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Estimation of unknown function of nonlinear weakly singular integral inequalities of Gronwall–Bellman–Pachpatte type

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Abstract

Some new nonlinear weakly singular integral inequalities of Gronwall–Bellman–Pachpatte type are given. The estimations of unknown functions are obtained by analysis techniques. These estimates are very significant tools in the study of differential-integral equations.

Keywords: Weakly singular integral inequalities; Discrete Jensen inequality; Hölder integral inequality; Ordinary differential equation

1 Introduction

The integral inequality provides a useful tool for investigating the existence, uniqueness, boundedness and other qualitative properties of the solutions of differential equations and integral equations and provides an explicit bound for unknown functions. Gronwall established the essential integral inequality in 1919 [1]:

If u is a continuous function defined on the interval $D = [\alpha, \alpha + h]$ and

$$0 \le u(t) \le \int_0^t \bigl[bu(s) + a\bigr]\,ds, \quad \forall t \in D,$$

where a and b are nonnegative constants. Then

$$0 \le u(t) \le (ah)\exp(bh), \quad \forall t \in D.$$

Bellman proved the Gronwall–Bellman inequality [2] in 1943, which was one of the most useful inequalities in the study of differential and integral equations:

If u and f are nonnegative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)\,ds, \quad t \in [a, b],$$

for some constant $c \geq 0$, then

$$u(t) \leq c \exp\left(\int_a^t f(s)\,ds\right), \quad t \in [a,b].$$

In the past few years, many scholars have extended the Gronwall–Bellman inequality and applied it to many aspects. There can be found many generalizations and analogs of it in various cases from literature (see [3–6]). In 2007, Jiang Fangcui and Meng Fanwei [3] investigated the estimation of the unknown function of the integral inequality:

$$u^p(t) \leq c + \int_0^t f(s)u^p(s)\,ds + \int_0^t h(s)u^q\big(\sigma(s)\big)\,ds, \quad \forall t \in [0,\infty).$$

In recent years, many researchers have made a great contribution to studying weakly singular integral inequalities and their applications (see [7, 8]). In [7], Xu Run and Meng Fanwei studied the following weakly singular integral inequality:

$$\begin{cases} u^p(t) \leq a(t) + \int_{t_0}^t (t-s)^{\beta-1}b(s)u^q(s)\,ds + \int_{t_0}^t (t-s)^{\beta-1}c(s)u^l(s-\gamma)\,ds, \\ \quad t \in [t_0,T) \subset R_+, \\ u(t) \leq \varphi(t), \quad t \in [t_0-\gamma, t_0). \end{cases}$$

Ma and Pečairé [8] considered the following nonlinear singular inequalities:

$$u^p(t) \leq a(t) + b(t)\int_{t_0}^t \big(t^\alpha - s^\alpha\big)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds, \quad t \geq 0.$$

In this paper, we extend certain results that were proved in [9, 10]. Abdeldaim and Yakout [9] obtained the explicit bound to the unknown function of the following integral inequalities:

$$u(t) \leq u_0 + \int_0^t f(s)u(s)\left[u(s) + \int_0^s g(\lambda)u(\lambda)\,d\lambda\right]^r ds, \quad \forall t \in [0,\infty), \tag{1}$$

$$u^n(t) \leq u_0 + \int_0^t f(s)u^n(s)\,ds + \int_0^t h(s)u^l(s)\,ds, \quad \forall t \in [0,\infty), \tag{2}$$

$$\begin{aligned} u(t) \leq u_0 &+ \left[\int_0^t f(s)u(s)\,ds\right]^2 + \int_0^t f(s)u(s)\Big[u(s) \\ &+ 2\int_0^s f(\lambda)u(\lambda)\,d\lambda\Big]\,ds, \quad \forall t \in [0,\infty). \end{aligned} \tag{3}$$

In 2012, Wu-sheng Wang [10] studied the following integral inequality:

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\varphi_1\big(u(s)\big)\left[u(s) + \int_0^s g(\lambda)\varphi_2\big(u(\lambda)\big)\,d\lambda\right]^r ds. \tag{4}$$

In 2017, Wang and Huang [11] studied the weakly singular integral inequality:

$$\begin{aligned} u(t) \leq b(t) &+ \int_a^t (t-s)^{\beta-1}f_1(s)\varphi_1\big(u(s)\big)\bigg\{u(s) + \int_a^s (s-\tau)^{\beta-1}f_2(\tau) \\ &\times \varphi_2\big(u(\tau)\big)\left[u(\tau) + \int_a^\tau (\tau-\xi)^{\beta-1}f_3(\xi)\varphi_3\big(u(\xi)\big)\,d\xi\right]d\tau\bigg\}\,ds. \end{aligned} \tag{5}$$

The aim of this paper is to extend certain results that were proved in [9, 10], and generalize (1)–(4) to some weakly singular integral inequalities. The upper bound estimations of the unknown functions are given by means of discrete Jensen inequality, the Hölder integral inequality and amplification techniques. Furthermore, we apply our result to integral equations for estimation.

2 Preliminaries and basic lemmas

In this section,we give some lemmas.

Lemma 2.1 ([12]; Hölder integral inequality) *We assume that $f(x)$ and $g(x)$ are nonnegative continuous functions defined on $[c,d]$ and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\int_c^d f(x)g(x)\,dx \le \left(\int_c^d f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_c^d g^q(x)\,dx\right)^{\frac{1}{q}}. \tag{6}$$

Lemma 2.2 ([12]) *If $\beta\in(0,\frac{1}{2}]$, $p=1+\beta$, then*

$$\int_{t_0}^t (t-s)^{p(\beta-1)}e^{ps}\,ds \le \frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma\bigl(1+p(\beta-1)\bigr), \quad \forall t_0\in R_+, \tag{7}$$

where $\Gamma(\beta)=\int_0^\infty \tau^{\beta-1}e^{-\tau}\,d\tau$ is the Gamma function.

Lemma 2.3 ([13]; Discrete Jensen inequality) *Let $A_1,A_2,\ldots,A_n$, $l>1$ be nonnegative real-valued constants and n a constant natural number, then*

$$(A_1+A_2+\cdots+A_n)^l \le n^{l-1}\bigl(A_1^l+A_2^l+\cdots+A_n^l\bigr). \tag{8}$$

3 Main results

In this section, we discuss some nonlinear weakly singular integral inequalities. Throughout this paper, let $I=[t_0,\infty)$.

