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On HT-convexity and Hadamard-type inequalities

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Abstract

In the paper, the authors define a new notion of "HT-convex function", present some Hadamard-type inequalities for the new class of HT-convex functions and for the product of any two HT-convex functions, and derive some inequalities for the arithmetic mean and the p-logarithmic mean. These results generalize corresponding ones for HA-convex functions and MT-convex functions.

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1 Introduction

The classical convexity and concavity of functions are two fundamental notions in mathematics. Mathematicians and scientists can see them in research papers, monographs, and textbooks devoted to the theory of convex analysis [13]. The origin of the theory of convex functions is generally attributed to Jensen [10], although he was not the first person to deal with such functions. Among Jensen's predecessors, there were Hermite [6], Hölder [7], and Stolz [19], to name a few. The well-known book [5] played an indispensable role in the popularization of the theory of convex functions.

The famous Hermite–Hadamard integral inequality in Theorem 2.1 for convex functions below is the first fundamental conclusion for convex functions and has been attracting a lot of interest from mathematicians and other scientists. In recent years, the Hermite–Hadamard integral inequality has been the subject of very active research. Various improvements, generalizations, and variants of this inequality can be found in the papers [1, 3, 8, 9, 11, 12, 14, 16, 20, 23, 24, 27, 28, 32] and closely related references therein. In [15], the late Pachpatte established some Hadamard-type inequalities for the product of two convex functions. Alternative Hadamard-type inequalities for the product of two convex functions were also established in the papers [25, 30, 31]. In [22], the Hermite–Hadamard inequality was applied to generalize and refine Young's integral inequality in terms of higher order derivatives.

Nowadays, many mathematicians have devoted their efforts to generalizations, refinements, counterparts, and extensions of the convexity of functions for adapting to other geometries of the domain and/or for disclosing other laws of comparison of mathematical means. In [1, 4, 8], by replacing the weighted arithmetic means with the weighted harmonic means, the notion of HA-convex functions was introduced. In [21], Tunç and Yildrim defined the MT-convexity and obtained some new Hadamard-type inequalities for MT-convex functions. The theory of various classes of convex functions is similar to that of classical convex functions. Some inequalities are easier to state using these new convex functions and some are easier to state using the classical convex functions. In many cases the inequalities based on these new convexity notions are better than direct applications of inequalities of the classical convexity. Consequently, there is a strong interest in investigating different classes of convex functions.

In Sect. 2, we will mathematically and technically quote some definitions of several classes of convex functions and some inequalities of Hadamard-type. In Sect. 3, by using the weighted arithmetic and harmonic means, we will introduce the notion of "HT-convex functions", which is a generalization of the HA-convex functions defined in [1, 4, 8]. In Sect. 4, we will present some Hadamard-type inequalities for HT-convex functions and for the product of two HT-convex functions. In Sect. 5, we will apply newly-established inequalities to derive some inequalities for the arithmetic mean and the p-logarithmic mean.

2 Preliminaries

In this section, we will mathematically quote some preliminary notations, definitions, and known results about several classes of convex functions and the Hadamard-type inequalities.

Definition 2.1 ([2]) A function Q defined on a nonempty interval $J \subseteq \mathbb{R}$ is said to be convex if

$$Q\bigl(\lambda\tau + (1-\lambda)\mu\bigr) \le \lambda Q(\tau) + (1-\lambda)Q(\mu)$$

is true for all $\tau, \mu \in J$ and $\lambda \in [0,1]$.

Theorem 2.1 ([17]) *Let Q be a convex function defined on a nonempty interval $J \subseteq \mathbb{R}$ and $\tau, \mu \in J$ with $\tau < \mu$. Then*

$$Q\left(\frac{\tau+\mu}{2}\right) \le \frac{1}{\mu-\tau}\int_{\tau}^{\mu} Q(x)\,\mathrm{d}x \le \frac{Q(\tau)+Q(\mu)}{2}. \tag{1}$$

If Q is concave, the above double inequality holds reversed.

In the mathematics community, the double inequality (1) is known as the Hermite–Hadamard integral inequality.

