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On the spectrum of the weighted p -Laplacian under the Ricci-harmonic flow

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Abstract

This paper studies the behaviour of the spectrum of the weighted p -Laplacian on a complete Riemannian manifold evolving by the Ricci-harmonic flow. Precisely, the first eigenvalue diverges in a finite time along this flow. It is further shown that the same divergence result holds on gradient shrinking and steady almost Ricci-harmonic solitons under the condition that the soliton function is nonnegative and superharmonic. We also continue the program in (Abolarinwa, Adebimpe and Bakare in *J. Ineq. Appl.* 2019:10, 2019) to the case of volume-preserving Ricci-harmonic flow.

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1 Introduction

In this paper we aim at studying the properties of the spectrum of the weighted p -Laplacian on a complete Riemannian manifold with evolving geometry. It is a well known feature that spectrum as an invariant quantity evolves as the domain does under any geometric flow. Throughout, we will consider an n -dimensional complete Riemannian manifold $(M, g, d\mu)$ equipped with weighted measure $d\mu = e^{-\phi} dv$ and potential function $\phi \in C^\infty(M, d\mu)$, whose metric $g = g(t)$ evolves along either the Ricci-harmonic flow or volume-preserving Ricci-harmonic flow. Firstly, we extend results in [8] to the case of volume-preserving Ricci-harmonic flow. We will obtain a variation formula for the first eigenvalue and show that it is monotonically increasing under this setup. Secondly, we study maximal time behaviour of the first eigenvalue. It is found that the bottom of the spectrum diverges in a finite time of the flow existence. We observe the same result for the behaviour of the evolving spectrum on a class of self-similar solutions, called gradient almost Ricci-harmonic solitons.

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1.1 The Ricci-harmonic flow

The pair $(g = g(t), \phi = \phi(t))$ is said to be a Ricci-harmonic flow if it satisfies the system of quasilinear parabolic equations

$$\begin{cases} \frac{\partial}{\partial t} g = -2Rc + 2\alpha \nabla \phi \otimes \nabla \phi, \\ \frac{\partial}{\partial t} \phi = \Delta_g \phi \end{cases} \quad (1.1)$$

subject to the initial condition $(g(0), \phi(0)) = (g_0, \phi_0)$. Here $\phi : M \times [0, \infty) \rightarrow \mathbb{R}$ is a one-parameter family of smooth functions, at least C^2 in x and C^1 in t , \otimes is the tensor product, Rc is the Ricci curvature tensor of (M, g) , ∇ is the gradient operator, α is a nonincreasing constant function of time, bounded below by $\alpha_n > 0$ in time, and Δ is the Laplace–Beltrami operator on M . The system (1.1) was first studied by List [20] with a motivation coming from general relativity. It was generalized by Müller [21] to the situation where $\phi : (M, g) \rightarrow (N, h)$ ((N, h) is a compact Riemannian manifold endowed with a static metric h) and ϕ satisfies the heat flow for a harmonic map [15]. System (1.1) generalizes the Ricci flow [16] for the case ϕ is a constant. For a detailed discussion on the Ricci flow, see [12, 13].

Strictly related to (1.1) in applications is its normalized counterpart defined in [21] (see [2, 3] also) by the system

$$\begin{cases} \frac{\partial}{\partial t} g = -2Rc + 2\alpha \nabla \phi \otimes \nabla \phi + \frac{2r}{n} g, \\ \frac{\partial}{\partial t} \phi = \Delta_g \phi, \end{cases} \quad (1.2)$$

with initial data $(g(0), \phi(0)) = (g_0, \phi_0)$ and $r = \text{Vol}(M)^{-1} \int_M (R - \alpha |\nabla \phi|^2) d\mu$ being a constant, having a striking property of volume preservation all through the flow, though differing from (1.1) by changes of scale in space and time parametrization.

1.2 The almost Ricci-harmonic soliton

Let $\sigma = \sigma(x) : M \rightarrow \mathbb{R}$ be a smooth function. We call the tuple (g, ϕ, f, σ) a gradient almost Ricci-harmonic soliton if it satisfies the coupled system of nonlinear elliptic equations

$$\begin{cases} Rc - \alpha \nabla \phi \otimes \nabla \phi + \text{Hess} f = \sigma g, \\ \Delta \phi - \langle \nabla \phi, \nabla f \rangle = 0 \end{cases} \quad (1.3)$$

for some smooth function f on M . Here, we assume $\sigma \geq 0$ and the tuple (g, ϕ, f, σ) is said to be shrinking when σ is positive or steady when σ is null. If ∇f is a Killing vector field or f is a constant, we say that the soliton is trivial and the underlying metric is harmonic Einstein. If the soliton function σ is constant, we have Ricci-harmonic solitons, which are special solutions to (1.1) via scaling and diffeomorphism. These solutions occur as singularity models or blow-up limits for the flow. Almost Ricci-harmonic solitons are generalization of Ricci solitons, Einstein metrics, harmonic Einstein metrics, all of which are very useful in geometry and theoretical physics. For a detailed background on Ricci solitons, see [10]; for Ricci-harmonic solitons, see [3, 20, 21]; and for almost Ricci harmonic solitons, see [4, 6, 7, 9].

In recent years, obtaining information about behaviours of eigenvalues of geometric operators on evolving manifolds has become a topic of concern among geometers since this

information usually turns out to be useful in the study of geometry and topology of the underlying manifolds. Perelman's preprint [22] is fundamental in this respect. Cao [11] and Li [18] extended Perelman's result with or without any curvature assumption. Recently, [8] was motivated by the first author's papers [5] and [1] where he studied the evolution and monotonicity of the first eigenvalue of the p -Laplacian and weighted Laplacian, respectively. In [14], Di Cerbo proved that the first eigenvalue of Laplace–Beltrami operator on a 3-dimensional closed manifold with positive Ricci curvature diverges as $t \rightarrow T$ under the Ricci flow. The authors in [26] obtained a similar result under 3-dimensional Ricci–Bourguignon flow. In [5] the first author proved the same result for the weighted Laplacian under the Ricci-harmonic flow. Motivated by [14] and [5], we will show the same result for the eigenvalue of the weighted p -Laplacian under the Ricci-harmonic flow and on gradient almost Ricci-harmonic soliton for $p = 2$. Meanwhile, we will first extend the results of [8] to the case of a volume-preserving flow.

