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Lower and upper bounds for lifespan of solutions to viscoelastic hyperbolic equations with variable sources and damping terms

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Abstract

The aim of this paper is to study bounds for lifespan of solutions to the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + |u_t|^{m(x,t)-2}u_t = |u|^{p(x,t)-2}u$$

under homogeneous Dirichlet boundary conditions. It is worth pointing out that it is not a trivial generalization for constant-exponent problems because we have to face some essential difficulties in studying such problems. The first difficulty is that the monotonicity of the energy functional fails. Another one is that there exists a gap between the norm and the modular to the generalized function space, which leads to the failure of the Poincaré inequality for modular form. To overcome such difficulties, the authors construct control function and apply new energy estimates to establish the quantitative relationship between the source $\int_\Omega |u|^{p(x,t)}\,dx$ and the initial energy, and then obtain the finite-time blow-up of solutions for a positive initial energy, especially, the authors only assume that $p_t(x,t)$ is integrable rather than uniformly bounded. Such weak conditions are seldom seen for the variable exponent case. At last, an estimate of lower bound for lifespan is established by applying differential inequality argument and energy inequalities.

MSC: 35L20; 35L70; 35B40

Keywords: Variable source; Blow-up in finite time; Positive initial energy

1 Introduction and the main result

We consider the following semilinear hyperbolic equation with nonstandard growth condition:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\cdot,\tau)\,d\tau + |u_t|^{m(x,t)-2}u_t = |u|^{p(x,t)-2}u, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$. It will be assumed throughout the paper that the exponents $p(x,t)$, $m(x,t)$ are continuous in $Q_T =$

$\Omega \times (0, T)$ and satisfy that

$$2 < p^- = \inf_{(x,t)\in Q_T} p(x,t) \le p(x,t) \le p^+ = \sup_{(x,t)\in Q_T} p(x,t) < \infty, \tag{2}$$

$$2 < m^- = \inf_{(x,t)\in Q_T} m(x,t) \le m(x,t) \le m^+ = \sup_{(x,t)\in Q_T} m(x,t) < \infty, \tag{3}$$

$$\begin{aligned} &\bigl|p(x,t) - p(y,s)\bigr| + \bigl|q(x,t) - q(y,s)\bigr| \le \omega\bigl(|x-y| + \sqrt{|t-s|}\bigr), \\ &\quad \forall x, y \in \Omega, t, s > 0, |x-y| + \sqrt{|t-s|} < 1, \end{aligned} \tag{4}$$

where $\omega(r)$ satisfies

$$\limsup_{r\to 0^+} \omega(r) \ln\left(\frac{1}{r}\right) = C < +\infty.$$

Problem (1) may describe many phenomena of applied science such as electro-rheological fluids, viscoelastic fluids, processes of filtration through a porous media, and fluids with temperature-dependent viscosity; the interested readers may refer to [1, 2, 7, 9, 21] and the references therein. As far as we know, when p and m are fixed constants, many authors discussed the existence, uniqueness, blowing-up, and global existence of solutions to Problem (1). For example, in the absence of the viscoelastic term ($g = 0$), Georgiev and Todorova in [8] studied the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \ge 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{5}$$

They applied the Galerkin approximation method and the contraction mapping theorem to prove that Problem (5) had a unique global solution for small initial data and $1 < p \le m$, whereas for $p > m$, they obtained that the solution of Problem (5) blew up in finite time for a negative initial energy by applying energy estimate methods and Gronwall's inequality. Later, Messaoudi in [16] improved the above results. Roughly speaking, he proved that the solution blew up in finite time for a positive initial energy. However, it is well known that the source term causes finite-time blow-up of solutions and drives the equation to possible instability, while the damping term prevents finite-time blow-up of the solution and drives the equation toward stability. So, it is of interest to explore the mechanism of how the sources dominate the two types of dissipation (the finite-time memory term $\int_0^t g(t-\tau)\Delta u(\cdot,\tau)\,d\tau$ and the weak damping term $|u_t|^{m-2}u_t$), which attracts considerable attention. The interaction between the damping term and the source term makes the problem more interesting. In the presence of the viscoelastic term ($g \ne 0$), Cavalcanti and Soriano [5] obtained a rate of exponential decay to the solution of Problem (1) with the assumption that the kernel g is of exponential decay and $m = 2$ (a localized damping mechanism $a(x)u_t$). Later, Cavalcanti in [6] and Berrimi and Messaoudi in [3] improved this work by using different methods. In addition, Messaoudi in [18] generalized the results in [3, 5]. For more works, the interested readers may refer to [4, 14–19] and the references therein. However, there are few results about lower bound for lifespan. Sun, Guo, and Gao in [22] considered some estimates of the lower bound of blow-up time for the following problem:

$$\begin{cases} u_{tt} - \triangle u - \omega \triangle u_t + \mu u_t = |u|^{p-2}u, & (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{6}$$