Theorem 3.1 *We assume that $\beta\in(0,\frac{1}{2}]$ is a constant, $u(t)$, $f(t)$ and $g(t)$ are nonnegative and nondecreasing real-valued continuous functions defined on I and satisfy the inequality*

$$\begin{aligned} u(t) \le{}& u_0+\int_{t_0}^t (t-s)^{\beta-1}f(s)u(s)\biggl[u(s) \\ &+\int_{t_0}^s (s-\lambda)^{\beta-1}g(\lambda)u(\lambda)\,d\lambda\biggr]^r ds, \quad \forall t\in I, \end{aligned} \tag{9}$$

$$2^{(q-1)(2r+1)}u_0^{rq}rP(t)\int_{t_0}^t f^q(s)e^{-qs}C_1(s)\,ds<1, \quad \forall t\in I,$$

where u_0 and r are positive constants. Then

$$u(t)\le 2^{\frac{q-1}{q}}u_0\left[\exp\left(\int_{t_0}^t 2^{(q-1)(r+1)}P(s)f^q(s)e^{-qs}B_1(s)\,ds\right)\right]^{\frac{1}{q}}, \quad \forall t\in I, \tag{10}$$

where

$$P(t) = \left[\frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma\big(1 + p(\beta - 1)\big) \right]^{\frac{q}{p}}, \tag{11}$$

$$B_1(t) = \frac{(2^{q-1}u_0^q)^r C_1(t)}{1 - 2^{(q-1)(2r+1)} u_0^{rq} r P(t) \int_{t_0}^t f^q(s) e^{-qs} C_1(s)\, ds}, \quad \forall t \in I, \tag{12}$$

$$C_1(t) = \exp\left(r P(t) \int_{t_0}^t g^q(s) e^{-qs}\, ds \right), \quad \forall t \in I, \tag{13}$$

and $p = 1 + \beta$, $q = \frac{1+\beta}{\beta}$.

Proof Using (6) and (7), from (9) we get

$$\begin{aligned} u(t) &\le u_0 + \int_{t_0}^t (t-s)^{\beta-1} e^s f(s) e^{-s} u(s) \bigg[u(s) \\ &\quad + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda) u(\lambda)\, d\lambda \bigg]^r ds \\ &\le u_0 + \left[\frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma\big(1 + p(\beta-1)\big) \right]^{\frac{1}{p}} \bigg\{ \int_{t_0}^t f^q(s) e^{-qs} u^q(s) \bigg[u(s) \\ &\quad + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda) u(\lambda)\, d\lambda \bigg]^{rq} ds \bigg\}^{\frac{1}{q}}, \quad \forall t \in I. \end{aligned} \tag{14}$$

Using (8), from (14) we have

$$\begin{aligned} u^q(t) &\le 2^{q-1} u_0^q + 2^{q-1} P(t) \int_{t_0}^t f^q(s) e^{-qs} u^q(s) \bigg[u(s) \\ &\quad + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda) u(\lambda)\, d\lambda \bigg]^{rq} ds, \quad \forall t \in I, \end{aligned} \tag{15}$$

where $P(t)$ is defined by (11).

Also using (6), (7) and (8), $\forall t \in I$, we have

$$\begin{aligned} &\left[u(s) + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda) u(\lambda)\, d\lambda \right]^{rq} \\ &\quad \le \left[2^{q-1} u^q(s) + 2^{q-1} P(s) \int_{t_0}^s g^q(\lambda) e^{-q\lambda} u^q(\lambda)\, d\lambda \right]^r. \end{aligned} \tag{16}$$

Substituting (16) into (15) we get

$$\begin{aligned} u^q(t) &\le 2^{q-1} u_0^q + 2^{q-1} P(t) \int_{t_0}^t f^q(s) e^{-qs} u^q(s) \bigg[2^{q-1} u^q(s) \\ &\quad + 2^{q-1} P(s) \int_{t_0}^s g^q(\lambda) e^{-q\lambda} u^q(\lambda)\, d\lambda \bigg]^r ds \\ &\le 2^{q-1} u_0^q + 2^{(q-1)(r+1)} P(T) \int_{t_0}^t f^q(s) e^{-qs} u^q(s) \bigg[u^q(s) \end{aligned}$$

$$+ P(T) \int_{t_0}^{s} g^q(\lambda) e^{-q\lambda} u^q(\lambda)\, d\lambda \Bigg]^r ds, \tag{17}$$

for all $t \in [t_0, T]$, where $t_0 < T < \infty$ is chosen arbitrarily.

Let $z_1(t)$ equal the right hand side in (17), we have $z_1(t_0) = 2^{q-1} u_0^q$ and

$$u(t) \leq z_1^{\frac{1}{q}}(t). \tag{18}$$

Differentiating $z_1(t)$ with respect to t, and using (18) we obtain

$$\begin{aligned}
\frac{dz_1(t)}{dt} &= 2^{(q-1)(r+1)} P(T) f^q(t) e^{-qt} u^q(t) \Bigg[u^q(t) \\
&\quad + P(T) \int_{t_0}^{t} g^q(\lambda) e^{-q\lambda} u^q(\lambda)\, d\lambda \Bigg]^r \\
&\leq 2^{(q-1)(r+1)} P(T) f^q(t) e^{-qt} z_1(t) \Bigg[z_1(t) \\
&\quad + P(T) \int_{t_0}^{t} g^q(\lambda) e^{-q\lambda} z_1(\lambda)\, d\lambda \Bigg]^r, \quad \forall t \in [t_0, T].
\end{aligned} \tag{19}$$

Letting $z_2(t) = z_1(t) + P(T) \int_{t_0}^t g^q(\lambda) e^{-q\lambda} z_1(\lambda)\, d\lambda$, we have $z_2(t_0) = 2^{q-1} u_0^q$ and

$$z_1(t) \leq z_2(t). \tag{20}$$

Differentiating $z_2(t)$ with respect to t, and using (19) and (20) we have

$$\begin{aligned}
\frac{dz_2(t)}{dt} &= \frac{dz_1(t)}{dt} + P(T) g^q(t) e^{-qt} z_1(t) \\
&\leq 2^{(q-1)(r+1)} P(T) f^q(t) e^{-qt} z_2^{r+1}(t) + P(T) g^q(t) e^{-qt} z_2(t),
\end{aligned}$$

for all $t \in [t_0, T]$, but $z_2(t) > 0$ (where $u_0 > 0$), then we have

$$z_2^{-(r+1)}(t) \frac{dz_2(t)}{dt} - P(T) g^q(t) e^{-qt} z_2^{-r}(t) \leq 2^{(q-1)(r+1)} P(T) f^q(t) e^{-qt}. \tag{21}$$