1.3 Preliminaries

Throughout this paper, (M, g) will be taken to be a closed Riemannian manifold. The Riemannian metric $g(x)$ at any point $x \in M$ is a bilinear symmetric positive definite matrix. As in [8], we denote a symmetric 2-tensor by $Sc := Rc - \alpha \nabla \phi \otimes \nabla \phi$ and its metric trace by $S := R - \alpha |\nabla \phi|^2$, where R is the scalar curvature of (M, g) and $\nabla_i \phi = \frac{\partial}{\partial x^i} \phi$. We denote the Laplace–Beltrami operator on (M, g) by Δ . We denote $d\nu$ as the Riemannian volume measure on (M, g) and $d\mu := e^{-\phi(x)} d\nu$, the weighted volume measure, where $\phi \in C^\infty(M)$.

1.3.1 The weighted p -Laplacian

Let $f : M \rightarrow \mathbb{R}$ be a smooth function, for $p \in [1, +\infty)$. The weighted p -Laplacian on smooth functions is defined by

$$\Delta_{p,\phi} := e^\phi \operatorname{div}(e^{-\phi} |\nabla f|^{p-2} \nabla f) = \Delta_p f - |\nabla f|^{p-2} \langle \nabla \phi, \nabla f \rangle.$$

When $p = 2$, this is just the Witten Laplacian, and when ϕ is a constant, it is just the p -Laplacian. See [8] for detailed descriptions of Witten Laplacian and p -Laplacian.

1.3.2 The minimax principle

The minimax principle also holds for the weighted p -Laplacian where its first nonzero eigenvalue is characterized as follows:

$$\lambda_1(t) = \inf_f \left\{ \int_M |\nabla f|^p d\mu : \int_M |f|^p d\mu = 1, f \neq 0, f \in W^{1,p}(M, g, d\mu) \right\} \quad (1.4)$$

satisfying the constraints $\int_M |f|^{p-2} f d\mu = 0$, where $W^{1,p}(M, g, d\mu)$ is the completion of $C^\infty(M, g, d\mu)$ with respect to the norm

$$\|f\|_{W^{1,p}} = \left(\int_M |f|^p d\mu + \int_M |\nabla f|^p d\mu \right)^{\frac{1}{p}}.$$

The infimum in (1.4) is achieved by $f \in W^{1,p}$ satisfying the Euler–Lagrange equation

$$\Delta_{p,\phi} f = -\lambda_1 |f|^{p-2} f, \quad (1.5)$$

or equivalently,

$$\int_M |\nabla f|^{p-2} \langle \nabla f, \nabla \psi \rangle d\mu - \lambda_1 \int_M |f|^{p-2} \langle f, \psi \rangle d\mu = 0 \quad (1.6)$$

for all $\psi \in C_0^\infty(M)$ in the sense of distributions. In other words, we say that λ is an eigenvalue of $\Delta_{p,\phi}$ and $f \in W^{1,p}$ is the corresponding eigenfunction if the pair (λ, f) satisfies (1.5). Then (1.6) implies

$$\int_M |\nabla f|^p d\mu = \lambda \int_M |f|^p d\mu, \quad (1.7)$$

implying $\lambda = \int_M |\nabla f|^p d\mu$ since $\int_M |f|^p d\mu = 1$. Interested readers can see the book [23] for a detailed discussion on the spectral theory.

1.3.3 Linearized operator

As in [8, Sect. 3], we define the linearized operator of the weighted p -Laplacian on a function $h \in C^\infty(M)$ pointwise at the points $\nabla h \neq 0$, which is strictly elliptic in general at these points

$$\begin{aligned} \mathcal{L}_\phi(\tilde{f}) &:= e^\phi \operatorname{div}(e^{-\phi} |\nabla h|^{p-2} G(\nabla \tilde{f})) \\ &= |\nabla h|^{p-2} \Delta_\phi \tilde{f} + (p-2) |\nabla h|^{p-4} \operatorname{Hess} \tilde{f}(\nabla h, \nabla h) + (p-2) \Delta_{p,\phi} h \frac{\langle \nabla h, \nabla \tilde{f} \rangle}{|\nabla h|^2} \\ &\quad + 2(p-2) |\nabla h|^{p-4} \operatorname{Hess} \tilde{f} \left(\nabla h, \nabla \tilde{f} - \frac{\nabla h}{|\nabla h|} \left\langle \frac{\nabla h}{|\nabla h|}, \nabla \tilde{f} \right\rangle \right) \end{aligned}$$

for a smooth function \tilde{f} on M , where G can be viewed as a tensor defined as $G := \operatorname{Id} + (p-2) \frac{\nabla h \otimes \nabla h}{|\nabla h|^2}$. Note that the sum of the second-order parts of \mathcal{L}_ϕ is

$$\mathbb{L}_\phi \tilde{f} := |\nabla h|^{p-2} \Delta_\phi \tilde{f} + (p-2) |\nabla h|^{p-4} \operatorname{Hess} \tilde{f}(\nabla h, \nabla h) = \Delta_{p,\phi} \tilde{f}.$$

The weighted p -Laplacian degenerates at points $\nabla f = 0$ for $p \neq 2$. In this case the ε -regularization technique is usually applied by replacing the linearized operator with its approximate operator.