They applied an energy estimate method and the Sobolev inequalities to give an estimate of the lower bound for the blow-up time when $2 < p \le \frac{2(N-1)}{N-2}$, and later Guo and Liu in [13] obtained an estimate of the lower bound for the blow-up time in the supercritical case $\frac{2(N-1)}{N-2} < p < \frac{2(N^2-2)}{N^2-2N}$. For more works, the interested readers may refer to [23, 24]. When p is a function, the authors in [2, 20] applied Kaplan's method to establish the nonglobal existence and global existence results for Problem (1) in the absence of the viscoelastic term and the damping term. As far as we know, in the presence of the viscoelastic term ($g \not\equiv 0$) and the damping mechanism $|u_t|^{m-2}u_t$, the results of blow-up of solutions with positive initial energy are seldom seen for the case with variable exponents. Different from the case with constant exponents, the variable exponent brings us some essential difficulties.

- How to overcome the lack of the monotonicity of the energy functional constructed in [18] with respect to time variable?
- Owing to the existence of a gap between the norm and the modular (that is, $\int_\Omega |u|^{p(\cdot)}\,dx \not\equiv \|u\|_{p(\cdot)}^{p(\cdot)}$), it is not easy to obtain the results similar to those of Lemmas 2.2–2.4 in [18]. In fact, the proof of Theorem 1.2 of [18] depends strongly on the conclusions of Lemmas 2.2–2.4 and the monotonicity of the energy functional. It is unfortunate that we cannot obtain such results in the case with variable exponents.

To bypass the difficulties mentioned above, we have to look for some new methods or techniques to discuss some properties of solutions to the above problem. In this paper, we construct a new control function and apply suitable embedding theorems to prove that the solution blows up in finite time for a positive initial energy. At the same time, we apply the energy estimate method to establish a differential inequality and then obtain an explicit lower bound for blow-up time.

Before stating our main result, we first define some energy functionals. Denote by $L^{p(\cdot)}(\Omega)$ the space of measurable functions $f(x)$ on Q_T such that

$$A_{p(\cdot)}(f) = \iint_{Q_T} \bigl|f(x)\bigr|^{p(\cdot)}\,dx\,dt < \infty.$$

The norm of $f(x)$ in space $L^{p(\cdot)}(Q_T)$ is defined as follows:

$$\|f\|_{p(\cdot),Q_T} \equiv \|f\|_{L^{p(\cdot)}(Q_T)} = \inf\left\{\lambda > 0 : A_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

It is obvious that $L^{p(\cdot)}(Q_T)$ is a Banach space [7]. It follows directly from the definition that

$$\min\bigl\{\|f\|_{p(\cdot),\Omega}^{p^-}, \|f\|_{p(\cdot),\Omega}^{p^+}\bigr\} \le A_{p(\cdot)}(f) \le \max\bigl\{\|f\|_{p(\cdot),\Omega}^{p^-}, \|f\|_{p(\cdot),\Omega}^{p^+}\bigr\}. \tag{7}$$

By Corollary 3.34 in [7], we have that

$$\|u\|_{p(\cdot)} \leq B\|\nabla u\|_2, \tag{8}$$

where $1 < p^- \leq p(\cdot) \leq p^+ \leq \frac{2N}{N-2}$ $(N \geq 3)$ and B is the embedding constant.

Before proving the main results of this paper, we first state a local existence theorem.

Theorem 1.1 *Suppose that the exponents $p(x,t)$, $m(x,t)$ satisfy (2)–(4), and the following conditions hold:*

$$(H_1) \quad \max\{m^+ \cdot p^+\} \leq \frac{2(N-1)}{N-2}, \quad p_t \geq 0, \quad \frac{p_t(x,t)}{p^2(x,t)} \in L^1_{\text{loc}}\big((0,\infty); L^1(\Omega)\big);$$

$$(H_2) \quad g(t) > 0, \quad g'(t) < 0, \quad t \geq 0, \quad 1 - \int_0^\infty g(s)\,ds = k > 0;$$

$$(H_3) \quad (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega).$$

Then Problem (1) has a unique local solution

$$u \in C\big([0,T]; H_0^1(\Omega)\big) \cap L^{p^-}\big(0,T; L^{p(x,t)}(\Omega)\big),$$
$$u_t \in C\big([0,T]; L^2(\Omega)\big) \cap L^m\big(\Omega \times (0,T)\big)$$

for some T.

The proof of the existence of solutions relays on Galerkin approximation technique and the contraction mapping theorem. For more details, we may refer to [7, 13, 23].

Set

$$E_1 = \left(\frac{1}{2} - \frac{1}{p^-}\right)\beta_1^{\frac{p^-}{2}}, \quad \beta_1 = \left(\frac{k}{B_1^2}\right)^{\frac{2}{p^- - 2}}, \tag{9}$$

where $B_1 = \max\{B, \sqrt{k}\}$.

