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On n-quasi-$[m, C]$-isometric operators

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Abstract

For positive integers m and n, an operator $T \in B(H)$ is said to be an n-quasi-$[m, C]$-isometric operator if there exists some conjugation C such that $T^{*n}(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j})T^n = 0$. In this paper, some basic structural properties of n-quasi-$[m, C]$-isometric operators are established with the help of operator matrix representation. As an application, we obtain that a power of an n-quasi-$[m, C]$-isometric operator is again an n-quasi-$[m, C]$-isometric operator. Moreover, we show that the class of n-quasi-$[m, C]$-isometric operators is norm closed. Finally, we examine the stability of n-quasi-$[m, C]$-isometric operator under perturbation by nilpotent operators commuting with T.

MSC: 47B20; 47A05

Keywords: n-quasi-$[m, C]$-isometric operator; Perturbation; Nilpotent operator

1 Introduction

Let $\mathbb{N}$ and $\mathbb{C}$ be the sets of natural numbers and complex numbers, respectively, and let $B(H)$ denote the algebra of all bounded linear operators on a separable complex Hilbert space H. If $T \in B(H)$, we shall write $N(T)$, $R(T)$, and $\sigma(T)$ for the null space, range space, and the spectrum of T, respectively. The closure of a set M will be denoted by $\overline{M}$.

An antilinear operator C on H is said to be conjugation if C satisfies $C^2 = I$ and $(Cx, Cy) = (y, x)$ for all $x, y \in H$. In 1990s, Agler and Stankus [1] studied the theory of m-isometric operators which are connected to Toeplitz operators, ordinary differential equations, classical function theory, classical conjugate point theory, distributions, Fejer–Riesz factorization, stochastic processes, and other topics. For fixed $m \in \mathbb{N}$, an operator $T \in B(H)$ is said to be an m-isometric operator if it satisfies the identity

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}T^{m-j} = 0,$$

where $\binom{m}{j}$ is the binomial coefficient. Several authors have studied the m-isometric operator. We refer the reader to [2–6, 9, 11] for further details.

In [7], Chō, Ko, and Lee introduced (m, C)-isometric operators with conjugation C as follows: For an operator $T \in B(H)$ and an integer $m \geq 1$, T is said to be an (m, C)-isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}.CT^{m-j}C = 0.$$

In [8], Chō, Lee, and Motoyoshi introduced $[m, C]$-isometric operators with conjugation C as follows: For an operator $T \in B(H)$ and an integer $m \geq 1$, T is said to be an $[m, C]$-isometric operator if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j} = 0.$$

For an operator $T \in B(H)$ and a conjugation C, define the operator $\lambda_m(T)$ by

$$\lambda_m(T) = \sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j}.$$

Then T is an $[m, C]$-isometric operator if and only if

$$\lambda_m(T) = 0.$$

Moreover,

$$CTC.\lambda_m(T).T - \lambda_m(T) = \lambda_{m+1}(T)$$

holds. Hence, an $[m, C]$-isometric operator is an $[n, C]$-isometric operator for every $n \geq m$.

In [13], Mahmoud Sid Ahmed, Chō, and Lee introduced n-quasi-(m, C)-isometric operators, which generalize (m, C)-isometric operators. For positive integers m and n, an operator $T \in B(H)$ is said to be an n-quasi-(m, C)-isometric operator if there exists some conjugation C such that

$$T^{*n}\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j}.CT^{m-j}C\right)T^n = 0.$$

In [10], Duggal studied n-quasi-$[m, C]$-isometric operators and gave some properties of them. For positive integers m and n, an operator $T \in B(H)$ is said to be an n-quasi-$[m, C]$-isometric operator if there exists some conjugation C such that

$$T^{*n}\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j}\right)T^n = 0.$$

It is clear that every $[m, C]$-isometric operator is an n-quasi-$[m, C]$-isometric operator.

The following example provides an operator which is an n-quasi-$[2, C]$-isometric operator, but not a $[2, C]$-isometric operator.

Example 1.1 Let $H = \mathbb{C}^2$ and let C be a conjugation on H given by $C(x,y) = (\overline{y},\overline{x})$. If $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ on H, then $CTC = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Hence

$$\begin{aligned}
&(CTC)^2.T^2 - 2CTC.T + I \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 - 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned}$$

i.e., T is not a $[2,C]$-isometric operator.

On the other hand, since

$$\begin{aligned}
&T^{*2}\big((CTC)^2.T^2 - 2CTC.T + I\big)T^2 \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \\
&= 0.
\end{aligned}$$

Hence T is a 2-quasi-$[2,C]$-isometric operator.

Remark 1.1 Let $T \in B(H)$ and let C be a conjugation on H.