If we let

$$S_1(t) = z_2^{-r}(t), \quad \forall t \in [t_0, T], \tag{22}$$

then we get $S_1(t_0) = z_2^{-r}(t_0) = (2^{q-1} u_0^q)^{-r}$, thus from (21) and (22) we obtain

$$\frac{dS_1(t)}{dt} + rP(T) g^q(t) e^{-qt} S_1(t) \geq -r 2^{(q-1)(r+1)} P(T) f^q(t) e^{-qt}, \quad \forall t \in [t_0, T].$$

The above inequality implies the following estimation for $S_1(t)$:

$$S_1(t) \geq \frac{(2^{q-1} u_0^q)^{-r} - \int_{t_0}^t r 2^{(q-1)(r+1)} P(T) f^q(s) e^{-qs} c_1(s)\, ds}{c_1(t)}, \quad \forall t \in [t_0, T],$$

where $c_1(t) = \exp \int_{t_0}^t rP(T) g^q(s) e^{-qs}\, ds$.

Let $t = T$, because $t_0 < T < \infty$ is chosen arbitrarily, we get

$$S_1(t) \geq \frac{1 - 2^{(q-1)(2r+1)} u_0^{rq} r P(t) \int_{t_0}^t f^q(s) e^{-qs} C_1(s)\, ds}{(2^{q-1} u_0^q)^r C_1(t)}, \quad \forall t \in I, \tag{23}$$

where $C_1(t)$ is defined by (13). Then from (23) in (22), we have

$$z_2^r(t) \leq \frac{(2^{q-1} u_0^q)^r C_1(t)}{1 - 2^{(q-1)(2r+1)} u_0^{rq} r P(t) \int_{t_0}^t f^q(s) e^{-qs} C_1(s)\, ds} = B_1(t), \quad \forall t \in I,$$

where $B_1(t)$ is defined by (12), thus from (19) we have

$$\frac{dz_1(t)}{dt} \leq 2^{(q-1)(r+1)} P(t) f^q(t) e^{-qt} z_1(t) B_1(t), \quad \forall t \in I,$$

the above inequality implies an estimation for $z_1(t)$ as in the following:

$$z_1(t) \leq 2^{q-1} u_0^q \exp\left(\int_{t_0}^t 2^{(q-1)(r+1)} P(s) f^q(s) e^{-qs} B_1(s)\, ds \right), \quad \forall t \in I, \tag{24}$$

and from (18) and (24) we have

$$u(t) \leq 2^{\frac{q-1}{q}} u_0 \left[\exp\left(\int_{t_0}^t 2^{(q-1)(r+1)} P(s) f^q(s) e^{-qs} B_1(s)\, ds \right) \right]^{\frac{1}{q}}, \quad \forall t \in I. \qquad \square$$

Theorem 3.2 *We assume that $\beta \in (0, \frac{1}{2}]$ is a constant, $u(t), f(t)$ and $g(t)$ are nonnegative and nondecreasing real-valued continuous functions defined on I. Suppose $\varphi_1, \varphi_2 \in C^1(I, I)$ are increasing functions with $\varphi_i(t) > 0$, $\forall t > t_0$, $i = 1, 2$, and $\frac{\varphi_1}{\varphi_2}$ is increasing and nonnegative function too. If the inequality*

$$\begin{aligned} u(t) \leq u_0 &+ \int_{t_0}^t (t-s)^{\beta-1} f(s) \varphi_1\big(u(s)\big) \bigg[u(s) \\ &+ \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda) \varphi_2\big(u(\lambda)\big)\, d\lambda \bigg]^r ds, \quad \forall t \in I, \end{aligned} \tag{25}$$

is satisfied, where u_0 and r are positive constants, then

$$u(t) \leq \left\{ \Phi_1^{-1}\left[\Phi_2^{-1}\left[U(t) \right] \right] \right\}^{\frac{1}{q}}, \quad \forall t \in I, \tag{26}$$

where $P(t)$ is defined as (11), $p = 1 + \beta$, $q = \frac{1+\beta}{\beta}$ and

$$\begin{aligned} U(t) = \Phi_2 &\left[\Phi_1\big(2^{q-1} u_0^q\big) + \int_{t_0}^t P(s) g^q(s) e^{-qs}\, ds \right] \\ &+ \int_{t_0}^t 2^{(q-1)(r+1)} P(s) f^q(s) e^{-qs}\, ds, \quad \forall t \in I, \end{aligned} \tag{27}$$

$$\Phi_1 = \int_1^k \frac{ds}{\varphi_2^q(s^{\frac{1}{q}})}, \quad k > 0, \tag{28}$$

$$\Phi_2 = \int_1^k \frac{\varphi_2^q[[\Phi_1^{-1}(s)]^{\frac{1}{q}}]}{\varphi_1^q[[\Phi_1^{-1}(s)]^{\frac{1}{q}}][\Phi_1^{-1}(s)]^r}\,ds, \quad k > 0, \tag{29}$$

and satisfied $U(t) \le \Phi_2(\infty)$, $\Phi_2^{-1}[U(t)] \le \Phi_1(\infty)$.

Proof Using (6) and (7), from (25) we get

$$\begin{aligned} u(t) &\le u_0 + \int_{t_0}^t (t-s)^{\beta-1} e^s f(s) e^{-s} \varphi_1\big(u(s)\big)\Big[u(s) \\ &\quad + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda)\varphi_2\big(u(\lambda)\big)\,d\lambda\Big]^r\,ds \\ &\le u_0 + \Big[\frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma\big(1+p(\beta-1)\big)\Big]^{\frac{1}{p}}\Big\{\int_{t_0}^t f^q(s)e^{-qs}\varphi_1^q\big(u(s)\big)\Big[u(s) \\ &\quad + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda)\varphi_2\big(u(\lambda)\big)\,d\lambda\Big]^{rq}\,ds\Big\}^{\frac{1}{q}}, \quad \forall t \in I. \end{aligned} \tag{30}$$

Using (8), from (30) we have

$$\begin{aligned} u^q(t) &\le 2^{q-1}u_0^q + 2^{q-1}P(t)\int_{t_0}^t f^q(s)e^{-qs}\varphi_1^q\big(u(s)\big)\Big[u(s) \\ &\quad + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda)\varphi_2\big(u(\lambda)\big)\,d\lambda\Big]^{rq}\,ds, \quad \forall t \in I, \end{aligned} \tag{31}$$

where $P(t)$ is defined by (11).