Define

$$\begin{aligned} E(t) = {}& \frac{1}{2}\int_\Omega \left|u_t(x,t)\right|^2 dx + \frac{1}{2}\left(1 - \int_0^t g(\tau)\,d\tau\right)\int_\Omega \left|\nabla u(x,t)\right|^2 dx \\ & + \frac{1}{2}\int_0^t g(t-\tau)\left\|\nabla u(x,\tau) - \nabla u(x,t)\right\|^2_{L^2(\Omega)} d\tau - \int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}(x,t)\,dx, \end{aligned} \tag{10}$$

where $(g \diamond u)(t) = \int_0^t g(t-\tau)\|u(x,\tau) - u(x,t)\|^2_{L^2(\Omega)}\,d\tau$.

Our main result is as follows.

Theorem 1.2 *Assume that (H_1)–(H_3) of Theorem 1.1 hold, and that the following conditions are satisfied:*

$$(H_4) \quad E(0) + \frac{|\Omega|}{p^-} < E_1, \quad \|\nabla u_0\|_2^2 > \frac{k}{B_1^2}\beta_1,$$

$$(H_5) \quad m^+ < p^-,$$

(H_6) *there exists a sufficiently small* $0 < \varepsilon_0 < 1$ *such that* $1 - \varepsilon_0 \le k < 1$.

Then the solution of Problem (1) *blows up in finite time* T^* *satisfying the following estimate*:

$$\int_{J(0)}^{\infty} \frac{1}{C_6 y^q + y + C_7} \, dy \le T^* \le \frac{C_5(1-\lambda)}{F^{\frac{\lambda}{1-\lambda}}(0) C_4 \lambda},$$

where the coefficients C_4, $F(0)$, C_5 *are defined in* (31) *and* (34), *respectively, the exponents* $q = p^+ - 1 > 1$, $0 < \lambda \le \frac{p^- - m^+}{(m^+-1)p^-}$, $J(0) = \int_\Omega |u_0|^{p^+} \, dx$ *and the coefficients* C_6, C_7 *are defined in* (42).

2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemmas.

Lemma 2.1 *Assume that* (H_1)–(H_3) *of Theorem* 1.1 *hold, then* $E(t)$ *defined in* (10) *satisfies the following estimate*:

$$E(t) + \int_0^t \int_\Omega |u_t|^{m(x,s)} \, dx \, ds \le E(0) + \frac{|\Omega|}{p^-}. \tag{11}$$

Proof Following the lines of the proof of Lemma 2.1 in [18], we get $E(t) \in C[0,T) \cap C^1(0,T)$ and

$$\begin{aligned} E'(t) = &-\int_\Omega |u_t|^{m(x,t)} \, dx - \frac{g(t)}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \frac{p_t}{p^2} |u|^p \big(\ln |u|^p - 1\big) \, dx \\ &+ \frac{1}{2} \int_0^t g'(t-\tau) \int_\Omega \big|\nabla\big(u(\cdot,\tau) - u(\cdot,t)\big)\big|^2 \, dx \, d\tau. \end{aligned}$$

The above identity and conditions $g(t) > 0$, $g'(t) < 0$ show that

$$E'(t) + \int_\Omega |u_t|^{m(x,t)} \le -\int_\Omega \frac{p_t}{p^2} |u|^p \big(\ln |u|^p - 1\big) \, dx := J. \tag{12}$$

Next, we estimate the value of J.

$$\begin{aligned} J &\le -\int_{\{|u|^p \le e\}} \frac{|u|^{p(x,t)}}{p^2(x,t)} \big(\ln |u|^{p(x,t)} - 1\big) p_t(x,t) \, dx \\ &\le \int_{\{|u|^p \le e\}} \frac{p_t(x,t)}{p^2(x,t)} \, dx \le \int_\Omega \frac{p_t(x,t)}{p^2(x,t)} \, dx. \end{aligned} \tag{13}$$

In the second inequality of (13), we have used the following facts:

$$-\frac{1}{e} \le s \ln s \le 0, \quad 0 \le s \le 1.$$

Inequality (11) follows from (12) and (13). □

Due to the lack of homogeneity and the existence of the gap between the norm and the modular, the control function constructed by Messaoudi in [18] fails in our problem, so we have to look for a new control function to establish the relations between the term $\int_\Omega \frac{u^{p(x,t)}(x,t)}{p(x,t)} \, dx$ and the value of E_1, the following lemma helps us solve the problem.