Note that

$$\begin{aligned}
&T^{*n}\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j}\right)T^n \\
&= C\big(CT^{*n}C\big)\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{m-j}.CT^{m-j}C\right)CT^nC)C.
\end{aligned}$$

It is clear that T is an n-quasi-$[m,C]$-isometric operator if and only if CTC is an n-quasi-$[m,C]$-isometric operator.

Remark 1.2 It is clear that every quasi-$[m,C]$-isometric operator is an n-quasi-$[m,C]$-isometric operator for $n \geq 2$. The converse is not true in general as shown in the following example.

Example 1.2 Let $T = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \in B(\mathbb{C}^3)$, and let $C : \mathbb{C}^3 \to \mathbb{C}^3$ satisfy $C(x_1,x_2,x_3) = (-\overline{x_3},\overline{x_2}, -\overline{x_1})$. We have $CTC = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and

$$\begin{aligned}
T^*(CTC.T - I)T &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Hence T is not a quasi-$[1, C]$-isometric operator.

On the other hand, since

$$
\begin{aligned}
&T^{*2}(CTCT - I)T^2 \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 \\
&= 0,
\end{aligned}
$$

it follows that T is a 2-quasi-$[1, C]$-isometric operator.

2 *n*-quasi-[*m*, *C*]-isometric operators

In this section we give some basic properties of n-quasi-$[m, C]$-isometric operators. We begin with the following theorem, which is a structural theorem for n-quasi-$[m, C]$-isometric operators.

Theorem 2.1 *Let $C = C_1 \oplus C_2$ be a conjugation on H where C_1 and C_2 are conjugations on $\overline{R(T^n)}$ and $N(T^{*n})$, respectively. If $T^n \neq 0$ does not have a dense range, then the following statements are equivalent*:

(1) *T is an n-quasi-$[m, C]$-isometric operator*;

(2) *$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^n)} \oplus N(T^{*n})$, where T_1 is an $[m, C_1]$-isometric operator and $T_3^n = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof (1) $\Rightarrow$ (2) Consider the matrix representation of T with respect to the decomposition $H = \overline{R(T^n)} \oplus N(T^{n*})$:

$$
T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.
$$

Let P be the projection onto $\overline{R(T^n)}$. Since T is an n-quasi-$[m, C]$-isometric operator, it follows that

$$
P\left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j} \right) P = 0.
$$

This means that

$$
\sum_{j=0}^{m} (-1)^j \binom{m}{j} C_1 T_1^{m-j} C_1.T_1^{m-j} = 0.
$$

Hence T_1 is an $[m, C_1]$-isometric operator on $\overline{R(T^n)}$. On the other hand, for any $x = (x_1, x_2) \in \overline{R(T^n)} \oplus N(T^{*n}) = H$, we have

$$
\left(T_3^n x_2, x_2\right) = \left(T^n(I-P)x, (I-P)x\right) = \left((I-P)x, T^{*n}(I-P)x\right) = 0,
$$

which implies $T_3^n = 0$. Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where M is the union of the holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ by Corollary 7 of [12], and $\sigma(T_1) \cap \sigma(T_3)$ has no interior point since T_3 is nilpotent, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^n)} \oplus N(T^{*n})$, where $\sum_{j=0}^{m}(-1)^j \binom{m}{j} C_1 \times T_1^{m-j} C_1.T_1^{m-j} = 0$ and $T_3^n = 0$. Since

$$T^n = \begin{pmatrix} T_1^n & \sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned}
&T^{*n}\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j}\right)T^n \\
&= \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*n} \\
&\quad \times \left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{m-j} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{m-j}\right) \\
&\quad \times \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{n} \\
&= \begin{pmatrix} T_1^{*n} F T_1^n & T_1^{*n} F \sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \\ (\sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j})^* F T_1^n & (\sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j})^* F \sum_{j=0}^{n-1} T_1^j T_2 T_3^{n-1-j} \end{pmatrix},
\end{aligned}$$

where $F = \sum_{j=0}^{m}(-1)^j \binom{m}{j} C_1 T_1^{m-j} C_1.T_1^{m-j}$. Hence

$$T^{*n}\left(\sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C.T^{m-j}\right)T^n = 0,$$

i.e., T is an n-quasi-$[m,C]$-isometric operator. □

As a consequence, we obtain the following corollaries.