Given $\varepsilon > 0$, an approximate operator $\mathcal{L}_{\phi,\varepsilon} := \Delta_{p,\phi,\varepsilon}$ for a smooth function f_ε is defined by

$$\Delta_{p,\phi,\varepsilon} f_\varepsilon = e^\phi \operatorname{div}(e^{-\phi} A_\varepsilon^{\frac{p-2}{2}} \nabla f_\varepsilon) = \Delta_{p,\varepsilon} f_\varepsilon - A_\varepsilon^{\frac{p-2}{2}} \langle \nabla \phi, \nabla f_\varepsilon \rangle,$$

where $A_\varepsilon = |\nabla f_\varepsilon|^2 + \varepsilon$. Define the G_ε norm $\|\cdot\|_{G_\varepsilon}$ for every smooth 2-symmetric tensor V_{ij} by

$$\|V_{ij}\|^2 = \left(g^{ij} + (p-2) \frac{\nabla f_\varepsilon^i \nabla f_\varepsilon^j}{A_\varepsilon} \right) \left(g^{kl} + (p-2) \frac{\nabla f_\varepsilon^k \nabla f_\varepsilon^l}{A_\varepsilon} \right) V_{ik} V_{jl}.$$

Then the G_ε trace of V_{ij} is

$$\operatorname{Tr}_{G_\varepsilon}(V_{ij}) = \left(g^{ij} + (p-2) \frac{\nabla f_\varepsilon^i \nabla f_\varepsilon^j}{A_\varepsilon} \right) V_{ij}.$$

In particular,

$$\|\text{Hess}f_\varepsilon\|^2 = |\text{Hess}f_\varepsilon|^2 + (p-2)\frac{|\nabla A_\varepsilon|^2}{2A_\varepsilon} + (p-2)^2\frac{|\langle \nabla A_\varepsilon, \nabla f_\varepsilon \rangle|^2}{A_\varepsilon^2}$$

and

$$\text{Tr}_{G_\varepsilon}(\text{Hess}f_\varepsilon) = \Delta f_\varepsilon + (p-2)\frac{\langle \nabla A_\varepsilon, \nabla f_\varepsilon \rangle}{2A_\varepsilon}.$$

Hence,

$$A_\varepsilon^{\frac{p-2}{2}} \text{Tr}_{G_\varepsilon}(\text{Hess}f_\varepsilon) = \Delta_{p,\varepsilon} f_\varepsilon = \Delta_{p,\phi} f_\varepsilon + A_\varepsilon^{\frac{p-2}{2}} \langle \nabla \phi, \nabla f_\varepsilon \rangle. \quad (1.8)$$

This regularization procedure has also been used in [17, 24].

2 Variation of $\lambda_1(t)$ under volume-preserving flow

This section uses the variation formula for $\lambda_1(t)$ to establish its monotonicity under the normalized Ricci-harmonic flow. Based on the argument in [8, Sect. 4], we shall assume that $\lambda_1(f(t), t) = \lambda_1(t)$ and that $f(t)$ and $\lambda_1(f(t), t)$ are smooth. Supposing $(M, g(t), \phi(t), d\mu)$, $t \in [0, \infty)$ solves (1.2) on a closed manifold, define a general smooth function as follows:

$$\lambda_1(f(t), t) := \int_M |\nabla f(t)|^p d\mu, \quad (2.1)$$

where $f(t)$ is a smooth function satisfying the normalization condition

$$\int_M |f(t)|^p d\mu = 1 \quad \text{and} \quad \int_M |f(t)|^{p-2} f(t) d\mu = 0.$$

Proposition 2.1 *Let $(M, g(t), \phi(t), d\mu)$, $t \in [0, \infty)$ solve the normalized Ricci-harmonic flow (1.2) on a closed Riemannian manifold, M , with $R - \alpha|\nabla|^2 > 0$. Let $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p -Laplacian, $\Delta_{p,\phi}$, and $f(x, t)$ its corresponding eigenfunction. Then $\lambda_1(t)$ evolves by*

$$\begin{aligned} \frac{d}{dt} \lambda_1(t) &= \lambda_1(t) \int_M (S + \phi_t) |f|^p d\mu - \int_M (S + \phi_t) |\nabla f|^p d\mu \\ &\quad + p \int_M |\nabla f|^{p-2} S^{ij} \nabla_i f \nabla_j f d\mu - \frac{pr}{n} \lambda_1(t) \end{aligned} \quad (2.2)$$

for all times $t \in [0, \infty)$.

Proof Proposition 2.1 is a counterpart of [8, Theorem 4.1] and the proof follows directly. The formulas in the next lemma are applied instead of those in [8, Lemma 3.1]. \square

Lemma 2.2 *Suppose $(M, g(t), \phi(t), d\mu)$, $t \in [0, \infty)$ solves normalized Ricci-harmonic flow (1.2). Then for any $f \in C^\infty(M)$, we have the following formulas:*

1. $\frac{\partial}{\partial t} |\nabla f|^p = p |\nabla f|^{p-2} (S^{ij} \nabla_i f \nabla_j f + g^{ij} \nabla_i f \nabla_j f_t) - \frac{pr}{n} |\nabla f|^p,$
2. $\frac{\partial}{\partial t} |\nabla f|^{p-2} = (p-2) |\nabla f|^{p-4} (S^{ij} \nabla_i f \nabla_j f + g^{ij} \nabla_i f \nabla_j f_t) - \frac{(p-2)r}{n} |\nabla f|^p,$

3.

$$\begin{aligned} \frac{\partial}{\partial t}(\Delta_{p,\phi} f) &= 2S^{ij}\nabla_i(Z\nabla_j f) + g^{ij}\nabla_i(Z_t\nabla_j f) + g^{ij}\nabla_i(Z\nabla_j f_t) \\ &\quad - 2\alpha Z(\Delta\phi)g^{ij}\nabla_i\phi\nabla_j f - Z_t\langle\nabla\phi, \nabla f\rangle - Z\langle\nabla\phi_t, \nabla f\rangle \\ &\quad - Z\langle\nabla\phi, \nabla f_t\rangle - 2S^{ij}Z\nabla_i\phi\nabla_j f - \frac{pr}{n}\Delta_{p,\phi} f, \end{aligned}$$

where $Z := |\nabla f|^{p-2}$ and $f_t = \frac{\partial}{\partial t}f$.