Theorem 2.2 ([15, Theorem 1]) *Suppose that P, Q are two positive and convex functions on a nonempty interval $J \subseteq \mathbb{R}$, $\tau, \mu \in J$ with $\tau < \mu$, and the product of P and Q is Lebesgue* *integrable on* $[\tau,\mu]$. *Then*

$$2P\left(\frac{\tau+\mu}{2}\right)Q\left(\frac{\tau+\mu}{2}\right)-\frac{1}{6}M(\tau,\mu)-\frac{1}{3}N(\tau,\mu) \\ \leq \frac{1}{\mu-\tau}\int_{\tau}^{\mu}P(x)Q(x)\,\mathrm{d}x \leq \frac{1}{3}M(\tau,\mu)+\frac{1}{6}N(\tau,\mu), \tag{2}$$

where

$$M(\tau,\mu)=P(\tau)Q(\tau)+P(\mu)Q(\mu) \quad \textit{and} \quad N(\tau,\mu)=Q(\tau)Q(\mu)+P(\mu)Q(\tau). \tag{3}$$

Definition 2.2 ([1, 4, 8]) Let $J\subseteq\mathbb{R}\setminus\{0\}$ be a nonempty interval. A function $Q:J\to\mathbb{R}$ is said to be HA-convex, denoted by $Q\in\mathrm{HA}(J)$, if the inequality

$$Q\left(\frac{\tau\mu}{\lambda\tau+(1-\lambda)\mu}\right)\leq\lambda Q(\tau)+(1-\lambda)f(\mu)$$

holds for all $\tau,\mu\in J$ and $\lambda\in[0,1]$.

Definition 2.3 ([21]) A nonnegative function Q on a nonempty interval J is said to be MT-convex, denoted by $Q\in\mathrm{MT}(J)$, if

$$Q\big(\lambda\tau+(1-\lambda)\mu\big)\leq\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}Q(\tau)+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}Q(\mu)$$

is valid for all $\tau,\mu\in J$ and $\lambda\in(0,1)$.

Theorem 2.3 ([21]) *Let* $Q\in\mathrm{MT}(J)$, $\tau,\mu\in J$ *with* $\tau<\mu$, *and* Q *be Lebesgue integrable on* $[\tau,\mu]$. *Then*

$$\frac{\pi}{2}Q\left(\frac{\tau+\mu}{2}\right)\leq Q(\tau)+Q(\mu), \qquad Q\left(\frac{\tau+\mu}{2}\right)\leq\frac{1}{\mu-\tau}\int_{\tau}^{\mu}Q(x)\,\mathrm{d}x,$$

and

$$\frac{2}{\mu-\tau}\int_{\tau}^{\mu}\nu(x)Q(x)\,\mathrm{d}x\leq\frac{Q(\tau)+Q(\mu)}{2},$$

where

$$\nu(x)=\frac{2\sqrt{(\mu-x)(x-\tau)}}{\mu-\tau}, \quad x\in[\tau,\mu].$$

Lemma 2.1 ([4]) *Suppose that* $J\subseteq(0,\infty)$ *is a nonempty interval and* $\tau,\mu\in J$ *with* $\tau<\mu$. *Let* $Q:[\tau,\mu]\to\mathbb{R}$ *be Lebesgue integrable on* $[\tau,\mu]$. *Then, for* $\lambda\in[0,1]$,

$$\int_0^1 Q\left(\frac{\tau\mu}{(1-s)\tau+s\mu}\right)\mathrm{d}s=(1-\lambda)\int_0^1 Q\left(\frac{\tau\mu}{(1-s)[(1-\lambda)\tau+\lambda\mu]+s\mu}\right)\mathrm{d}s \\ +\lambda\int_0^1 Q\left(\frac{\tau\mu}{(1-s)\tau+s[(1-\lambda)\tau+\lambda\mu]}\right)\mathrm{d}s.$$

Definition 2.4 ([18]) Two functions $P, Q : J \subseteq \mathbb{R} \to \mathbb{R}$ are said to be similarly ordered if

$$\big[P(\tau) - P(\mu)\big]\big[Q(\tau) - Q(\mu)\big] \geq 0, \quad \tau, \mu \in J.$$

3 HT-convexity

We now define the concept of the HT-convexity and give several basic properties.

Definition 3.1 Let $J \subseteq \mathbb{R} \setminus \{0\}$ be a nonempty interval. A function Q is called HT-convex on J, denoted by $Q \in \mathrm{HT}(J)$, if the inequality

$$Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} Q(\mu) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} Q(\tau) \tag{4}$$

holds for all $\tau, \mu \in J$ and $\lambda \in (0,1)$. If the inequality in (4) is reversed, then Q is called an HT-concave function.