Corollary 2.1 *Let $T \in B(H)$ and let C be a conjugation on H. If T is an n-quasi-$[m,C]$-isometric operator and T^n has a dense range, then T is an $[m,C]$-isometric operator.*

Corollary 2.2 *Let $T \in B(H)$ and let C be a conjugation on H. If T is an invertible n-quasi-$[m,C]$-isometric operator, then T^{-1} is an n-quasi-$[m,C]$-isometric operator.*

Proof Suppose that T is an invertible n-quasi-$[m,C]$-isometric operator. Then T is an $[m,C]$-isometric operator, and so is T^{-1} by [8]. Hence T^{-1} is an n-quasi-$[m,C]$-isometric operator. □

Corollary 2.3 *Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^n)} \oplus N(T^{*n})$, and let $C = C_1 \oplus C_2$ be a conjugation on H where C_1 and C_2 are conjugations on $\overline{R(T^n)}$ and $N(T^{*n})$, respectively. If T is an n-quasi-$[m,C]$-isometric operator and T_1 is invertible, then T is similar to a direct sum of an $[m,C_1]$-isometric operator and a nilpotent operator.*

Proof Since T_1 is invertible, we have $\sigma(T_1) \cap \sigma(T_3) = \phi$. Then there exists an operator S such that $T_1 S - S T_3 = T_2$ [14]. Since

$$\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix},$$

it follows that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

□

Corollary 2.4 *Let $T \in B(H)$ and let $C = C_1 \oplus C_2$ be a conjugation on H where C_1 and C_2 are conjugations on $\overline{R(T^n)}$ and $N(T^{*n})$, respectively. If T is an n-quasi-$[m, C]$-isometric operator, then T^k is also an n-quasi-$[m, C]$-isometric operator for any $k \in \mathbb{N}$.*

Proof If T^n has a dense range, then T is an $[m, C]$-isometric operator, and so is T^k for any $k \in \mathbb{N}$ by [8, Theorem 3.4]. If T^n does not have a dense range, we decompose T as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^n)} \oplus N(T^{*n})$, where T_1 is an $[m, C_1]$-isometric operator, and so is T_1^k. Since

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix} \quad \text{on } H = \overline{R\bigl(T^k\bigr)} \oplus N\bigl(T^{*k}\bigr),$$

it follows from Theorem 2.1 that T^k is an n-quasi-$[m, C]$-isometric operator for any $k \in \mathbb{N}$. □

Remark 2.1 The converse of Corollary 2.4 is not true in general as shown in the following example.

Example 2.1 Let $T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in B(\mathbb{C}^3)$, and let $C : \mathbb{C}^3 \to \mathbb{C}^3$ satisfy $C(x_1, x_2, x_3) = (-\overline{x_3}, \overline{x_2}, -\overline{x_1})$. A simple calculation shows that $T^{*2}(CT^2C.T^2 - I)T^2 = 0$ and $T^*(CTC.T - I)T \neq 0$. So, we obtain that T^2 is a quasi-$[1, C]$-isometric operator, but T is not a quasi-$[1, C]$-isometric operator.

Theorem 2.2 *Let $T \in B(H)$ and let $C = C_1 \oplus C_2$ be a conjugation on H where C_1 and C_2 are conjugations on $\overline{R(T^n)}$ and $N(T^{*n})$, respectively. If T is an n-quasi-$[m, C]$-isometric operator, then T is an n-quasi-$[k, C]$-isometric operator for every positive integer $k \geq m$.*

Proof If T^n has a dense range, then T is an $[m, C]$-isometric operator, and hence T is a $[k, C]$-isometric operator for every positive integer $k \geq m$. If T^n does not have a dense range, we decompose T as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{R(T^n)} \oplus N(T^{*n})$, where T_1 is an $[m, C_1]$-isometric operator and $T_3^n = 0$. Hence T_1 is a $[k, C_1]$-isometric operator for every positive integer $k \geq m$. It follows from Theorem 2.1 that T is an n-quasi-$[k, C]$-isometric operator. □

Theorem 2.3 *Let $T \in B(H)$ and let C be a conjugation on H. If $\{T_k\}$ is a sequence of n-quasi-$[m, C]$-isometric operators such that $\lim_{k \to \infty} \|T_k - T\| = 0$, then T is an n-quasi-$[m, C]$-isometric operator.*

Proof Suppose that $\{T_k\}$ is a sequence of n-quasi-$[m,C]$-isometric operators such that $\lim_{n\to\infty}\|T_k - T\| = 0$. Then

$$\begin{aligned}
&\left\| T_k^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT_k^{m-j}C.T_k^{m-j}\right)T_k^n - T^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j}\right)T^n\right\| \\
&\quad\le \left\| T_k^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT_k^{m-j}C.T_k^{m-j}\right)T_k^n \right. \\
&\qquad \left. - T_k^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j}\right)T^n\right\| \\
&\qquad + \left\| T_k^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j}\right)T^n \right. \\
&\qquad \left. - T^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j}\right)T^n\right\| \\
&\quad\le \|T_k^{*n}\|\sum_{j=0}^{m}\binom{m}{j}\|T_k^{m-j}C.T_k^{m-j+n} - T^{m-j}C.T^{m-j+n}\| \\
&\qquad + \|T_k^n - T^n\|\sum_{j=0}^{m}\binom{m}{j}\|T^{m-j}C.T^{m-j+n}\| \to 0.
\end{aligned}$$