Proof The proof follows from [8, Lemma 3.1] by using $\frac{\partial}{\partial t}g^{ij} = 2S^{ij} - \frac{2r}{n}g^{ij}$. \square

The following is the main theorem of this section.

Theorem 2.3 *Let $(M, g(t), \phi(t), d\mu), t \in [0, \infty)$ solve the normalized Ricci-harmonic flow (1.2) on a closed Riemannian manifold, M , with $S_{\min}(g_0) > 0$. Let $\lambda_1(t)$ be the first nonzero eigenvalue of the weighted p -Laplacian $\Delta_{p,\phi}$ and $f(t)$ its corresponding eigenfunction. Suppose $S_{ij} \geq \beta(S + \Delta\phi)g_{ij}$ with $\frac{1}{p} \leq \beta \leq \frac{1}{n}$, $p \geq n$, $\Delta\phi \geq 0$. Then, the quantity $\lambda_1(t)(e^{-\frac{n}{2}rt} - (1 - rS_{\min}(0)^{-1})^{\frac{n\beta p}{2}}e^{\frac{pr}{n}t})$ is monotonically nondecreasing along (1.1). Consequently, $\lambda_1(t)$ is increasing and differentiable almost everywhere along the flow.*

Proof By [21], $S(t)$ evolves by $\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 + 2\alpha|\Delta\phi|^2 - \frac{2r}{n}S$ under (1.2). Applying the inequality $|S_{ij}|^2 \geq \frac{1}{n}S^2$, we have

$$\frac{\partial S}{\partial t} \geq \Delta S + \frac{2}{n}S(S - r).$$

By the maximum principle, comparing its solution with that of the ODE

$$\frac{dy}{dt} = \frac{2}{n}y(y - r), \quad y(0) = y_0 = S_{\min}(0),$$

we obtain

$$S(t) \geq y(t) = \frac{r}{1 - (1 - \frac{r}{y_0})e^{\frac{2}{n}rt}}.$$

Now we apply the assumptions $S_{ij} \geq \beta(S + \Delta\phi)g_{ij}$ and $\Delta\phi \geq 0$, and use the condition that $S_{\min}(0) > 0$ in the variation formula (2.2) to get

$$\begin{aligned} \left. \frac{d\lambda_1(t)}{dt} \right|_{t_0} &\geq \lambda_1(t_0) \int_M (S + \Delta\phi)|f|^p d\mu + (\beta p - 1) \int_M (S + \Delta\phi)|\nabla f|^p d\mu - \frac{pr}{n}\lambda_1(t_0) \\ &\geq \lambda_1(t)S_{\min}(t_0) + \beta p - 1)S_{\min}(t_0) - \frac{pr}{n}\lambda_1(t_0) \\ &= p\left(\beta S_{\min} - \frac{r}{n}\right)\lambda_1(t). \end{aligned}$$

By a similar argument as in [8], we obtain

$$\frac{d\lambda_1(t)}{dt} \geq p\left(\beta S_{\min} - \frac{r}{n}\right)\lambda_1(t_0)$$

in any sufficiently small neighbourhood of t_0 . Integrating with respect to $t \in [t_1, t_2]$, $t_1 < t_2$ yields

$$\ln \frac{\lambda_1(t_2)}{\lambda_1(t_1)} \geq p\beta \int_{t_1}^{t_2} S_{\min}(t) dt - \frac{pr}{n}(t_2 - t_1).$$

A simple calculus gives

$$\int_{t_1}^{t_2} S_{\min}(t) dt = \int_{t_1}^{t_2} \frac{re^{-\frac{2}{n}rt} dt}{e^{-\frac{2}{n}rt} - (1 - \frac{r}{y_0})} = \ln \frac{[e^{-\frac{2}{n}rt_1} - (1 - \frac{r}{y_0})]^{\frac{n}{2}}}{[e^{-\frac{2}{n}rt_2} - (1 - \frac{r}{y_0})]^{\frac{n}{2}}}.$$

Therefore

$$\ln \frac{\lambda_1(t_2)}{\lambda_1(t_1)} \geq p\beta \ln \frac{[e^{-\frac{2}{n}rt_1} - (1 - \frac{r}{y_0})]^{\frac{n}{2}}}{[e^{-\frac{2}{n}rt_2} - (1 - \frac{r}{y_0})]^{\frac{n}{2}}} - \frac{pr}{n}(t_2 - t_1).$$

Exponentiating we have

$$\lambda_1(t_2) \left[e^{-\frac{2}{n}rt_2} - \left(1 - \frac{r}{y_0}\right) \right]^{\frac{n}{2}p\beta} e^{\frac{pr}{n}t_2} \geq \lambda_1(t_1) \left[e^{-\frac{2}{n}rt_1} - \left(1 - \frac{r}{y_0}\right) \right]^{\frac{n}{2}p\beta} e^{\frac{pr}{n}t_1},$$

implying that the quantity $\lambda_1(t)[e^{-\frac{2}{n}rt} - (1 - ry_0^{-1})]^{\frac{n}{2}p\beta} e^{\frac{pr}{n}t}$ is monotonically nondecreasing along (1.2). By Lebesgue's theorem, λ_1 is differentiable almost everywhere along the flow for all $t \in [0, \infty)$. \square

3 Behaviour of $\lambda_1(t)$ at the maximal time

In [14], Di Cerbo proved that the first eigenvalue of Laplace–Beltrami operator on a 3-dimensional closed manifold with positive Ricci curvature diverges as $t \rightarrow T$ under the Hamilton's Ricci flow. In [5] the author proved the same result for the weighted Laplacian under Ricci-harmonic flow. Motivated by [14] and [5], we will show the same result for the eigenvalue of the weighted p -Laplacian under the Ricci-harmonic flow. Our derivation will be via weighted p -Reilly formula.