Lemma 2.2 *Assume that* (H_4) *and* $p^- > 2$ *hold, then there exists a positive constant* $\beta_2 > \beta_1$ *such that*

$$k\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t) \geq \frac{k}{B_1^2}\beta_2, \quad \forall t \geq 0 \tag{14}$$

and

$$\int_\Omega \frac{1}{p(x,t)} u^{p(x,t)}(x,t)\,dx \geq \frac{1}{p^-}\max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\}. \tag{15}$$

Proof We borrow some ideas from [10]–[9]. First, we can derive from (7), (8), and (10) that

$$\begin{aligned} E(t) &\geq \frac{1}{2}\left[\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\right] - \frac{1}{p^-}\int_\Omega u^{p(x,t)}(\cdot,t)\,dx \\ &\geq \frac{1}{2}\left[k\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\right] - \frac{1}{p^-}\max\{\|u(\cdot,t)\|_{p(\cdot),\Omega}^{p^+}, \|u(\cdot,t)\|_{p(\cdot),\Omega}^{p^-}\} \\ &\geq \frac{1}{2}\left[k\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\right] - \frac{1}{p^-}\max\{B_1^{p^+}\|\nabla u\|_2^{p^+}, B_1^{p^-}\|\nabla u\|_2^{p^-}\} \\ &= \frac{k}{2B_1^2}\beta - \frac{1}{p^-}\max\{\beta^{\frac{p^+}{2}}, \beta^{\frac{p^-}{2}}\} := h(\beta(t)), \end{aligned} \tag{16}$$

where $\beta(t) = \frac{B_1^2}{k}[k\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)]$.

Next, we analyze the properties of the function $h(\beta)$. By calculating directly, we know that $h(\beta)$ satisfies the following properties:

$$\begin{aligned} &h(\beta) \in C[0,+\infty); \\ &h'(\beta) = \begin{cases} \frac{k}{2B_1^2} - \frac{p^+}{2p^-}\beta^{\frac{p^+-2}{2}} < 0, & \beta > 1; \\ \frac{k}{2B_1^2} - \frac{1}{2}\beta^{\frac{p^--2}{2}}, & 0 < \beta < 1; \end{cases} \\ &h'_+(1) = \frac{k}{2B_1^2} - \frac{p^+}{2p^-} < 0, \qquad h'_-(1) = \frac{k}{2B_1^2} - \frac{1}{2} < 0; \\ &h'(\beta_1) = 0, \quad 0 < \beta_1 = \left(\frac{k}{B_1^2}\right)^{\frac{2}{p^--2}} < 1. \end{aligned} \tag{17}$$

Although the function $h(\beta)$ is not differentiable at $\beta = 1$, a simple analysis shows that $h(\beta)$ is increasing for $0 < \beta < \beta_1$, while $h(\beta)$ is decreasing for $\beta \geq \beta_1$, and $\lim_{\beta\to\infty} h(\beta) = -\infty$. Due to $E(0) + \frac{|\Omega|}{p^-} < E_1$, there exists a positive constant $\beta_2 > \beta_1$ such that $h(\beta_2) = E(0) + \frac{|\Omega|}{p^-}$. Let $\beta_0 = \|\nabla u_0\|_2^2$, then we have $h(\frac{B_1^2\beta_0}{k}) \leq E(0) + \frac{|\Omega|}{p^-} = h(\beta_2)$. By (11) with $\frac{B_1^2\beta_0}{k} > \beta_1$, we have $\beta_2 > \beta_1$.

To prove (14), we suppose that $k\|\nabla u(t_0)\|_2^2 + (g \diamond \nabla u)(t_0) < \frac{k}{B_1^2}\beta_2$ for some $t_0 > 0$. By the continuity of $k\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)$, we may choose $t_1 > 0$ such that

$$\frac{k}{B_1^2}\beta_1 < k\|\nabla u(t_1)\|_2^2 + (g \diamond \nabla u)(t_1) < \frac{k}{B_1^2}\beta_2.$$

And then, combining the monotonicity of $h(\beta)$ with the above inequalities, we have

$$E(0)+\frac{|\Omega|}{p^-}=h(\beta_2)<h\left(\frac{B_1^2}{k}\left(k\|\nabla u(t_1)\|_2^2+(g\diamond\nabla u)(t_1)\right)\right)\leq E(t_1),$$

which contradicts (11).

From (10) we can see that

$$\frac{1}{2}\left[k\|\nabla u(t)\|_2^2+(g\diamond\nabla u)(t)\right]\leq E(0)+\frac{|\Omega|}{p^-}+\int_\Omega\frac{1}{p(x,t)}u^{p(x,t)}(t)\,dx,$$

which implies that

$$\begin{aligned}\int_\Omega\frac{1}{p(x,t)}u^{p(x,t)}(x,t)\,dx&\geq\frac{1}{2}\left[k\|\nabla u(t)\|_2^2+(g\diamond\nabla u)(t)\right]-E(0)-\frac{|\Omega|}{p^-}\\&\geq\frac{k}{2B_1^2}\beta_2-h(\beta_2)=\frac{1}{p^-}\max\left\{\beta_2^{\frac{p^+}{2}},\beta_2^{\frac{p^-}{2}}\right\}.\end{aligned}$$ □

Let $H(t)=E_1-E(t)$, $t\geq 0$, we have the following.