Since $\{T_k\}$ is an n-quasi-$[m,C]$-isometric operator,

$$T_k^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT_k^{m-j}C.T_k^{m-j}\right)T_k^n = 0,$$

we have

$$T^{*n}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}C.T^{m-j}\right)T^n = 0,$$

i.e., T is an n-quasi-$[m,C]$-isometric operator. □

Theorem 2.4 *Let $T,N\in B(H)$ and let C be a conjugation on H. Assume that $T^*CNC = CNCT^*$ and $T^*CTC = CTCT^*$. If T is an n-quasi-$[m,C]$-isometric operator and N is a nilpotent operator of order p such that $TN = NT$, then $T+N$ is an $n+p$-quasi-$[m+2p-2,C]$-isometric operator.*

Proof Since

$$\lambda_m(T+N) = \sum_{i+j+k=m}\binom{m}{i,j,k}C(T+N)^iC.CN^jC.\lambda_k(T).T^j.N^i,$$

where $\binom{m}{i,j,k} = \frac{m!}{i!.j!,k!}$ and $\lambda_0(*) = I$ by [8].

We have

$$
\begin{aligned}
&(T+N)^{*n+p}\lambda_{m+2p-2}(T+N)(T+N)^{n+p}\\
&=\left(\sum_{s=0}^{n+p}\binom{n+p}{s}T^{*n+p-s}N^{*s}\right)\\
&\quad\times\left(\sum_{i+j+k=m+2p-2}\binom{m+2p-2}{i,j,k}C(T+N)^iC.CN^jC.\lambda_k(T).T^j.N^i\right)\\
&\quad\times\left(\sum_{t=0}^{n+p}\binom{n+p}{t}T^{n+p-t}N^t\right).
\end{aligned}
$$

(i) If $\max\{i,j\}\geq p$, then $CN^jC=0$ or $N^i=0$.

(ii) If $\max\{i,j\}\leq p-1$, then $k\geq m$. Since T is an n-quasi-$[m,C]$-isometric operator, $T^*CNC=CNCT^*$ and $T^*CTC=CTCT^*$, we obtain

$$T^{*n+p-s}\lambda_k(T)T^{n+p-t}=0 \quad \text{for } 0\leq s,t\leq p$$

and

$$N^{*s}=0 \quad \text{or} \quad N^t=0 \quad \text{for } p+1\leq s\leq n+p \text{ or } p+1\leq t\leq n+p.$$

By (i) and (ii), $(T+N)^{*n+p}\lambda_{m+2p-2}(T+N)(T+N)^{n+p}=0$. Therefore $T+N$ is an $n+p$-quasi-$[m+2p-2,C]$-isometric operator. □

Example 2.2 Let C be a conjugation given by $C(z_1,z_2,z_3)=(\overline{z_3},\overline{z_2},\overline{z_1})$ on $\mathbb{C}^3$. If $T=\begin{pmatrix}1&m&m\\0&1&0\\0&0&1\end{pmatrix}$ on $\mathbb{C}^3$, we have $CTC=\begin{pmatrix}1&0&0\\0&1&0\\\overline{m}&\overline{m}&1\end{pmatrix}$, then

$$
\begin{aligned}
&I-3CTC.T+3(CTC)^2.T^2-(CTC)^3T^3\\
&=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-3\begin{pmatrix}1&0&0\\0&1&0\\\overline{m}&\overline{m}&1\end{pmatrix}\begin{pmatrix}1&m&m\\0&1&0\\0&0&1\end{pmatrix}\\
&\quad+3\begin{pmatrix}1&0&0\\0&1&0\\\overline{m}&\overline{m}&1\end{pmatrix}^2\begin{pmatrix}1&m&m\\0&1&0\\0&0&1\end{pmatrix}^2-\begin{pmatrix}1&0&0\\0&1&0\\\overline{m}&\overline{m}&1\end{pmatrix}^3\begin{pmatrix}1&m&m\\0&1&0\\0&0&1\end{pmatrix}^3\\
&=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.
\end{aligned}
$$

Hence $T^{*2}(I-3CTC.T+3(CTC)^2.T^2-(CTC)^3T^3)T^2=0$, i.e., T is a 2-quasi-$[3,C]$-isometric operator with conjugation C.

On the other hand, since $T=I+N$, where $N=\begin{pmatrix}0&m&m\\0&0&0\\0&0&0\end{pmatrix}$, $N^2=0$, it follows from Theorem 2.4 that T is a 2-quasi-$[3,C]$-isometric operator with conjugation C.

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