Remark 3.1 Taking $\lambda = \frac{1}{2}$ in inequality (4) yields

$$Q\left(\frac{2\tau\mu}{\tau + \mu}\right) \leq \frac{Q(\tau) + Q(\mu)}{2}.$$

Remark 3.2 Suppose that $J \subseteq \mathbb{R} \setminus \{0\}$ is a nonempty interval and $Q : J \to \mathbb{R}$ is an HT-convex function.

1 If $J \subseteq (0, \infty)$ and Q is an MT-convex and nondecreasing function, then Q is HT-convex.
2 If $J \subseteq (0, \infty)$ and Q is an HT-convex and nonincreasing function, then Q is MT-convex.
3 If $J \subseteq (-\infty, 0)$ and Q is an MT-convex and nonincreasing function, then Q is HT-convex.
4 If $J \subseteq (-\infty, 0)$ and Q is an HT-convex and nondecreasing function, then Q is MT-convex.

Proposition 3.1 *Suppose that $J \subseteq (0, \infty)$ is a nonempty interval. For $\tau, \mu \in J$ with $\tau < \mu$, if $P : [\frac{1}{\mu}, \frac{1}{\tau}] \to \mathbb{R}$ is MT-convex, then $Q : [\tau, \mu] \to \mathbb{R}$, $Q(t) = P(\frac{1}{t})$, is HT-convex.*

Proof Let $s, t \in [\tau, \mu]$ and $\lambda \in (0,1)$. Then

$$\begin{aligned} Q\left(\frac{st}{t\lambda + (1-\lambda)t}\right) &= Q\left(\frac{1}{\lambda\frac{1}{t} + (1-\lambda)\frac{1}{s}}\right) = P\left(\lambda\frac{1}{t} + (1-\lambda)\frac{1}{s}\right) \\ &\leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} P\left(\frac{1}{t}\right) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} P\left(\frac{1}{s}\right) = \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} Q(t) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} Q(s), \end{aligned}$$

which shows that Q is HT-convex on $[\tau, \mu]$. □

Proposition 3.2 *All nonnegative HA-convex functions are HT-convex.*

Proof Suppose that Q is HA-convex. By the fact that $\lambda \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}$ and $1-\lambda \leq \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}$, it is easy to obtain

$$Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right) \leq \lambda Q(\mu) + (1-\lambda)Q(\tau) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}Q(\mu) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}Q(\tau),$$

where $\tau, \mu \in J \subseteq \mathbb{R} \setminus \{0\}$ and $\lambda \in (0,1)$. This means that each HA-convex function is HT-convex. □

4 Hadamard-type inequalities for HT-convex functions

Now we are in a position to establish some Hadamard-type inequalities for HT-convex functions.

Theorem 4.1 *Suppose that $J \subseteq \mathbb{R} \setminus \{0\}$ is a nonempty interval, $Q \in \mathrm{HT}(J)$, $\tau, \mu \in J$ with $\tau < \mu$, and Q is Lebesgue integrable on $[\tau, \mu]$. Then*

$$Q\left(\frac{2\tau\mu}{\tau + \mu}\right) \leq \frac{\tau\mu}{\mu - \tau}\int_\tau^\mu \frac{Q(x)}{x^2}\,\mathrm{d}x \leq \frac{\pi}{4}\big[Q(\tau) + Q(\mu)\big]. \tag{5}$$

Proof Because Q is an HT-convex function, for $\tau, \mu \in J$ with $\tau < \mu$ and $\lambda \in (0,1)$, we have

$$Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right) \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}Q(\mu) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}Q(\tau). \tag{6}$$

Integrating the above inequality over $\lambda \in (0,1)$ and replacing $\frac{\tau\mu}{\lambda\tau+(1-\lambda)\mu}$ by x yield easily the right inequality of (5).

For any $\lambda \in (0,1)$, we have

$$Q\left(\frac{2\tau\mu}{\tau + \mu}\right) \leq \frac{1}{2}\left[Q\left(\frac{\tau\mu}{\lambda\mu + (1-\lambda)\tau}\right) + Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)\right].$$

Integrating the above inequality with respect to $\lambda \in [0,1]$ gives

$$Q\left(\frac{2\tau\mu}{\tau + \mu}\right) \leq \frac{1}{2}\left[\int_0^1 Q\left(\frac{\tau\mu}{\lambda\mu + (1-\lambda)\tau}\right)\mathrm{d}\lambda + \int_0^1 Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)\mathrm{d}\lambda\right].$$

Making use of the fact that

$$\int_0^1 Q\left(\frac{\tau\mu}{\lambda\mu + (1-\lambda)\tau}\right)\mathrm{d}\lambda = \int_0^1 Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)\mathrm{d}\lambda$$

and replacing $\frac{\tau\mu}{\lambda\tau+(1-\lambda)\mu}$ by x result in the left inequality of (5). □

Multiplying both sides of inequality (6) by $2\sqrt{t(1-t)}$ for $t \in (0,1)$ and simultaneously using the HT-convexity of f, we easily obtain the following corollaries.