Also using (6), (7) and (8), $\forall t \in I$, we have

$$\begin{aligned} &\Big[u(s) + \int_{t_0}^s (s-\lambda)^{\beta-1} g(\lambda)\varphi_2\big(u(\lambda)\big)\,d\lambda\Big]^{rq} \\ &\quad \le \Big[2^{q-1}u^q(s) + 2^{q-1}P(s)\int_{t_0}^s g^q(\lambda)e^{-q\lambda}\varphi_2^q\big(u(\lambda)\big)\,d\lambda\Big]^r. \end{aligned} \tag{32}$$

Substituting (32) into (31) we get

$$\begin{aligned} u^q(t) &\le 2^{q-1}u_0^q + 2^{q-1}P(t)\int_{t_0}^t f^q(s)e^{-qs}\varphi_1^q\big(u(s)\big)\Big[2^{q-1}u^q(s) \\ &\quad + 2^{q-1}P(s)\int_{t_0}^s g^q(\lambda)e^{-q\lambda}\varphi_2^q\big(u(\lambda)\big)\,d\lambda\Big]^r\,ds \\ &\le 2^{q-1}u_0^q + 2^{(q-1)(r+1)}P(T)\int_{t_0}^t f^q(s)e^{-qs}\varphi_1^q\big(u(s)\big)\Big[u^q(s) \\ &\quad + P(T)\int_{t_0}^s g^q(\lambda)e^{-q\lambda}\varphi_2^q\big(u(\lambda)\big)\,d\lambda\Big]^r\,ds, \end{aligned} \tag{33}$$

for all $t \in [t_0, T]$, where $t_0 < T < \infty$ is chosen arbitrarily.

Let $z_3(t)$ equal the right hand side in (33), we have $z_3(t_0) = 2^{q-1}u_0^q$ and

$$u(t) \le z_3^{\frac{1}{q}}(t). \tag{34}$$

Differentiating $z_3(t)$ with respect to t, and using (34) we obtain

$$\begin{aligned}\frac{dz_3(t)}{dt} &= 2^{(q-1)(r+1)}P(T)f^q(t)e^{-qt}\varphi_1^q\big(u(t)\big)\bigg[u^q(t)\\&\quad + P(T)\int_{t_0}^t g^q(\lambda)e^{-q\lambda}\varphi_2^q\big(u(\lambda)\big)\,d\lambda\bigg]^r\\&\le 2^{(q-1)(r+1)}P(T)f^q(t)e^{-qt}\varphi_1^q\big(z_3^{\frac{1}{q}}(t)\big)\bigg[z_3(t)\\&\quad + P(T)\int_{t_0}^t g^q(\lambda)e^{-q\lambda}\varphi_2^q\big(z_3^{\frac{1}{q}}(\lambda)\big)\,d\lambda\bigg]^r, \quad \forall t\in[t_0,T].\end{aligned}\tag{35}$$

Letting $z_4(t) = z_3(t) + P(T)\int_{t_0}^t g^q(\lambda)e^{-q\lambda}\varphi_2^q(z_3^{\frac{1}{q}}(\lambda))\,d\lambda$, then we have $z_4(t_0) = 2^{q-1}u_0^q$ and

$$z_3(t) \le z_4(t).\tag{36}$$

Differentiating $z_4(t)$ with respect to t, and using (35) and (36) we have

$$\begin{aligned}\frac{dz_4(t)}{dt} &= \frac{dz_3(t)}{dt} + P(T)g^q(t)e^{-qt}\varphi_2^q\big(z_4^{\frac{1}{q}}(t)\big)\\&\le 2^{(q-1)(r+1)}P(T)f^q(t)e^{-qt}\varphi_1^q\big(z_4^{\frac{1}{q}}(t)\big)z_4^r(t)\\&\quad + P(T)g^q(t)e^{-qt}\varphi_2^q\big(z_4^{\frac{1}{q}}(t)\big), \quad \forall t\in[t_0,T].\end{aligned}\tag{37}$$

Since $\varphi_2^q(z_4^{\frac{1}{q}}(t)) > 0$, $\forall t > t_0$, we have

$$\begin{aligned}\frac{\frac{dz_4(t)}{dt}}{\varphi_2^q(z_4^{\frac{1}{q}}(t))} &\le P(T)g^q(t)e^{-qt}\\&\quad + \frac{2^{(q-1)(r+1)}P(T)f^q(t)e^{-qt}\varphi_1^q(z_4^{\frac{1}{q}}(t))z_4^r(t)}{\varphi_2^q(z_4^{\frac{1}{q}}(t))}, \quad \forall t\in[t_0,T].\end{aligned}\tag{38}$$

By taking $t = s$ in (38) and integrating it from t_0 to t, and using (28) we get

$$\begin{aligned}\Phi_1\big(z_4(t)\big) &\le \Phi_1\big(z_4(t_0)\big) + \int_{t_0}^t P(T)g^q(s)e^{-qs}\,ds\\&\quad + \int_{t_0}^t \frac{2^{(q-1)(r+1)}P(T)f^q(s)e^{-qs}\varphi_1^q(z_4^{\frac{1}{q}}(s))z_4^r(s)}{\varphi_2^q(z_4^{\frac{1}{q}}(s))}\,ds\\&\le \Phi_1\big(z_4(t_0)\big) + \int_{t_0}^T P(T)g^q(s)e^{-qs}\,ds\\&\quad + \int_{t_0}^t \frac{2^{(q-1)(r+1)}P(T)f^q(s)e^{-qs}\varphi_1^q(z_4^{\frac{1}{q}}(s))z_4^r(s)}{\varphi_2^q(z_4^{\frac{1}{q}}(s))}\,ds\end{aligned}\tag{39}$$

for all $t\in[t_0,T]$, where Φ_1 is defined by (28).

Let $z_5(t)$ equal the right hand side in (39), we have $z_5(t_0) = \Phi_1(2^{q-1}u_0^q) + \int_{t_0}^T P(T)g^q(s) \times e^{-qs}\,ds$, and

$$z_4(t) \le \Phi_1^{-1}\bigl(z_5(t)\bigr). \tag{40}$$

Differentiating $z_5(t)$ with respect to t, and using (40) we get

$$\begin{aligned} \frac{dz_5(t)}{dt} &= \frac{2^{(q-1)(r+1)}P(T)f^q(t)e^{-qt}\varphi_1^q(z_4^{\frac{1}{q}}(t))z_4^r(t)}{\varphi_2^q(z_4^{\frac{1}{q}}(t))} \\ &\le \frac{2^{(q-1)(r+1)}P(T)f^q(t)e^{-qt}\varphi_1^q[(\Phi_1^{-1}(z_5(t)))^{\frac{1}{q}}][\Phi_1^{-1}(z_5(t))]^r}{\varphi_2^q[(\Phi_1^{-1}(z_5(t)))^{\frac{1}{q}}]}, \end{aligned} \tag{41}$$

for all $t \in [t_0, T]$.