Lemma 2.3 *For all $t>0$,*

$$0<H(0)-\frac{|\Omega|}{p^-}<H(t)\leq\int_\Omega\frac{u^{p(x,t)}(x,t)}{p(x,t)}\,dx.\tag{18}$$

Proof By Lemma 2.1, we have $H'(t)=-E'(t)$. Inequalities (12) and (13) show that

$$H'(t)\geq-\int_\Omega\frac{p_t(x,t)}{p^2(x,t)}\,dx.\tag{19}$$

The left inequality in (18) follows from (19) and $E(0)+\frac{|\Omega|}{p^-}<E_1$.

On the other hand, (10) and (15) yield

$$\begin{aligned}H(t)=E_1&-\frac{1}{2}\int_\Omega|u_t(x,t)|^2\,dx-\frac{1}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)\int_\Omega|\nabla u(x,t)|^2\,dx\\&-\frac{1}{2}\int_0^t g(t-\tau)\|\nabla u(x,\tau)-\nabla u(x,t)\|_{L^2(\Omega)}^2\,d\tau+\int_\Omega\frac{1}{p(x,t)}|u|^{p(x,t)}(x,t)\,dx.\end{aligned}$$

From (11) and (13), it is easy to verify that

$$\begin{aligned}&E_1-\frac{1}{2}\int_\Omega|u_t(x,t)|^2\,dx-\frac{1}{2}\left(1-\int_0^t g(\tau)\,d\tau\right)\int_\Omega|\nabla u(x,t)|^2\,dx\\&\quad-\frac{1}{2}\int_0^t g(t-\tau)\|\nabla u(x,\tau)-\nabla u(x,t)\|_{L^2(\Omega)}^2\,d\tau\\&\leq E_1-\frac{\alpha_2}{2}\leq E_1-\frac{\alpha_1}{2}\leq 0,\quad t>0.\end{aligned}$$

This completes the proof of Lemma 2.3. □

Proof of Theorem 1.2 This proof will be divided into two steps.

Step 1. The idea of this proof mainly comes from [18]. Choose $0 < \lambda \leq \min\{\frac{p^- - m}{(m-1)p^-}, \frac{p^- - 2}{2p^-}\} < \frac{1}{2}$ and define $F(t) = H^{1-\lambda}(t) + \varepsilon \int_\Omega u_t(x,t)u(x,t)\,dx$. Then

$$\begin{aligned} F'(t) &= (1-\lambda)H^{-\lambda}(t)H'(t) + \varepsilon \int_\Omega u_{tt}(x,t)u(x,t)\,dx + \varepsilon \int_\Omega u_t^2(x,t)\,dx \\ &= (1-\lambda)H^{-\lambda}(t)\biggl[\int_\Omega |u_t|^{m(x,t)}\,dx + \frac{g(t)}{2}\bigl\|\nabla u(t)\bigr\|_2^2 - \frac{1}{2}\bigl(g' \diamond \nabla u\bigr)(t)\biggr] \\ &\quad + \varepsilon \int_\Omega u_{tt}u\,dx + \varepsilon \int_\Omega u_t^2\,dx := J_1 + J_2 + J_3. \end{aligned} \tag{20}$$

According to the first identity in Problem (1) and Cauchy inequality with δ, we obtain the following inequalities:

$$\begin{aligned} J_2 &= \varepsilon \int_\Omega u_{tt}u\,dx = \varepsilon \int_\Omega u\biggl(\Delta u - \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - |u_t|^{m(x,t)-2}u_t + |u|^{p(x,t)-2}u\biggr)\,dx \\ &= \varepsilon\biggl[-\biggl(1-\int_0^t g(s)\,ds\biggr)\int_\Omega |\nabla u|^2\,dx + \int_0^t g(t-\tau)\int_\Omega \bigl(\nabla u(\tau)-\nabla u(t)\bigr)\nabla u(t)\,dx\,d\tau \\ &\quad - \int_\Omega |u_t|^{m(x,t)-2}u_t u\,dx + \int_\Omega |u|^{p(x,t)}\,dx\biggr] \\ &\geq -\varepsilon\biggl[1 - \int_0^t g(s)\,ds + \frac{1}{4\delta}\int_0^t g(s)\,ds\biggr]\int_\Omega |\nabla u|^2\,dx - \varepsilon \int_\Omega |u_t|^{m(x,t)-2}u_t u\,dx \\ &\quad + \varepsilon \int_\Omega |u|^{p(x,t)}\,dx - \varepsilon\delta \int_0^t g(t-\tau)\int_\Omega \bigl|\nabla\bigl(u(\tau)-\nabla u(t)\bigr)\bigr|^2\,dx\,d\tau, \end{aligned} \tag{21}$$

where the coefficient δ will be determined later.