Theorem 3.1 (Weighted p -Reilly formula [25, Theorem 2.2]) *Let $(M, g, d\mu)$ be a compact smooth metric measure space. Then*

$$\int_M (\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} \|\text{Hess} f\|_G^2 d\mu = \int_M |\nabla f|^{2p-4} (Rc + \nabla^2 \phi)(\nabla f, \nabla f) d\mu \quad (3.1)$$

for $f \in C^\infty(M)$ and

$$\|\text{Hess} f\|_G^2 = |\text{Hess} f|^2 + \frac{p-2}{2} \frac{|\nabla |\nabla f|^2|^2}{|\nabla f|^2} + \frac{(p-2)^2}{4} \frac{\langle \nabla f, \nabla |\nabla f|^2 \rangle^2}{|\nabla f|^4}.$$

Before we state the main result of the section, we remark that it has been proved in [19, Theorem 1.1] that either

$$\limsup_{t \rightarrow T} \left(\max_M R(t) \right) = \infty \quad (3.2)$$

or

$$\limsup_{t \rightarrow T} \left(\max_M R(t) \right) < \infty, \quad \text{but} \quad \limsup_{t \rightarrow T} \left(\max_M \frac{|W(t)|_{g(t)} + |\nabla^2 \phi(t)|_{g(t)}^2}{R(t)} \right) = \infty, \quad (3.3)$$

where $W(t)$ is the Weyl part of the Riemannian tensor, under the extended Ricci flow for the case $n \geq 3$ and $T < \infty$. Also in this case, $|\nabla \phi|^2$ is uniformly bounded. Observe that if one assumes (3.2), then one can easily deduce that

$$\lim_{t \rightarrow T} S_{\min}(t) = \infty \quad (3.4)$$

without any additional assumption.

Finally, we use the estimate (3.4) together with (3.1) to prove that the eigenvalues of the weighted p -Laplacian diverge as t approaches the maximal time. The main result is the following.

Theorem 3.2 *Let $\lambda_1(t)$ be the first eigenvalue of the weighted p -Laplacian for $p \geq 2$ under the Ricci-harmonic flow $(M, g(t), \phi(t), d\mu)$, $t \in [0, T]$, $T < \infty$ with $S(0) > 0$. Then*

$$\lim_{t \rightarrow T} \lambda_1(t) = +\infty, \quad (3.5)$$

where $S_{ij} - \beta S g_{ij} > 0$ in $M \times [0, T]$, $\beta \in [0, \frac{1}{n}]$.

Proof By the Weighted p -Reilly formula (3.1), we have

$$\begin{aligned} & \int_M (\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} \|\text{Hess} f\|_G^2 d\mu \\ &= \int_M |\nabla f|^{2p-4} S_G(\nabla f, \nabla f) d\mu \\ &+ \int_M |\nabla f|^{2p-4} (\alpha \nabla \phi \otimes \nabla \phi + \nabla^2 \phi)(\nabla f, \nabla f) d\mu. \end{aligned} \quad (3.6)$$

Since $\Delta_{p,\phi} f = \Delta_p f - |\nabla f|^{p-2} \langle \nabla \phi, \nabla f \rangle$, and using an elementary inequality of the form $(a + b)^2 \geq \frac{1}{1+s} a^2 - \frac{1}{s} b^2$ for $s > 0$, we obtain the following inequality:

$$\begin{aligned} (\Delta_p f)^2 &= (\Delta_{p,\phi} f + |\nabla f|^{p-2} \langle \nabla \phi, \nabla f \rangle)^2 \\ &\geq \frac{1}{1+s} (\Delta_{p,\phi} f)^2 - \frac{1}{s} |\nabla f|^{2p-4} |\langle \nabla \phi, \nabla f \rangle|^2. \end{aligned} \quad (3.7)$$

Hence by (1.8) as $\varepsilon \searrow 0$, we have

$$\begin{aligned} |\nabla f|^{2p-4} \|\text{Hess} f\|_G^2 &\geq \frac{1}{n} (|\nabla f|^{p-2} \text{Tr}_G(\text{Hess} f))^2 = \frac{1}{n} (\Delta_p f)^2 \\ &\geq \frac{1}{n(1+s)} (\Delta_{p,\phi} f)^2 - \frac{1}{ns} |\nabla f|^{2p-4} |\langle \nabla \phi, \nabla f \rangle|^2. \end{aligned} \quad (3.8)$$

Using (3.8) in the formula below,

$$\int_M (\Delta_{p,\phi} f)^2 d\mu = \lambda_1^2 \int_M |f|^{2p-2} d\mu,$$

yields

$$\begin{aligned} \int_M (\Delta_{p,\phi} f)^2 - |\nabla f|^{2p-4} \|\text{Hess} f\|_G^2 d\mu &= \left(1 - \frac{1}{n(1+s)}\right) \lambda_1^2 \int_M |f|^{2p-2} d\mu \\ &\quad + \frac{1}{ns} |\nabla f|^{2p-4} |\langle \nabla \phi, \nabla f \rangle|^2. \end{aligned} \quad (3.9)$$

Putting (3.9) into (3.6) gives

$$\begin{aligned} &\left(1 - \frac{1}{n(1+s)}\right) \lambda_1^2 \int_M |f|^{2p-2} d\mu + \frac{1}{ns} \int_M |\nabla f|^{2p-4} |\langle \nabla \phi, \nabla f \rangle|^2 d\mu \\ &\geq \int_M |\nabla f|^{2p-4} S \langle \nabla f, \nabla f \rangle d\mu + \alpha \int_M |\nabla f|^{2p-4} \nabla \phi \otimes \nabla \phi \langle \nabla f, \nabla f \rangle d\mu \\ &\quad + \int_M |\nabla f|^{2p-4} \nabla^2 \phi \langle \nabla f, \nabla f \rangle d\mu. \end{aligned} \quad (3.10)$$