Corollary 4.1 *Suppose that $J \subseteq \mathbb{R} \setminus \{0\}$ is a nonempty interval, $Q \in \mathrm{HT}(J)$, $\tau, \mu \in J$ with $\tau < \mu$, and Q is Lebesgue integrable on $[\tau, \mu]$. Then*

$$Q\left(\frac{2}{\frac{1}{\mu} + \frac{1}{\tau}}\right) \leq \int_0^1 Q\left(\frac{1}{\lambda\frac{1}{\tau} + (1-\lambda)\frac{1}{\mu}}\right)\mathrm{d}\lambda \leq \frac{\pi}{4}\big[Q(\tau) + Q(\mu)\big]. \tag{7}$$

Corollary 4.2 *Suppose that $J \subseteq \mathbb{R} \setminus \{0\}$ is a nonempty interval, $Q \in \mathrm{HT}(J)$, $\tau, \mu \in J$ with $\tau < \mu$, and Q is Lebesgue integrable on $[\tau, \mu]$. Then*

$$\frac{\pi}{4} Q\left(\frac{2\tau\mu}{\tau+\mu}\right) \le \frac{\tau\mu}{\mu-\tau} \int_\tau^\mu \tau(x) \frac{Q(x)}{x^2}\,\mathrm{d}x \le \frac{Q(\tau)+Q(\mu)}{2},$$

where

$$\tau(x) = 2\frac{\sqrt{\tau\mu(x-\tau)(\mu-x)}}{(\mu-\tau)x}. \tag{8}$$

Corollary 4.3 *Suppose that $J \subseteq \mathbb{R} \setminus \{0\}$ is a nonempty interval, $Q \in \mathrm{HT}(J)$, $\tau, \mu \in J$ with $\tau < \mu$, and Q is Lebesgue integrable on $[\tau, \mu]$. Then*

$$\frac{\pi}{2} Q\left(\frac{2\tau\mu}{\tau+\mu}\right) \le Q(\tau) + Q(\mu).$$

Theorem 4.2 *Suppose that $J \subseteq (0,\infty)$ is a nonempty interval, $Q \in \mathrm{HT}(J)$, $\tau, \mu \in J$ with $\tau < \mu$, and Q is Lebesgue integrable on $[\tau, \mu]$. Then, for $\lambda \in (0,1)$,*

$$\begin{aligned}
2\sqrt{\lambda(1-\lambda)}Q\left(\frac{2\tau\mu}{\tau+\mu}\right) &\le (1-\lambda)Q\left(\frac{2\tau\mu}{(1-\lambda)\tau+(1+\lambda)\mu}\right) \\
&\quad + \lambda Q\left(\frac{2\tau\mu}{(2-\lambda)\tau+\lambda\mu}\right) \\
&\le \frac{\tau\mu}{\mu-\tau}\int_\tau^\mu \frac{Q(x)}{x^2}\,\mathrm{d}x \\
&\le \frac{\pi}{4}\left[(1-\lambda)Q(\tau)+\lambda Q(\mu)+Q\left(\frac{\tau\mu}{(1-\lambda)\tau+\lambda\mu}\right)\right] \\
&\le \frac{\pi}{8\sqrt{\lambda(1-\lambda)}}\big[Q(\tau)+Q(\mu)\big].
\end{aligned}$$