Moreover, using the condition $g(t) > 0$, $g'(t) < 0$, we have

$$J_1 \geq (1-\lambda)H^{-\lambda}(t)\int_\Omega |u_t|^{m(x,t)}\,dx. \tag{22}$$

Applying Young's inequality with $\eta > 1$ and the conditions $m^+ < p^-$, $0 < \lambda \leq \frac{p^- - m^+}{(m^+-1)p^-}$ and Lemma 2.3, we have

$$\begin{aligned} &\biggl|\int_\Omega |u_t|^{m(x,t)-1}|u|\,dx\biggr| \\ &\quad \leq \eta^{\frac{m^-}{m^- -1}}H^{-\lambda}(t)\int_\Omega |u_t|^{m(x,t)}\,dx + \frac{1}{\eta^{m^-}}C_1^{\lambda(m^- - m^+)}H^{\lambda(m^+-1)}(t)\int_\Omega |u|^{m(x,t)}\,dx \\ &\quad \leq \eta^{\frac{m^-}{m^- -1}}H^{-\lambda}(t)\int_\Omega |u_t|^{m(x,t)}\,dx + \frac{C_2}{\eta^{m^-}}H^{\lambda(m^+-1)}(t)\max\bigl\{\|u\|_{p(\cdot)}^{m^+}, \|u\|_{p(\cdot)}^{m^-}\bigr\}, \end{aligned} \tag{23}$$

where η will be determined later and the constants are defined as follows:

$$C_1 := \min\biggl\{\biggl(H(0) - \frac{|\Omega|}{p^-}\biggr), 1\biggr\}, \qquad C_2 = \bigl(1+|\Omega|\bigr)^{m^+}C_1^{\lambda(m^- - m^+)}.$$

According to (18) and (7), we have

$$\begin{aligned}
\|u\|_{p(\cdot)}^{m^+} &\le \max\left\{\left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^+}{p^+}}, \left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^+}{p^-}}\right\} \\
&\le \max\left\{1, H^{\frac{m^+}{p^+}-\frac{m^+}{p^-}}(t)\right\}\left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^+}{p^-}}; \\
\|u\|_{p(\cdot)}^{m^-} &\le \max\left\{\left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^-}{p^+}}, \left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^-}{p^-}}\right\} \\
&\le \max\left\{H^{\frac{m^- - m^+}{p^-}}(t), H^{\frac{m^-}{p^+}-\frac{m^+}{p^-}}(t)\right\}\left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^+}{p^-}}.
\end{aligned} \tag{24}$$

From (18) and (24), it follows easily

$$\max\left\{\|u\|_{p(\cdot)}^{m^+}, \|u\|_{p(\cdot)}^{m^-}\right\} \le C_3\left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{m^+}{p^-}}, \tag{25}$$

where $C_3 = 2C_1^{\frac{m^-}{p^+}-\frac{m^+}{p^-}}$.

Furthermore, by (20)–(22) and (25), we have

$$\begin{aligned}
F'(t) \ge{}& \left(1 - \lambda - \varepsilon\eta^{\frac{m^-}{m^- - 1}}\right)H^{-\lambda}(t)\int_\Omega |u_t|^{m(x,t)}\,dx + \varepsilon\int_\Omega u_t^2\,dx \\
&+ \varepsilon\int_\Omega |u|^{p(x,t)}\,dx - \varepsilon\delta(g\diamond\nabla u)(t)) \\
&- \varepsilon\left[1 - \int_0^t g(s)\,ds + \frac{1}{4\delta}\int_0^t g(s)\,ds\right]\int_\Omega |\nabla u|^2\,dx \\
&- \frac{\varepsilon C_2 C_3 C_1^{\lambda(m^+ - 1) + \frac{m^+}{p^-} - 1}}{\eta^{m^-}}\int_\Omega |u|^{p(x,t)}\,dx.
\end{aligned} \tag{26}$$

Utilizing the definition of $H(t)$ and $E(t)$, we get

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}\,dx \\
&\quad = H(t) - E_1 + \frac{1}{2}\left[\|u_t\|_2^2 + \left(1 - \int_0^t g(s)\,ds\right)\int_\Omega \left|\nabla u(t)\right|^2\,dx + (g\diamond\nabla u)(t)\right].
\end{aligned} \tag{27}$$

Inequality (15) and the definition of E_1 show that the following inequality holds:

$$E_1 \le \frac{(p^- - 2)\beta_1^{\frac{p^-}{2}}}{2\max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\}}\int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}\,dx. \tag{28}$$