Choosing $s := \frac{1}{\alpha_n}$, $\alpha \geq \alpha_n > 0$, we have

$$1 - \frac{1}{n(1+s)} = \frac{n(1+\alpha_n) - \alpha_n}{n(1+\alpha_n)} \quad \text{and} \quad \frac{1}{ns} = \frac{\alpha_n}{n},$$

and observe that

$$\alpha \int_M |\nabla f|^{2p-4} \nabla \phi \otimes \nabla \phi \langle \nabla f, \nabla f \rangle d\mu \geq \frac{\alpha_n}{n} \int_M |\nabla f|^{2p-4} |\langle \nabla \phi, \nabla f \rangle|^2 d\mu$$

by identifying $\nabla \phi \otimes \nabla \phi \langle \nabla f, \nabla f \rangle$ with $|\langle \nabla \phi, \nabla f \rangle|^2$. Hence (3.10) reads

$$\begin{aligned} &\left(\frac{n(1+\alpha_n) - \alpha_n}{n(1+\alpha_n)}\right) \lambda_1^2 \int_M |f|^{2p-2} d\mu \geq \int_M |\nabla f|^{2p-4} S \langle \nabla f, \nabla f \rangle d\mu \\ &\quad + \int_M |\nabla f|^{2p-4} \nabla^2 \phi \langle \nabla f, \nabla f \rangle d\mu. \end{aligned} \quad (3.11)$$

Since ϕ solves the heat equation, we observe that $|\nabla^2 \phi| \geq \frac{1}{\sqrt{n}} |\Delta \phi| = \frac{1}{\sqrt{n}} |\phi_t|$. Using the condition $\mathcal{S}_{ij} - \beta S g \geq 0$, (3.11) implies

$$\begin{aligned} &\left(\frac{n(1+\alpha_n) - \alpha_n}{n(1+\alpha_n)}\right) \lambda_1^2 \int_M |f|^{2p-2} d\mu \\ &\geq \beta \int_M S |\nabla f|^{2p-2} d\mu \frac{1}{\sqrt{n}} \min_M |\phi_t| \int_M |\nabla f|^{2p-2} d\mu \\ &\geq \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t|\right) \int_M |\nabla f|^{2p-2} d\mu. \end{aligned} \quad (3.12)$$

Multiplying both sides of (1.5) by the quantity $|f|^{p-2} f$ and integrating over M , then using integration by parts formulas, we arrive at

$$\lambda_1 \int_M |\nabla f|^{2p-2} d\mu = (p-1) \int_M |\nabla f|^p |f|^{p-2} d\mu.$$

Applying the Hölder inequality for any $p > 2$, we have

$$\lambda_1 \int_M |f|^{2p-2} d\mu \leq (p-1) \left(\int_M |\nabla f|^{2p-2} d\mu \right)^{\frac{p}{2p-2}} \left(\int_M |f|^{2p-2} d\mu \right)^{\frac{p-2}{2p-2}},$$

hence

$$\int_M |\nabla f|^{2p-2} d\mu \geq \left(\frac{\lambda_1}{p-1} \right)^{\frac{2p-2}{p}} \int_M |f|^{2p-2} d\mu. \quad (3.13)$$

Putting (3.13) into (3.12), we arrive at

$$\begin{aligned} & \left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)} \right) \lambda_1^2 \int_M |f|^{2p-2} d\mu \\ & \geq \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t| \right) \left(\frac{\lambda_1}{p-1} \right)^{\frac{2p-2}{p}} \int_M |f|^{2p-2} d\mu. \end{aligned} \quad (3.14)$$

For $p > 2$, we can conclude that

$$\left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)} \right) \lambda_1^2 \geq \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t| \right) \left(\frac{\lambda_1}{p-1} \right)^{\frac{2p-2}{p}} \quad (3.15)$$

and then

$$\left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)} \right) \lambda_1^{\frac{2}{p}} \geq \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t| \right) \left(\frac{1}{p-1} \right)^{\frac{2p-2}{p}}, \quad (3.16)$$

which finally implies

$$\lambda_1(t) \geq \left[\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)-\alpha_n} \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t| \right) \right]^{\frac{p}{2}} \cdot (p-1)^{1-p}. \quad (3.17)$$

Since $S_{\min}(t) \rightarrow +\infty$ as $t \rightarrow T$ and $\min_M |\phi_t|$ is finite, $\lim_{t \nearrow T} = +\infty$. This completes the proof of the theorem. \square

Remark 3.3 The above result also holds for the case $p = 2$. Indeed, (3.14) reduces to

$$\begin{aligned} & \left(\frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)} \right) \lambda_1^2 \int_M |f|^2 d\mu \\ & \geq \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t| \right) \left(\frac{\lambda_1}{p-1} \right) \int_M |f|^2 d\mu \end{aligned} \quad (3.18)$$

for $p = 2$, and consequently,

$$\lambda_1(t) \geq \frac{n(1+\alpha_n)-\alpha_n}{n(1+\alpha_n)-\alpha_n} \left(\beta S_{\min}(t) + \frac{1}{\sqrt{n}} \min_M |\phi_t| \right). \quad (3.19)$$

Then, Theorem 3.2 reduces to [5, Theorem 2.5].

4 The spectrum on almost Ricci-harmonic solitons

In this section, we study the behaviour of the evolving spectrum on a class of self-similar solutions called almost Ricci-harmonic solitons. In what follows, we assume that soliton potential function f is the eigenfunction corresponding to the first nonzero eigenvalue λ_1 of the weighted p -Laplacian with $p = 2$. Our main result in this section is the following.