Proof Using inequality (7), for $\lambda \in (0,1)$, we have

$$\begin{aligned}
Q\left(\frac{2\tau\mu}{(1-\lambda)\tau+(1+\lambda)\mu}\right) &= Q\left(\frac{2}{(1-\lambda)\frac{1}{\mu}+\lambda\frac{1}{\tau}+\frac{1}{\tau}}\right) \\
&\le \int_0^1 Q\left(\frac{1}{(1-s)[(1-\lambda)\frac{1}{\mu}+\lambda\frac{1}{\tau}]+s\frac{1}{\tau}}\right)\mathrm{d}s \\
&= \int_0^1 Q\left(\frac{\tau\mu}{(1-s)[(1-\lambda)\tau+\lambda\mu]+s\mu}\right)\mathrm{d}s \\
&\le \frac{\pi}{4}\left[Q\left(\frac{\tau\mu}{(1-\lambda)\tau+\lambda\mu}\right)+Q(\tau)\right]
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
Q\left(\frac{2\tau\mu}{(2-\lambda)\tau+\lambda\mu}\right) &= Q\left(\frac{2}{\frac{1}{\mu}+(1-\lambda)\frac{1}{\mu}+\lambda\frac{1}{\tau}}\right) \\
&\le \int_0^1 Q\left(\frac{1}{(1-s)\frac{1}{\mu}+s[(1-\lambda)\frac{1}{\mu}+\lambda\frac{1}{\tau}]}\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 Q\left(\frac{\tau\mu}{(1-s)\tau + s[(1-\lambda)\tau + \lambda\mu]}\right) \mathrm{d}s \\
&\leq \frac{\pi}{4}\left[Q(\mu) + Q\left(\frac{\tau\mu}{(1-\lambda)\tau + \lambda\mu}\right)\right]. \qquad (10)
\end{aligned}
$$

Multiplying both sides of (9) and (10) by $1-\lambda$ and λ, respectively, adding the obtained inequalities, and making use of Lemma 2.1, we arrive at

$$
\begin{aligned}
&(1-\lambda)Q\left(\frac{2\tau\mu}{(1-\lambda)\tau + (1+\lambda)\mu}\right) + \lambda Q\left(\frac{2\tau\mu}{(2-\lambda)\tau + \lambda\mu}\right) \\
&\quad\leq \int_0^1 Q\left(\frac{\tau\mu}{(1-\lambda)\tau + \lambda\mu}\right) \mathrm{d}\lambda \\
&\quad\leq \frac{\pi}{4}(1-\lambda)\left[Q\left(\frac{\tau\mu}{(1-\lambda)\tau + \lambda\mu}\right) + Q(\tau)\right] + \frac{\pi}{4}\lambda\left[Q(\mu) + Q\left(\frac{\tau\mu}{(1-\lambda)\tau + \lambda\mu}\right)\right] \\
&\quad= \frac{\pi}{4}\left[(1-\lambda)Q(\tau) + \lambda Q(\mu) + Q\left(\frac{\tau\mu}{(1-\lambda)\tau + \lambda\mu}\right)\right],
\end{aligned}
$$

which proves the second and third inequalities in (9).

By the HT-convexity of Q, we have

$$
\begin{aligned}
&(1-\lambda)Q\left(\frac{2\tau\mu}{(1-\lambda)\tau + (1+\lambda)\mu}\right) + \lambda Q\left(\frac{2\tau\mu}{(2-\lambda)\tau + \lambda\mu}\right) \\
&\quad= 2\sqrt{\lambda(1-\lambda)}\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}Q\left(\frac{2\tau\mu}{(1-\lambda)\tau + (1+\lambda)\mu}\right) \\
&\qquad + 2\sqrt{\lambda(1-\lambda)}\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}Q\left(\frac{2\tau\mu}{(2-\lambda)\tau + \lambda\mu}\right) \\
&\quad\geq 2\sqrt{\lambda(1-\lambda)}Q\left(\frac{2\tau\mu}{(1-\lambda)[(1-\lambda)\tau + (1+\lambda)\mu] + \lambda[(2-\lambda)\tau + \lambda\mu]}\right) \\
&\quad= 2\sqrt{\lambda(1-\lambda)}Q\left(\frac{2\tau\mu}{\tau + \mu}\right),
\end{aligned}
$$

which proves the first inequality in (10).

Similarly, we obtain

$$
\begin{aligned}
&\frac{\pi}{4}\left[(1-\lambda)Q(\tau) + \lambda Q(\mu) + Q\left(\frac{\tau\mu}{(1-\lambda)\tau + \lambda\mu}\right)\right] \\
&\quad\leq \frac{\pi}{4}\left[(1-\lambda)Q(\tau) + \lambda Q(\mu) + \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}Q(\tau) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}Q(\mu)\right] \\
&\quad\leq \frac{\pi}{4}\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\right)\left[Q(\tau) + Q(\mu)\right] \\
&\quad= \frac{\pi}{8\sqrt{\lambda(1-\lambda)}}\left[Q(\tau) + Q(\mu)\right],
\end{aligned}
$$

which proves the last inequality in (10). □

Remark 4.1 With the assumptions of Theorem 4.2, taking $\lambda = \frac{1}{2}$ leads to

$$Q\left(\frac{2\tau\mu}{\tau+\mu}\right) \le \frac{1}{2}\left[Q\left(\frac{4\tau\mu}{\tau+3\mu}\right) + Q\left(\frac{4\tau\mu}{3\tau+\mu}\right)\right] \le \frac{\tau\mu}{\mu-\tau}\int_\tau^\mu \frac{Q(x)}{x^2}\,\mathrm{d}x$$
$$\le \frac{\pi}{8}\left[Q(\tau) + Q(\mu) + 2Q\left(\frac{2\tau\mu}{\tau+\mu}\right)\right] \le \frac{\pi}{4}\left[Q(\tau) + Q(\mu)\right].$$