By taking $t = s$ in (41) and integrating it from t_0 to t, and using (29) we get

$$\Phi_2\bigl(z_5(t)\bigr) \le \Phi_2\bigl(z_5(t_0)\bigr) + \int_{t_0}^t 2^{(q-1)(r+1)}P(T)f^q(s)e^{-qs}\,ds, \quad \forall t \in [t_0, T]. \tag{42}$$

Let $t = T$, from (42) we have

$$\Phi_2\bigl(z_5(T)\bigr) \le \Phi_2\bigl(z_5(t_0)\bigr) + \int_{t_0}^T 2^{(q-1)(r+1)}P(T)f^q(s)e^{-qs}\,ds, \quad \forall t \in [t_0, T]. \tag{43}$$

Because $t_0 < T < \infty$ is chosen arbitrarily, from (34), (36) and (40), we have

$$u(t) \le \bigl\{\Phi_1^{-1}\bigl[\Phi_2^{-1}\bigl[U(t)\bigr]\bigr]\bigr\}^{\frac{1}{q}}, \quad \forall t \in I,$$

where $U(t)$ is defined by (27). □

Remark It is interesting to note that in the special case when $\varphi_1(t) = t$ and $\varphi_2(t) = t$ the inequality given in Theorem 3.2 reduces to Theorem 3.1.

Theorem 3.3 *We assume that $\beta \in (0, \frac{1}{2}]$ is a constant, $u(t)$, $f(t)$ and $h(t)$ are nonnegative real-valued continuous functions defined on I, and they satisfy the inequality*

$$u^n(t) \le u_0 + \int_{t_0}^t (t-s)^{\beta-1}f(s)u^n(s)\,ds + \int_{t_0}^t (t-s)^{\beta-1}h(s)u^l(s)\,ds, \quad \forall t \in I, \tag{44}$$

where $u_0 > 0$, and $n > l \ge 0$ are constants. Then

$$\begin{aligned} u(t) \le{}& \exp\biggl(\frac{1}{nq}\omega_1(t)\biggr)\biggl\{\bigl(3^{q-1}u_0^q\bigr)^{n_1} + 3^{q-1}P(t)n_1 \\ &\times \int_{t_0}^t h^q(s)e^{-qs}\bigl[\exp\bigl(-n_1\omega_1(s)\bigr)\bigr]\biggr\}^{\frac{1}{(n-l)q}}, \quad \forall t \in I, \end{aligned} \tag{45}$$

where $P(t)$ is defined by (11), $n_1 = \frac{n-l}{n}$, $p = 1 + \beta$, $q = \frac{1+\beta}{\beta}$ and

$$\omega_1(t) = 3^{q-1}P(t)\int_{t_0}^t f^q(s)e^{-qs}\,ds, \quad \forall t \in I. \tag{46}$$

Proof Using (6) and (7), from (44) we get

$$\begin{aligned}
u^n(t) &\le u_0 + \int_{t_0}^{t} (t-s)^{\beta-1} e^s f(s) e^{-s} u^n(s)\,ds \\
&\quad + \int_{t_0}^{t} (t-s)^{\beta-1} e^s h(s) e^{-s} u^l(s)\,ds \\
&\le u_0 + \left[\frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma\bigl(1+p(\beta-1)\bigr)\right]^{\frac{1}{p}} \left[\int_{t_0}^{t} f^q(s) e^{-qs} u^{nq}(s)\,ds\right]^{\frac{1}{q}} \\
&\quad + \left[\frac{e^{pt}}{p^{1+p(\beta-1)}} \Gamma\bigl(1+p(\beta-1)\bigr)\right]^{\frac{1}{p}} \left[\int_{t_0}^{t} h^q(s) e^{-qs} u^{lq}(s)\,ds\right]^{\frac{1}{q}}, \\
&\quad \forall t \in I.
\end{aligned} \tag{47}$$

Using (8) and from (47) we have

$$\begin{aligned}
u^{nq}(t) &\le 3^{q-1} u_0^q + 3^{q-1} P(t) \int_{t_0}^{t} f^q(s) e^{-qs} u^{nq}(s)\,ds \\
&\quad + 3^{q-1} P(t) \int_{t_0}^{t} h^q(s) e^{-qs} u^{lq}(s)\,ds \\
&\le 3^{q-1} u_0^q + 3^{q-1} P(T) \left[\int_{t_0}^{t} f^q(s) e^{-qs} u^{nq}(s)\,ds \right. \\
&\quad \left. + \int_{t_0}^{t} h^q(s) e^{-qs} u^{lq}(s)\,ds\right],
\end{aligned} \tag{48}$$

for all $t \in [t_0, T]$, where $t_0 < T < \infty$ is chosen arbitrarily, and $P(t)$ is defined by (11).

Let $z_6^n(t)$ equal the right hand side in (48), we have $z_6(t_0) = (3^{q-1} u_0^q)^{\frac{1}{n}}$ and

$$u(t) \le z_6^{\frac{1}{q}}(t). \tag{49}$$

Differentiating $z_6^n(t)$ with respect to t, and using (48) and (49) we have

$$\begin{aligned}
n z_6^{n-1}(t) \frac{dz_6(t)}{dt} &= 3^{q-1} P(T) \bigl[f^q(t) e^{-qt} u^{nq}(t) + h^q(t) e^{-qt} u^{lq}(t)\bigr] \\
&\le 3^{q-1} P(T) \bigl[f^q(t) e^{-qt} z_6^n(t) + h^q(t) e^{-qt} z_6^l(t)\bigr], \quad \forall t \in [t_0, T].
\end{aligned} \tag{50}$$