Proposition 4.1 *Suppose $(M, g(t), \phi(t), d\mu)$, $t \in [0, T]$, $T < \infty$ solves (1.1) and (g, ϕ, f, σ) is either a gradient shrinking or steady almost Ricci-harmonic soliton with soliton function σ satisfying $\langle \nabla \sigma, \nabla f \rangle \geq 0$ and $\Delta \sigma \leq 0$. Let $\lambda_{1,\phi}(t)$ be the first eigenvalue of Δ_ϕ with f being the associated eigenfunction. Then*

$$\lim_{t \rightarrow T} \lambda_{1,\phi}(t) = +\infty,$$

where $S_{ij} - \beta S g_{ij} \geq 0$, $\beta > 0$.

Before we give the proof of the above proposition, we discuss some basic formulas that will be useful in the proof.

Lemma 4.2 *Let (g, ϕ, f, σ) be a gradient almost $(RH)_\alpha$ soliton, then the following equations hold:*

$$\frac{1}{2} \nabla S = Sc(\nabla f, \cdot) + (n-1) \nabla \sigma, \quad (4.1)$$

$$\begin{aligned} \frac{1}{2} \int_M (\Delta f)^2 d\mu &= \int_M \left[Sc(\nabla f, \nabla f) + \frac{n-2}{2} \langle \nabla \sigma, \nabla f \rangle \right] d\mu \\ &\quad - \frac{1}{2} \int_M (S - n\sigma) \langle \nabla \phi, \nabla f \rangle d\mu. \end{aligned} \quad (4.2)$$

Proof The proofs of the above formulas follow from standard computation. For formula (4.1), see [6, Proposition 3.1]. The proof for formula (4.2) is given here. Taking the metric trace of the first equation in (1.3), we have

$$S + \Delta f = n\sigma.$$

Therefore,

$$\begin{aligned} \int_M (\Delta f)^2 d\mu &= \int_M (n\sigma - S) \Delta f d\mu \\ &= n \int_M \sigma \Delta f e^{-\phi} dv - \int_M S \Delta f e^{-\phi} dv \\ &= \int_M \langle \nabla f, \nabla S \rangle d\mu - \int_M \langle \nabla \phi, \nabla f \rangle (S - n\sigma) d\mu \\ &\quad - n \int_M \langle \nabla \sigma, \nabla f \rangle d\mu, \end{aligned}$$

where we have used integration by parts. Using formula (4.1), we have

$$\begin{aligned}\int_M (\Delta f)^2 d\mu &= 2 \int_M \left[Sc(\nabla f, \nabla f) + (n-1) \langle \nabla \sigma, \nabla f \rangle \right] d\mu \\ &\quad - \int_M \langle \nabla \phi, \nabla f \rangle (S - n\sigma) d\mu - n \int_M \langle \nabla \sigma, \nabla f \rangle d\mu \\ &= 2 \int_M \left[Sc(\nabla f, \nabla f) + \frac{n-2}{2} \langle \nabla \sigma, \nabla f \rangle \right] d\mu - \int_M \langle \nabla \phi, \nabla f \rangle (S - n\sigma) d\mu. \quad \square\end{aligned}$$

Another fact that we need is [6, Theorem 2.1], where we have proved that if (g, ϕ, f, σ) is a gradient almost Ricci-harmonic soliton, then it is shrinking, provided $0 \leq S_{\min} \leq n\sigma^*$ with $0 < \sigma \leq \sigma^*$, and it is steady, provided $S_{\min} = 0$ with $\sigma = 0$ under the condition $\Delta\sigma \leq 0$, where $\sigma^* := \sup_M \sigma$.

We are now set to give the proof of the proposition.

Proof of Proposition 4.1 By formula (4.2) of Lemma 4.2, we have for gradient steady almost Ricci-harmonic soliton

$$\frac{1}{2} \int_M (\Delta f)^2 d\mu = \int_M \left[Sc(\nabla f, \nabla f) + \frac{n-2}{2} \langle \nabla \sigma, \nabla f \rangle \right] d\mu, \quad (4.3)$$

since $S = S_{\min} = 0$ and $\sigma = 0$. Similarly, for gradient shrinking almost Ricci-harmonic soliton,

$$\frac{1}{2} \int_M (\Delta f)^2 d\mu \geq \int_M \left[Sc(\nabla f, \nabla f) + \frac{n-2}{2} \langle \nabla \sigma, \nabla f \rangle \right] d\mu. \quad (4.4)$$

Using an elementary inequality for $q \geq 1$, we have

$$\begin{aligned}(\Delta f)^2 &= (\Delta_\phi f + \langle \nabla \phi, \nabla f \rangle)^2 \\ &= (\Delta_\phi f)^2 + |\langle \nabla \phi, \nabla f \rangle|^2 + 2\Delta_\phi f \langle \nabla \phi, \nabla f \rangle \\ &\leq (\Delta_\phi f)^2 + |\langle \nabla \phi, \nabla f \rangle|^2 + \frac{1}{q} (\Delta_\phi f)^2 - q |\langle \nabla \phi, \nabla f \rangle|^2 \\ &= \left(1 + \frac{1}{q}\right) (\Delta_\phi f)^2 + (1-q) |\langle \nabla \phi, \nabla f \rangle|^2.\end{aligned}$$

Combining the above inequality with the condition $\langle \nabla \sigma, \nabla f \rangle \geq 0$ (or $\Delta\sigma \leq 0$) in (4.3) (steady case) or (4.4) (shrinking case), we have

$$\begin{aligned}\left(1 + \frac{1}{q}\right) \int_M (\Delta_\phi f)^2 d\mu &\geq \int_M (\Delta f)^2 d\mu + (q-1) \int_M |\langle \nabla \phi, \nabla f \rangle|^2 d\mu \\ &\geq 2 \int_M \left[Sc(\nabla f, \nabla f) + \frac{n-2}{2} \langle \nabla \sigma, \nabla f \rangle \right] d\mu \\ &\quad + (q-1) \int_M |\langle \nabla \phi, \nabla f \rangle|^2 d\mu \\ &\geq 2\beta \int_M S |\nabla f|^2 d\mu,\end{aligned}$$

since $q \geq 1$ and $S_{ij} - \beta S g_{ij} \geq 0$.