Theorem 4.3 *Suppose that $J \subseteq \mathbb{R}\setminus\{0\}$ is a nonempty interval, $P, Q \in \mathrm{HT}(J)$, $\tau, \mu \in J$ with $\tau < \mu$, and PQ is Lebesgue integrable on $[\tau,\mu]$. If P, Q are nonnegative, then*

$$\frac{\tau\mu}{\mu-\tau}\int_\tau^\mu \frac{\tau^2(x)}{x^4}P(x)Q(x)\,\mathrm{d}x \le \frac{2M(\tau,\mu) + N(\tau,\mu)}{6}, \tag{11}$$

where $\tau(x)$ is defined in (8) and $M(\tau,\mu)$ and $N(\tau,\mu)$ are defined in (3).

Proof Because $P, Q \in \mathrm{HT}(J)$, for $\lambda \in (0,1)$, we have

$$Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right) \le \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}Q(\mu) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}Q(\tau)$$

and

$$P\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right) \le \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}P(\mu) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}P(\tau).$$

Because P, Q are nonnegative, we obtain

$$Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)P\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)$$
$$\le \frac{\lambda Q(\mu)P(\mu)}{4(1-\lambda)} + \frac{(1-\lambda)Q(\tau)P(\tau)}{4\lambda} + \frac{Q(\tau)P(\mu) + Q(\mu)P(\tau)}{4},$$

that is,

$$\lambda(1-\lambda)Q\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)P\left(\frac{\tau\mu}{\lambda\tau + (1-\lambda)\mu}\right)$$
$$\le \frac{\lambda^2 Q(\mu)P(\mu) + (1-\lambda)^2 Q(\tau)P(\tau) + \lambda(1-\lambda)[Q(\tau)P(\mu) + Q(\mu)P(\tau)]}{4}. \tag{12}$$

Integrating the above inequality with respect to $\lambda \in [0,1]$ and replacing $\frac{\tau\mu}{\lambda\tau+(1-\lambda)\mu}$ by x result in (11). □

Remark 4.2 Choosing $\lambda = \frac{1}{2}$ in inequality (12), we obtain

$$Q\left(\frac{2\tau\mu}{\tau+\mu}\right)P\left(\frac{2\tau\mu}{\tau+\mu}\right)$$
$$\le \frac{Q(\tau)P(\tau) + Q(\mu)P(\mu) + Q(\tau)P(\mu) + Q(\mu)P(\tau)}{4}.$$

Corollary 4.4 *Under conditions of Theorem 4.3, if P, Q are similarly ordered, then*

$$\frac{\tau\mu}{\mu-\tau}\int_\tau^\mu \frac{\tau^2(x)}{x^4}P(x)Q(x)\,\mathrm{d}x \le \frac{Q(\tau)P(\tau)+Q(\mu)P(\mu)}{2},$$

where $\tau(x)$ is defined in (8).

Corollary 4.5 *Under conditions of Theorem 4.3, if P, Q are nonnegative, then*

$$\begin{aligned}&\frac{\tau\mu P(\mu)}{\mu-\tau}\int_\tau^\mu \frac{\mu(x-\tau)}{x(\mu-\tau)}\frac{\tau(x)Q(x)}{x^2}\,\mathrm{d}x+\frac{\tau\mu P(\tau)}{\mu-\tau}\int_\tau^\mu \frac{\tau(\mu-x)}{x(\mu-\tau)}\frac{\tau(x)Q(x)}{x^2}\,\mathrm{d}x\\&\quad+\frac{\tau\mu Q(\mu)}{\mu-\tau}\int_\tau^\mu \frac{\mu(x-\tau)}{x(\mu-\tau)}\frac{\tau(x)P(x)}{x^2}\,\mathrm{d}x+\frac{\tau\mu Q(\tau)}{\mu-\tau}\int_\tau^\mu \frac{\tau(\mu-x)}{x(\mu-\tau)}\frac{\tau(x)P(x)}{x^2}\,\mathrm{d}x\\&\le\frac{\tau\mu}{\mu-\tau}\int_\tau^\mu \frac{\tau^2(x)}{x^4}P(x)Q(x)\,\mathrm{d}x+\frac{2M(\tau,\mu)+N(\tau,\mu)}{24},\end{aligned}$$

where $\tau(x)$ is defined in (8) and $M(\tau,\mu)$ and $N(\tau,\mu)$ are defined in (3).