Since $u_0 > 0$ we have $z_6(t) > 0$. Thus, we have

$$\begin{aligned}
n z_6^{n-l-1}(t) \frac{dz_6(t)}{dt} &\le 3^{q-1} P(T) f^q(t) e^{-qt} z_6^{n-l}(t) \\
&\quad + 3^{q-1} P(T) h^q(t) e^{-qt}, \quad \forall t \in [t_0, T],
\end{aligned} \tag{51}$$

if we let

$$z_6^{n-l}(t) = z_7(t), \quad \forall t \in [t_0, T], \tag{52}$$

then we have $z_7(t_0) = (3^{q-1}u_0^q)^{n_1}$, and $nz_6^{n-l-1}(t)\frac{dz_6(t)}{dt} = \frac{1}{n_1}\frac{dz_7(t)}{dt}$, thus from (51) we obtain

$$\frac{dz_7(t)}{dt} \le 3^{q-1}P(T)n_1 f^q(t)e^{-qt}z_7(t) + 3^{q-1}P(T)n_1 h^q(t)e^{-qt}, \quad \forall t \in [t_0, T]. \tag{53}$$

The inequality (53) implies the estimation for $z_7(t)$, as

$$\begin{aligned} z_7(t) \le{}& \exp\left(n_1 3^{q-1}P(T)\int_{t_0}^t f^q(s)e^{-qs}\,ds\right)\left\{\left(3^{q-1}u_0^q\right)^{n_1} + 3^{q-1}P(T)n_1\right.\\ &\left.\times \int_{t_0}^t h^q(s)e^{-qs}\left[\exp\left(-n_1 3^{q-1}P(T)\int_{t_0}^s f^q(\lambda)e^{-q\lambda}\,d\lambda\right)\right]ds\right\}, \end{aligned} \tag{54}$$

for all $t \in [t_0, T]$.

Letting $t = T$, because $t_0 < T < \infty$ is chosen arbitrarily, then we get

$$\begin{aligned} z_7(t) \le{}& \exp\bigl(n_1\omega_1(t)\bigr)\left\{\left(3^{q-1}u_0^q\right)^{n_1} + 3^{q-1}P(t)n_1\right.\\ &\left.\times \int_{t_0}^t h^q(s)e^{-qs}\bigl[\exp\bigl(-n_1\omega_1(s)\bigr)\bigr]ds\right\}, \quad \forall t \in I, \end{aligned} \tag{55}$$

where $\omega_1(t)$ is defined by (46).

Then from (55) in (52), we have

$$\begin{aligned} z_6(t) \le{}& \exp\left(\frac{1}{n}\omega_1(t)\right)\left\{\left(3^{q-1}u_0^q\right)^{n_1} + 3^{q-1}P(t)n_1\right.\\ &\left.\times \int_{t_0}^t h^q(s)e^{-qs}\bigl[\exp\bigl(-n_1\omega_1(s)\bigr)\bigr]ds\right\}^{\frac{1}{n-l}}, \quad \forall t \in I, \end{aligned} \tag{56}$$

thus from (49) we can get (45). □

Theorem 3.4 *We assume that $\beta \in (0, \frac{1}{2}]$ is a constant, $u(t)$ and $f(t)$ are nonnegative real-valued continuous functions defined on I, and they satisfy the inequality*

$$\begin{aligned} u(t) \le{}& u_0 + \left[\int_{t_0}^t (t-s)^{\beta-1}f(s)u(s)\,ds\right]^2 + \int_{t_0}^t (t-s)^{\beta-1}f(s)u(s)\left[u(s)\right.\\ &\left.+ 2\int_{t_0}^s (s-\lambda)^{\beta-1}f(\lambda)u(\lambda)\,d\lambda\right]ds, \quad \forall t \in I, \end{aligned} \tag{57}$$

$$108^{q-1}P^2(t)u_0^q\int_{t_0}^t f^q(s)e^{-qs}\bigl[\exp\bigl(\omega_2(s)\bigr)\bigr]ds < 1,$$

where $u_0 > 0$ is constant. Then

$$u(t) \le 3^{\frac{q-1}{q}}u_0\exp\left[\frac{1}{q}\int_{t_0}^t f^q(s)e^{-qs}B_2(s)\,ds\right], \quad \forall t \in I, \tag{58}$$

where

$$B_2(t) = \frac{18^{q-1}P(t)u_0^q\exp(\omega_2(t))}{1 - 108^{q-1}P^2(t)u_0^q\int_{t_0}^t f^q(s)e^{-qs}[\exp(\omega_2(s))]\,ds}, \quad \forall t \in I, \tag{59}$$

where $\omega_2(t) = 2 \times 3^{q-1}(1 + 2^{q-1})P^2(t)\int_{t_0}^t f^q(s)e^{-qs}\,ds$, $p = 1 + \beta$, $q = \frac{1+\beta}{\beta}$, *and* $P(t)$ *is defined by* (11).

Proof Using (6) and (7), from (57) we get

$$\begin{aligned}
u(t) &\le u_0 + \left[\int_{t_0}^t (t-s)^{\beta-1}e^s f(s)e^{-s}u(s)\,ds\right]^2 \\
&\quad + \int_{t_0}^t (t-s)^{\beta-1}e^s f(s)e^{-s}u(s)\left[u(s) + 2\int_{t_0}^s (s-\lambda)^{\beta-1}f(\lambda)u(\lambda)\,d\lambda\right]ds \\
&\le u_0 + \left[\left(\frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma\bigl(1+p(\beta-1)\bigr)\right)^{\frac{1}{p}}\left(\int_{t_0}^t f^q(s)e^{-qs}u^q(s)\,ds\right)^{\frac{1}{q}}\right]^2 \\
&\quad + \left(\frac{e^{pt}}{p^{1+p(\beta-1)}}\Gamma\bigl(1+p(\beta-1)\bigr)\right)^{\frac{1}{p}}\left\{\int_{t_0}^t f^q(s)e^{-qs}u^q(s)\left[u(s)\right.\right. \\
&\quad \left.\left. + 2\int_{t_0}^s (s-\lambda)^{\beta-1}f(\lambda)u(\lambda)\,d\lambda\right]^q ds\right\}^{\frac{1}{q}}, \quad \forall t \in I. \qquad (60)
\end{aligned}$$

Using (8) and from (60) we have

$$\begin{aligned}
u^q(t) &\le 3^{q-1}u_0^q + 3^{q-1}P^2(t)\left[\int_{t_0}^t f^q(s)e^{-qs}u^q(s)\,ds\right]^2 \\
&\quad + 3^{q-1}P(t)\int_{t_0}^t f^q(s)e^{-qs}u^q(s)\left[u(s)\right. \\
&\quad \left. + 2\int_{t_0}^s (s-\lambda)^{\beta-1}f(\lambda)u(\lambda)\,d\lambda\right]^q ds, \quad \forall t \in I, \qquad (61)
\end{aligned}$$

where $P(t)$ is defined by (11).