Using $\Delta_\phi f = -\lambda_{1,\phi} f$, $\lambda_{1,\phi} \int_M f^2 d\mu = \int_M |\nabla f|^2 d\mu$ and $\int_M f^2 d\mu = 1$, we arrive at

$$\lambda_{1,\phi}(t) \geq \frac{2q}{1+q} \beta \max_M S_{\min}(t), \quad (4.5)$$

and, by taking the limit as t grows to maximal time T , with the fact $S_{\min}(t) \rightarrow +\infty$ as $t \rightarrow T$, we obtain the desired result. \square

Remark 4.3 In the two cases of solitons discussed above, the quantity

$$\int_M \left[Sc(\nabla f, \nabla f) + \frac{n-2}{2} \langle \nabla \sigma, \nabla f \rangle \right] d\mu$$

is positive. If it were nonpositive then the soliton had to be trivial, in which case f would be constant. For instance, in [6, Theorem 2.3], we have shown that compact gradient almost Ricci-harmonic soliton is trivial for $n \geq 3$ if

$$\int_M [Sc(\nabla f, \nabla f) + (n-2) \langle \nabla \sigma, \nabla f \rangle] dv \leq 0.$$

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Authors' contributions

All authors contributed equally to the writing of this paper. The main idea of this paper was proposed by AA. All authors read and approved the final manuscript.

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References

- Abolarinwa, A.: Evolution and monotonicity of the first eigenvalue of p -Laplacian under the Ricci-harmonic flow. *J. Appl. Anal.* **21**(2), 147–160 (2015)
- Abolarinwa, A.: Gradient estimates for a nonlinear parabolic equation with potential under geometric flow. *Electron. J. Differ. Equ.* **2015**, 12 (2015)
- Abolarinwa, A.: Differential Harnack and logarithmic Sobolev inequalities along Ricci-harmonic map flow. *Pac. J. Math.* **278**(2), 257–290 (2015)
- Abolarinwa, A., Apata, T.: L^p -Hardy–Rellich and uncertainty principle inequalities on the sphere. *Adv. Geom.* **3**(4), 745–762 (2018)
- Abolarinwa, A.: Eigenvalues of weighted-Laplacian under the extended Ricci flow. *Adv. Geom.* **19**(1), 131–143 (2019)
- Abolarinwa, A.: Basic structural equations for almost Ricci-harmonic solitons and applications. *Differ. Geom. Dyn. Syst.* **21**, 1–13 (2019)
- Abolarinwa, A.: Gap theorems for compact almost Ricci-harmonic solitons. *Int. J. Math.* **30**(08), 1950040 (2019)
- Abolarinwa, A., Adebimpe, O., Bakare, E.A.: Monotonicity formulas for the first eigenvalue of the weighted p -Laplacian under the Ricci-harmonic flow. *J. Inequal. Appl.* **2019**, 10 (2019)
- Abolarinwa, A., Oladejo, N., Salawu, S.: Notes on entropy formulas and solitons for the Ricci-harmonic flow. *Bull. Iran. Math. Soc.* **45**(4), 1177–1192 (2019)

10. Cao, H.D.: Geometry of Ricci solitons. *Chin. Ann. Math.* **27B**(2), 121–142 (2006)
11. Cao, X.: Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator. *Math. Ann.* **337**(2), 435–442 (2007)
12. Chow, B., Knopf, D.: *The Ricci Flow: An Introduction*. AMS, Providence (2004)
13. Chow, B., Lu, P., Ni, L.: *Hamilton's Ricci Flow: An Introduction*. AMS, Providence (2006)
14. Di Cerbo, L.F.: Eigenvalues of the Laplacian under the Ricci flow. *Rend. Mat. Appl.* **27**(7), 183–195 (2007)
15. Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifold. *Am. J. Math.* **86**, 109–160 (1964)
16. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. *J. Differ. Geom.* **17**(2), 253–306 (1982)
17. Kotschwar, B., Ni, L.: Gradient estimate for p -harmonic functions, $1/H$ flow and an entropy formula. *Ann. Sci. Éc. Norm. Supér.* **42**(4), 1–36 (2009)
18. Li, J.-F.: Eigenvalues and energy functionals with monotonicity formulae under Ricci flow. *Math. Ann.* **338**(4), 927–946 (2007)
19. Li, Y.: Long time existence of Ricci-harmonic flow. *Front. Math. China* **11**(5), 1313–1334 (2016)
20. List, B.: Evolution of an extended Ricci flow system. *Commun. Anal. Geom.* **16**(5), 1007–1048 (2008)
21. Müller, R.: Ricci flow coupled with harmonic map flow. *Ann. Sci. Éc. Norm. Supér.* **4**(45), 101–142 (2012)
22. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. [arXiv:math/0211159](https://arxiv.org/abs/math/0211159)
23. Taheri, A.: *Function Spaces and Partial Differential Equations I & II*. Oxford Lecture Series in Mathematics and Its Applications, vol. 40 & 41. Oxford University Press, London (2015)
24. Wang, L.F.: Eigenvalue estimate for the weighted p -Laplacian. *Ann. Math.* **191**, 539–550 (2012)
25. Wang, Y.-Z., Li, H.-Q.: Lower bound estimates for the first eigenvalue of the weighted p -Laplacian on smooth metric measure spaces. *Differ. Geom. Appl.* **45**, 23–42 (2016)
26. Zeng, F., He, Q., Chen, B.: Monotonicity of eigenvalues of geometric operators along the Ricci–Bourguignon flow. *Pac. J. Math.* **296**(1), 1–2 (2018). [arxiv:1512.08158v1](https://arxiv.org/abs/1512.08158v1)

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