Again choosing $2 < \theta < \frac{2p^- \max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\}}{2\max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\} + (p^- - 2)\beta_1^{\frac{p^-}{2}}} < p^-$ and then applying (26)–(28), we have

$$\begin{aligned} F'(t) \geq{}& \bigl[1 - \lambda - \varepsilon \eta^{\frac{m^-}{m^- - 1}}\bigr] H^{-\lambda}(t) \int_\Omega |u_t|^{m(x,t)}\,dx + \varepsilon\left(1 + \frac{\theta}{2}\right)\|u_t\|_2^2 + \varepsilon\theta H(t) \\ &+ \varepsilon\left(\frac{\theta}{2} - \delta\right)(g \diamond \nabla u)(t) + \varepsilon\left(\frac{\theta - 2}{2} - \left(\frac{\theta - 2}{2} + \frac{1}{4\delta}\right)\int_0^\infty g(s)\,ds\right)\|\nabla u(t)\|_2^2 \\ &+ \varepsilon\left(p^- - \theta - \frac{p^+ C_2 C_3 C_1^{\lambda(m^+ - 1) + \frac{m^+}{p^-} - 1}}{\eta^{m^-}}\right. \\ &\left. - \frac{(p^- - 2)\beta_1^{\frac{p^-}{2}}\theta}{2\max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\}}\right) \int_\Omega \frac{1}{p(x,t)}|u|^{p(x,t)}. \end{aligned} \tag{29}$$

Choosing δ, η, ε such that

$$\begin{aligned} &0 < \delta < \frac{\theta}{2}, \qquad \frac{p^+ C_2 C_3 C_1^{\lambda(m^+ - 1) + \frac{m^+}{p^-} - 1}}{\eta^{m^-}} < p^- - \theta - \frac{(p^- - 2)\beta_1^{\frac{p^-}{2}}\theta}{2\max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\}}, \\ &0 < \varepsilon < (1 - \lambda)\eta^{\frac{m^-}{1 - m^-}} \end{aligned}$$

and dropping nonnegative terms, in which we use condition (H_6), we have

$$F'(t) \geq C_4\left[\|u_t\|_2^2 + H(t) + (g \diamond \nabla u)(t) + \int_\Omega |u|^{p(x,t)}\,dx\right], \tag{30}$$

where

$$\begin{aligned} &C_4 = \min\left\{\varepsilon\left(1 + \frac{\theta}{2}\right), \varepsilon\theta, \varepsilon\left(\frac{\theta}{2} - \delta\right), \varepsilon\left(p^- - \theta - \frac{C_2 p^+}{\eta^m} - \frac{(p^- - 2)\beta_1^{\frac{p^-}{2}}\theta}{2\max\{\beta_2^{\frac{p^+}{2}}, \beta_2^{\frac{p^-}{2}}\}}\right)\frac{1}{p^+}\right\}; \\ &F(0) = H^{1-\lambda}(0) + \varepsilon\int_\Omega u_0 u_1\,dx > 0. \end{aligned} \tag{31}$$

Applying Hölder's inequality, the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{p^-}(\Omega) \hookrightarrow L^2(\Omega)$ $(p^- > 2)$, Young's inequality, and inequality (7), we have that

$$\begin{aligned} &\left(\left|\int_\Omega u_t u\,dx\right|\right)^{\frac{1}{1-\lambda}} \\ &\quad\leq \bigl(\|u\|_2\|u_t\|_2\bigr)^{\frac{1}{1-\lambda}} \leq \bigl(1 + |\Omega|\bigr)^{\frac{1}{(1-\lambda)}} \|u\|_{p(\cdot)}^{\frac{1}{1-\lambda}} \|u_t\|_2^{\frac{1}{1-\lambda}} \\ &\quad\leq \bigl(1 + |\Omega|\bigr)^{\frac{1}{(1-\lambda)}}\bigl[\|u\|_{p(x,t)}^{\frac{2}{1-2\lambda}} + \|u_t\|_2^2\bigr] \\ &\quad\leq \bigl(1 + |\Omega|\bigr)^{\frac{1}{(1-\lambda)}} \max\left\{\left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{2}{(1-2\lambda)p^-}}, \left(\int_\Omega |u|^{p(x,t)}\,dx\right)^{\frac{2}{(1-2\lambda)p^+}}\right\} \\ &\qquad + \bigl(1 + |\Omega|\bigr)^{\frac{1}{(1-\lambda)}}\|u_t\|_2^2 \\ &\quad\leq \bigl(1 + |\Omega|\bigr)^{\frac{1}{(1-\lambda)}} C_1^{\frac{2-(1-2\lambda)p^+}{(1-2\lambda)p^+}} \int_\Omega |u|^{p(x,t)}\,dx + \bigl(1 + |\Omega|\bigr)^{\frac{1}{(1-\lambda)}}\|u_t\|_2^2. \end{aligned} \tag{32}$$