Also using (6), (7) and (8), $\forall t \in I$, we have

$$\begin{aligned}
&\left[u(s) + 2\int_{t_0}^s (s-\lambda)^{\beta-1}f(\lambda)u(\lambda)\,d\lambda\right]^q \\
&\quad \le 2^{q-1}u^q(s) + 2 \times 2^{q-1}P(s)\int_{t_0}^s f^q(\lambda)e^{-q\lambda}u^q(\lambda)\,d\lambda. \qquad (62)
\end{aligned}$$

Substituting (62) into (61) we get

$$\begin{aligned}
u^q(t) &\le 3^{q-1}u_0^q + 3^{q-1}P^2(t)\left[\int_{t_0}^t f^q(s)e^{-qs}u^q(s)\,ds\right]^2 + 6^{q-1}P(t) \\
&\quad \times \int_{t_0}^t f^q(s)e^{-qs}u^q(s)\left[u^q(s) + 2P(s)\int_{t_0}^s f^q(\lambda)e^{-q\lambda}u^q(\lambda)\,d\lambda\right]ds \\
&\le 3^{q-1}u_0^q + 3^{q-1}P^2(T)\left[\int_{t_0}^t f^q(s)e^{-qs}u^q(s)\,ds\right]^2 + 6^{q-1}P(T) \\
&\quad \times \int_{t_0}^t f^q(s)e^{-qs}u^q(s)\left[u^q(s) + 2P(T)\int_{t_0}^s f^q(\lambda)e^{-q\lambda}u^q(\lambda)\,d\lambda\right]ds, \qquad (63)
\end{aligned}$$

for all $t \in [t_0, T]$, where $t_0 < T < \infty$ is chosen arbitrarily.

Let $z_8(t)$ equal the right hand side in (63), we have $z_8(t_0) = 3^{q-1}u_0^q$ and

$$u(t) \le z_8^{\frac{1}{q}}(t). \tag{64}$$

Differentiating $z_8(t)$ with respect to t, and using (64) we obtain

$$\begin{aligned}
\frac{dz_8(t)}{dt} &= 2 \times 3^{q-1}P^2(T)f^q(t)e^{-qt}u^q(t)\int_{t_0}^{t} f^q(s)e^{-qs}u^q(s)\,ds \\
&\quad + 6^{q-1}P(T)f^q(t)e^{-qt}u^q(t)\left[u^q(t) + 2P(T)\int_{t_0}^{t} f^q(s)e^{-qs}u^q(s)\,ds\right] \\
&\le f^q(t)e^{-qt}z_8(t)\biggl[6^{q-1}P(T)z_8(t) + 2 \times 3^{q-1}\bigl(1 + 2^{q-1}\bigr)P^2(T) \\
&\quad \times \int_{t_0}^{t} f^q(s)e^{-qs}z_8(s)\,ds\biggr] \\
&= f^q(t)e^{-qt}z_8(t)Y(t), \quad \forall t \in [t_0, T],
\end{aligned} \tag{65}$$

where $Y(t) = 6^{q-1}P(T)z_8(t) + 2 \times 3^{q-1}(1 + 2^{q-1})P^2(T)\int_{t_0}^{t} f^q(s)e^{-qs}z_8(s)\,ds$, hence $Y(t_0) = 18^{q-1}P(T)u_0^q$, and $z_8(t) \le Y(t)$.

Differentiating $Y(t)$ with respect to t and using (65) we obtain

$$\begin{aligned}
\frac{dY(t)}{dt} &\le 6^{q-1}P(T)f^q(t)e^{-qt}Y^2(t) \\
&\quad + 2 \times 3^{q-1}\bigl(1 + 2^{q-1}\bigr)P^2(T)f^q(t)e^{-qt}Y(t), \quad \forall t \in [t_0, T].
\end{aligned} \tag{66}$$

Since $Y(t) > 0$, we have

$$\begin{aligned}
Y^{-2}(t)\frac{dY(t)}{dt} &\le 2 \times 3^{q-1}\bigl(1 + 2^{q-1}\bigr)P^2(T)f^q(t)e^{-qt}Y^{-1}(t) \\
&\quad + 6^{q-1}P(T)f^q(t)e^{-qt}, \quad \forall t \in [t_0, T].
\end{aligned} \tag{67}$$

If we let

$$S_2(t) = Y^{-1}(t), \quad \forall t \in [t_0, T], \tag{68}$$

then we get $S_2(t_0) = Y^{-1}(t_0) = (18^{q-1}P(T)u_0^q)^{-1}$ and $Y^{-2}(t)\frac{dY(t)}{dt} = -\frac{dS_2(t)}{dt}$, thus from (67) we obtain

$$\begin{aligned}
\frac{dS_2(t)}{dt} &\ge -2 \times 3^{q-1}\bigl(1 + 2^{q-1}\bigr)P^2(T)f^q(t)e^{-qt}S_2(t) \\
&\quad - 6^{q-1}P(T)f^q(t)e^{-qt}, \quad \forall t \in [t_0, T].
\end{aligned} \tag{69}$$

The above inequality implies the following estimation for $S_2(t)$:

$$S_2(t) \ge \frac{1 - 108^{q-1}P^2(T)u_0^q\int_{t_0}^{t} f^q(s)e^{-qs}\exp(c_2(s))\,ds}{18^{q-1}P(T)u_0^q\exp(c_2(t))}, \quad \forall t \in [t_0, T],$$

where $c_2(t) = 2 \times 3^{q-1}(1 + 2^{q-1})P^2(T)\int_{t_0}^{t} f^q(s)e^{-qs}\,ds$.

Let $t = T$, because $t_0 < T < \infty$ is chosen arbitrarily, then we get

$$S_2(t) \geq \frac{1 - 108^{q-1} P^2(t) u_0^q \int_{t_0}^t f^q(s) e^{-qs} \exp(\omega_2(s))\, ds}{18^{q-1} P(t) u_0^q \exp(\omega_2(t))}, \quad \forall t \in I, \tag{70}$$

then from (68) and (70) we obtain

$$Y(t) \leq \frac{18^{q-1} P(t) u_0^q \exp(\omega_2(t))}{1 - 108^{q-1} P^2(t) u_0^q \int_{t_0}^t f^q(s) e^{-qs} [\exp(\omega_2(s))]\, ds} = B_2(t), \quad \forall t \in I,$$

where $B_2(t)$ is defined by (59), thus from (65) we have

$$\frac{dz_8(t)}{dt} \leq f^q(t) e^{-qt} z_8(t) B_2(t), \quad \forall t \in I.$$

The above inequality implies the following estimation for $z_8(t)$:

$$z_8(t) \leq 3^{q-1} u_0^q \exp\left(\int_{t_0}^t f^q(s) e^{-qs} B_2(s)\, ds\right). \tag{71}$$

Then from (64) we get (58). □

4 Application

In this section, we present two applications of our results to the estimation of unknown functions of the integral equations.