Corollary 4.6 *Under conditions of Corollary 4.5, if P, Q are similarly ordered, then*

$$\begin{aligned}&\frac{\tau\mu P(\mu)}{\mu-\tau}\int_\tau^\mu \frac{\mu(x-\tau)}{x(\mu-\tau)}\frac{\tau(x)Q(x)}{x^2}\,\mathrm{d}x+\frac{\tau\mu P(\tau)}{\mu-\tau}\int_\tau^\mu \frac{\tau(\mu-x)}{x(\mu-\tau)}\frac{\tau(x)Q(x)}{x^2}\,\mathrm{d}x\\&\quad+\frac{\tau\mu Q(\mu)}{\mu-\tau}\int_\tau^\mu \frac{\mu(x-\tau)}{x(\mu-\tau)}\frac{\tau(x)P(x)}{x^2}\,\mathrm{d}x+\frac{\tau\mu Q(\tau)}{\mu-\tau}\int_\tau^\mu \frac{\tau(\mu-x)}{x(\mu-\tau)}\frac{\tau(x)P(x)}{x^2}\,\mathrm{d}x\\&\le\frac{\tau\mu}{\mu-\tau}\int_\tau^\mu \frac{\tau^2(x)}{x^4}P(x)Q(x)\,\mathrm{d}x+\frac{Q(\tau)P(\tau)+Q(\mu)P(\mu)}{8},\end{aligned}$$

where $\tau(x)$ is defined in (8).

Theorem 4.4 *Suppose that $J\subseteq\mathbb{R}\setminus\{0\}$ is a nonempty interval, $P,Q\in\mathrm{HT}(J)$, $\tau,\mu\in J$ with $\tau<\mu$, and PQ is Lebesgue integrable on $[\tau,\mu]$. If P, Q are nonnegative, then*

$$Q\left(\frac{2\tau\mu}{\tau+\mu}\right)P\left(\frac{2\tau\mu}{\tau+\mu}\right)\le\frac{\pi}{16}\bigl[M(\tau,\mu)+N(\tau,\mu)\bigr],$$

where $M(\tau,\mu)$ and $N(\tau,\mu)$ are defined in (3).

Proof Because P, Q are nonnegative and HT-convex, for $\lambda\in(0,1)$,

$$\begin{aligned}Q\left(\frac{2\tau\mu}{\tau+\mu}\right)&=Q\left(\frac{2\tau\mu}{\lambda\tau+(1-\lambda)\mu+(1-\lambda)\tau+\lambda\mu}\right)\\&\le\frac{1}{2}\left[Q\left(\frac{\tau\mu}{\lambda\tau+(1-\lambda)\mu}\right)+Q\left(\frac{\tau\mu}{(1-\lambda)\tau+\lambda\mu}\right)\right]\\&\le\frac{1}{2}\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\right)\bigl[Q(\tau)+Q(\mu)\bigr].\end{aligned}\tag{13}$$

Similarly, we have

$$P\left(\frac{2\tau\mu}{\tau+\mu}\right)\le\frac{1}{2}\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\right)\bigl[P(\tau)+P(\mu)\bigr].\tag{14}$$

Multiplying (13) and (14) reveals

$$\begin{aligned} Q\left(\frac{2\tau\mu}{\tau+\mu}\right)P\left(\frac{2\tau\mu}{\tau+\mu}\right) &\leq \frac{1}{16}\left(\frac{\sqrt{\lambda}}{\sqrt{1-\lambda}}+\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}}\right)^2 \\ &\quad \times \left[Q(\tau)+Q(\mu)\right]\left[P(\tau)+P(\mu)\right] \\ &= \frac{1}{16}\frac{1}{\lambda(1-\lambda)}\left[Q(\tau)+Q(\mu)\right]\left[P(\tau)+P(\mu)\right]. \end{aligned} \tag{15}$$

Integrating inequality (15) with respect to $\lambda \in (0,1)$ leads to the stated result. □

Again using inequalities (13) and (14), we derive the following corollaries.

