F_i$; $0 \leq \phi_i(x) \leq 1$, for all $x \in \mathbb{R}^N$. Let

$$S^{k-1} = \left\{(t_1, \ldots, t_k) \in \mathbb{R}^k : \max_{1 \leq i \leq k} |t_i| = 1\right\}, \tag{3.7}$$

$$W_k = \left\{\sum_{i=1}^{k} t_i\phi_i(x) : (t_1, \ldots, t_k) \in S^{k-1}\right\} \subset E.$$

According to the fact that the mapping $(t_1, \ldots, t_k) \to \sum_{i=1}^{k} t_i\phi_i$ from S^{k-1} to W_k is odd and homeomorphic, so $\gamma(W_k) = \gamma(S^{k-1}) = k$. Since W_k is compact in E, then $\exists \alpha_k > 0$ such that

$$\|u\|^2 \leq \alpha_k, \quad \forall u \in W_k.$$

On the other hand, by Hölder's inequality and Sobolev's embedding, we have that

$$\|u\|_2 \leq c\|u\|_0^r \|u\|_{p+1}^{1-r} \leq c\|u\|,$$

where $r = \frac{2_s^*(1-p)}{2(2_s^*-p-1)}$.

According to the above facts, there exists $c_k > 0$ such that

$$\|u\|_2^2 \leq c_k \quad \text{for all } u \in W_k.$$

Let $t > 0$ and $u = \sum_{=1}^{k} t_i\phi_i(x) \in W_k$,

$$\begin{aligned}\varphi(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\xi|^{2s}|\mathcal{F}u|^2\,d\xi + \frac{t^2}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx - \frac{1}{p+1}\sum_{i=1}^{k}\int_{D_i} K(x)|tt_i\phi_i|^{p+1}\,dx \\ &\leq \frac{t^2}{2}\alpha_k + \frac{t^2}{2}\|V\|_\infty c_k - \frac{1}{p+1}\sum_{i=1}^{k}\int_{D_i} K(x)|tt_i\phi_i|^{p+1}\,dx.\end{aligned} \tag{3.8}$$

From (3.7), there exists $j \in [1, k]$ such that $|t_j| = 1$ and $|t_i| \leq 1$ for $i \neq j$. So

$$\begin{aligned}\sum_{i=1}^{k}\int_{D_i} K(x)|tt_i\phi_i|^{p+1}\,dx &= \int_{F_j} K(x)|tt_j\phi_j|^{p+1}\,dx \\ &\quad + \int_{D_j\setminus F_j} K(x)\bigl|tt_j\phi_j(x)\bigr|^{p+1}\,dx + \sum_{i\neq j}\int_{D_i} K(x)|tt_i\phi_i|^{p+1}\,dx.\end{aligned} \tag{3.9}$$

According to $\phi_j(x) = 1$ for $x \in F_j$ and $|t_j| = 1$, one has that

$$\int_{F_j} K(x)|tt_j\phi_j|^{p+1}\,dx = |t|^{p+1}\int_{F_j} K(x)\,dx. \tag{3.10}$$

By (K), one has that

$$\int_{D_j \setminus F_j} K(x) \left| tt_j \phi_j(x) \right|^{p+1} dx + \sum_{i \neq j} \int_{D_i} K(x) |tt_i \phi_i|^{p+1} dx \geq 0. \tag{3.11}$$

According to (3.8), (3.9), (3.10), and (3.11), we have that

$$\frac{\varphi(tu)}{t^2} \leq \frac{1}{2}\alpha_k + \frac{1}{2}\|V\|_\infty c_k - \frac{|t|^{p+1}}{(p+1)t^2} \inf_{1 \leq i \leq k} \left(\int_{F_i} K(x)\, dx \right).$$

So,

$$\lim_{t \to 0} \sup_{u \in W_k} \frac{\varphi(tu)}{t^2} = -\infty.$$

Hence, we can fix t small enough such that $\sup\{\varphi(u), u \in A_k\} < 0$, where $A_k = tW_k \in \Gamma_k$. □

Lemma 3.4 *Assume that* (V_1)–(V_2) *and* (K) *hold. Then* φ *is bounded from below.*

Proof By (K), Hölder's inequality and Sobolev's embedding, as in the proof of Lemma 3.1, we have that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 \, d\xi + \int_{\mathbb{R}^N} V(x) u^2 \, dx \right) - \frac{1}{p+1} \int_{\mathbb{R}^N} K(x) |u|^{p+1} dx \\
&\geq \frac{1}{2}\left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 \, d\xi - \int_{\mathbb{R}^N} V^-(x) u^2 \, dx \right) - \frac{1}{p+1} \int_{\mathbb{R}^N} K^+(x) |u|^{p+1} dx \\
&\geq \left(\frac{1}{2} - \frac{S \|V^-\|_{\frac{N}{2s}}}{2} \right) \|u\|_0^2 - \frac{S^{\frac{p+1}{2}}}{p+1} \left\| K^+ \right\|_{\frac{2_s^*}{2_s^* - p - 1}} \|u\|_0^{p+1}.
\end{aligned}$$

Since $0 < p < 1$, we conclude the proof. □

Proof of Theorem 1.1 In fact, $\varphi(0) = 0$ and φ is an even functional. Then by Lemmas 3.2, 3.3, and 3.4, conditions (I_1) and (I_2) of Theorem 2.1 are satisfied. Therefore, by Theorem 2.1, problem (1.1) possesses infinitely many nontrivial solutions converging to 0 with negative energy. □

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