As an application of the inequality given in Theorem 3.1, we consider the following integral equation:

$$x(t) = x_0 + \int_{t_0}^t (t-s)^{\beta-1} F_1\left(s, x(s), \int_{t_0}^s (s-\tau)^{\beta-1} F_2\big(\tau, x(\tau)\big)\, d\tau\right) ds, \\ \forall t \in I, \tag{72}$$

where x_0 is a positive constant. We assume that $F_1 \in C([t_0,\infty) \times R^2, R)$, $F_2 \in C([t_0,\infty) \times R, R)$ satisfy the following conditions:

$$\big|F_1(t,x,y)\big| \leq f(t)|x|\big[|x| + |y|\big]^r, \tag{73}$$

$$\big|F_2(t,x)\big| \leq g(t)|x|, \tag{74}$$

where f, g are nonnegative and nondecreasing real-valued continuous functions defined on I.

Theorem 4.1 *Consider the integral equation* (72) *and suppose that* F_1, F_2 *satisfy the conditions* (73) *and* (74), *and* f, g *are nonnegative and nondecreasing real-valued continuous functions defined on* I. *Then*

$$\begin{aligned} &x(t) \leq 2^{\frac{q-1}{q}} x_0 \left[\exp\left(\int_{t_0}^t 2^{(q-1)(r+1)} P(t) f^q(s) e^{-qs} B_3(s)\, ds\right)\right]^{\frac{1}{q}}, \quad \forall t \in I, \\ &1 - 2^{(q-1)(r+1)} \big(2^{q-1} x_0^q\big)^r r P(t) \int_{t_0}^t f^q(s) e^{-qs} \exp\left(r P(t) \int_{t_0}^s g^q(\tau) e^{-q\tau}\, d\tau\right) ds < 1, \end{aligned} \tag{75}$$

where $P(t)$ is defined by (11), $p = 1 + \beta$, $q = \frac{1+\beta}{\beta}$ *and*

$$B_3(t) = \frac{(2^{q-1}x_0^q)^r \exp[rP(t)\int_{t_0}^t g^q(s)e^{-qs}\,ds]}{1 - 2^{(q-1)(r+1)}(2^{q-1}x_0^q)^r rP(t)\int_{t_0}^t f^q(s)e^{-qs}\exp[rP(t)\int_{t_0}^s g^q(\tau)e^{-q\tau}\,d\tau]\,ds}.$$

Proof Substituting (73) and (74) into (72), we get

$$\begin{aligned} |x(t)| \le |x_0| &+ \int_{t_0}^t (t-s)^{\beta-1} f(s)|x(s)|\biggl[|x(s)| \\ &+ \int_{t_0}^s (s-\tau)^{\beta-1} g(\tau)|x(\tau)|\,d\tau\biggr]^r ds, \quad \forall t \in I. \end{aligned} \tag{76}$$

Obviously, (76) satisfies the conditions of Theorem 3.1 and is of the form of (9). Applying Theorem 3.1 to (76), we can get the estimation (75).

As an application of the inequality given in Theorem 3.4, we consider the following integral equation:

$$\begin{aligned} x(t) = x_0 &+ \biggl[\int_{t_0}^t (t-s)^{\beta-1} F_2(s, x(s))\,ds\biggr]^2 + \int_{t_0}^t (t-s)^{\beta-1} F_1\biggl(s, x(s), \\ &\int_{t_0}^s (s-\lambda)^{\beta-1} F_2(\lambda, x(\lambda))\,d\lambda\biggr)\,ds, \quad \forall t \in I, \end{aligned} \tag{77}$$

where x_0 is a positive constant. We assume that $F_1 \in C([t_0,\infty) \times R^2, R)$, $F_2 \in C([t_0,\infty) \times R, R)$ satisfy the following conditions:

$$|F_1(t,x,y)| \le f(t)|x|\bigl[|x| + 2|y|\bigr], \tag{78}$$

$$|F_2(t,x)| \le f(t)|x|, \tag{79}$$

where f is a nonnegative real-valued continuous functions defined on I. □

Theorem 4.2 *Consider the integral equation* (77) *and suppose that F_1, F_2 satisfy the conditions* (78) *and* (79), *and f is a nonnegative and nondecreasing real-valued continuous function defined on I. Then*

$$\begin{aligned} &x(t) \le 3^{\frac{q-1}{q}} x_0 \exp\biggl[\frac{1}{q}\int_{t_0}^t f^q(s)e^{-qs}B_4(s)\,ds\biggr], \quad \forall t \in I, \\ &108^{q-1}P^2(t)x_0^q \int_{t_0}^t f^q(s)e^{-qs}\bigl[\exp(\omega_2(s))\bigr]\,ds < 1, \end{aligned} \tag{80}$$

where $P(t)$ is defined by (11), $p = 1 + \beta$, $q = \frac{1+\beta}{\beta}$ *and*

$$B_4(t) = \frac{18^{q-1}P(t)x_0^q \exp(\omega_2(t))}{1 - 108^{q-1}P^2(t)x_0^q\int_{t_0}^t f^q(s)e^{-qs}[\exp(\omega_2(s))]\,ds},$$

$$\omega_2(t) = 2 \times 3^{q-1}\bigl(1 + 2^{q-1}\bigr)P^2(t)\int_{t_0}^t f^q(s)e^{-qs}\,ds.$$

Proof Substituting (78) and (79) into (77), we get

$$\begin{aligned} |x(t)| \le{}& |x_0| + \left[\int_{t_0}^{t}(t-s)^{\beta-1}f(s)x(s)\,ds\right]^2 + \int_{t_0}^{t}(t-s)^{\beta-1}f(s)x(s)\biggl[x(s) \\ &+ 2\int_{t_0}^{s}(s-\lambda)^{\beta-1}f(\lambda)x(\lambda)\,d\lambda\biggr]\,ds, \quad \forall t \in I. \end{aligned} \tag{81}$$

Obviously, (81) satisfies the conditions of Theorem 3.4 and is of the form of (57). Applying Theorem 3.4 to (81), we can get the estimation (80). □

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Authors' contributions
The main idea of this paper was proposed by YR and MFW. YR prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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