Corollary 4.7 *Under conditions of Theorem 4.4, if P, Q are similarly ordered, then*

$$Q\left(\frac{2\tau\mu}{\tau+\mu}\right)P\left(\frac{2\tau\mu}{\tau+\mu}\right) \leq \frac{\pi}{8}\left[Q(\tau)P(\tau)+Q(\mu)P(\mu)\right].$$

Corollary 4.8 *Under conditions of Theorem 4.4, if P, Q are nonnegative, then*

$$\begin{aligned} &Q\left(\frac{2\tau\mu}{\tau+\mu}\right)\left[P(\tau)+P(\mu)\right]+P\left(\frac{2\tau\mu}{\tau+\mu}\right)\left[Q(\tau)+Q(\mu)\right] \\ &\quad \leq \frac{16}{3\pi}Q\left(\frac{2\tau\mu}{\tau+\mu}\right)P\left(\frac{2\tau\mu}{\tau+\mu}\right)+\frac{2}{\pi}\left[Q(\tau)+Q(\mu)\right]\left[P(\tau)+P(\mu)\right]. \end{aligned}$$

Corollary 4.9 *Under conditions of Corollary 4.8, if P, Q are similarly ordered, then*

$$\begin{aligned} &Q\left(\frac{2\tau\mu}{\tau+\mu}\right)\left[P(\tau)+P(\mu)\right]+P\left(\frac{2\tau\mu}{\tau+\mu}\right)\left[Q(\tau)+Q(\mu)\right] \\ &\quad \leq \frac{16}{3\pi}Q\left(\frac{2\tau\mu}{\tau+\mu}\right)P\left(\frac{2\tau\mu}{\tau+\mu}\right)+\frac{4}{\pi}\left[Q(\tau)P(\tau)+Q(\mu)P(\mu)\right]. \end{aligned}$$

5 Applications to some special means

In this section, we will consider applications of our newly-established results to the following special means.

For real numbers $\tau,\mu > 0$, the arithmetic mean and the p-logarithmic mean are respectively defined [26, 29] by

$$A = A(\tau,\mu) = \frac{\tau+\mu}{2}$$

and

$$L_p = L_p(\tau,\mu) = \begin{cases} \left[\frac{\mu^{p+1}-\tau^{p+1}}{(p+1)(\mu-\tau)}\right]^{1/p}, & \tau \neq \mu; \\ \tau, & \tau = \mu. \end{cases}$$

For the HT-convex function $Q:(1,\infty) \to \mathbb{R}$, $Q(x) = \frac{1}{x^p}$ for $p \geq 1$, applying Theorem 4.2 and Corollary 4.1, we derive the following inequalities involving A and L_p.

Theorem 5.1 *Let* $1 < \tau < \mu$ *and* $p \geq 1$. *Then*

$$\frac{\pi}{4} A^p\left(\frac{1}{\tau}, \frac{1}{\mu}\right) \leq \frac{\pi}{4\tau^p} F\left(-p, \frac{3}{2}, 3, \frac{\mu-\tau}{\mu}\right) \leq A\left(\frac{1}{\tau^{p+1}}, \frac{1}{\mu^{p+1}}\right)$$

and

$$\begin{aligned} A^p\left(\frac{1}{\tau}, \frac{1}{\mu}\right) &\leq \frac{1}{2^{p+1}}\left[A^p\left(\frac{3}{\tau}, \frac{1}{\mu}\right) + A^p\left(\frac{1}{\tau}, \frac{3}{\mu}\right)\right] \leq L_p^p\left(\frac{1}{\tau}, \frac{1}{\mu}\right) \\ &\leq \frac{\pi}{4}\left[A\left(\frac{1}{\tau}, \frac{1}{\mu}\right) + A^p\left(\frac{1}{\tau}, \frac{1}{\mu}\right)\right] \leq \frac{\pi}{2} A\left(\frac{1}{\tau^{p+1}}, \frac{1}{\mu^{p+1}}\right), \end{aligned}$$

where $F(\alpha, \beta, \gamma, x)$ *is the hypergeometric function which can be represented by*

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-xt)^{\alpha}}\, \mathrm{d}t.$$

6 Conclusions

In [15] and [23], the HA- and MT-convexity were defined and some Hadamard-type inequalities were obtained. As a generalization of these two convexity notions a new notion of "HT-convex functions" is introduced in this paper, some Hadamard-type inequalities for the new class of HT-convex functions and for the product of any two HT-convex functions are established, and, as applications, some inequalities for the arithmetic mean and the p-logarithmic mean are derived.

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