By (32) and the definition of $F(t)$, we get

$$\begin{aligned} F^{\frac{1}{1-\lambda}}(t) &\le 2^{\frac{1}{1-\lambda}}\left[H(t) + \varepsilon^{\frac{1}{1-\lambda}}\left|\int_\Omega u_t u\,dx\right|^{\frac{1}{1-\lambda}}\right] \\ &\le 2^{\frac{1}{1-\lambda}}\left[H(t) + \bigl(1+|\Omega|\bigr)^{\frac{1}{(1-\lambda)}} C_1^{\frac{2-(1-2\lambda)p^+}{(1-2\lambda)p^+}} \int_\Omega |u|^{p(x,t)}\,dx \right. \\ &\quad \left. + \bigl(1+|\Omega|\bigr)^{\frac{1}{(1-\lambda)}}\|u_t\|_2^2\right] \\ &\le C_5\left[H(t) + \int_\Omega |u|^{p(x,t)}\,dx + \|u_t\|_2^2\right] \end{aligned} \tag{33}$$

with

$$C_5 = 2^{\frac{1}{1-\lambda}}\Bigl(1 + \bigl(1+|\Omega|\bigr)^{\frac{1}{(1-\lambda)}} C_1^{\frac{2-(1-2\lambda)p^+}{(1-2\lambda)p^+}} + \bigl(1+|\Omega|\bigr)^{\frac{1}{(1-\lambda)}}\Bigr). \tag{34}$$

By (30) and (33), we obtain the following inequality:

$$F'(t) \ge \frac{C_4}{C_5}F^{\frac{1}{1-\lambda}}(t). \tag{35}$$

By Gronwall's inequality, we have

$$F^{\frac{\lambda}{1-\lambda}}(t) \ge \frac{1}{F^{\frac{-\lambda}{1-\lambda}}(0) - \frac{C_4\lambda}{C_5(1-\lambda)}t}. \tag{36}$$

Therefore, inequality (36) implies that $F(t)$ blows up in finite time

$$T^* \le \frac{C_5(1-\lambda)}{F^{\frac{\lambda}{1-\lambda}}(0)C_4\lambda}. \tag{37}$$

Step 2. We give a lower bound for blow-up time T^*.
Define $J(t) = \int_\Omega |u|^{p^+}\,dx$, then

$$J'(t) = p^+\int_\Omega |u|^{p^+-2}uu_t\,dx \le p^+\left[\int_\Omega |u|^{2p^+-2}\,dx + \int_\Omega |u_t|^2\,dx\right]. \tag{38}$$

By $2 < p^+ \le \frac{2N-2}{N-2}$ and (8), one has

$$\int_\Omega |u|^{2p^+-2}\,dx \le B^{2p^+-2}\|\nabla u\|_2^{2p^+-2}. \tag{39}$$

Recalling the definition of $E(t)$ and $E(t) \le E_1$, we get

$$\begin{aligned} k\|\nabla u\|_2^2 + \|u_t\|_2^2 &\le E_1 + \int_\Omega |u|^{p(x,t)}\,dx \\ &\le E_1 + \int_{\{|u|\ge 1\}} |u|^{p(x,t)}\,dx + \int_{\{|u|\le 1\}} |u|^{p(x,t)}\,dx \\ &\le E_1 + \int_\Omega |u|^{p^+}\,dx + |\Omega|. \end{aligned} \tag{40}$$

Combining (38)–(40) with the inequality $(|a| + |b|)^q \le 2^{q-1}(|a|^q + |b|^q)$ $(q > 1)$, we have

$$\begin{aligned} J'(t) &\le C_6 k^{1-p^+} J^q(t) + C_6 \left(\frac{E_1 + |\Omega|}{k}\right)^{p^+-1} + E_1 + |\Omega| + J(t) \\ &\le C_6 k^{1-p^+} J^q(t) + J(t) + C_7, \end{aligned} \tag{41}$$

with

$$C_6 = 2^{p^+-2} p^+ B^{2p^+-2}, \qquad C_7 = C_6 \left(\frac{E_1 + |\Omega|}{k}\right)^{p^+-1} + E_1 + |\Omega|. \tag{42}$$

Applying $\lim_{t\to T^*} F(t) = +\infty$, Lemma 2.3, and inequality (18), we have

$$\lim_{t\to T^*} \int_\Omega |u|^{p^+}\, dx = +\infty. \tag{43}$$

(41) and (43) yield

$$\int_{J(0)}^{\infty} \frac{1}{C_6 y^{p^+-1} + y + C_7}\, dy \le T^*.$$

This completes the proof of the main results. □

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Authors' contributions
ZZ analyzed and interpreted the data regarding the viscoelastic hyperbolic equations with variable sources and damping terms. LD performed the method to lower and upper bounds for lifespan of solutions to viscoelastic hyperbolic equations, and was the major contributor in writing the manuscript. All authors read and approved the final manuscript.

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