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Exponential stability analysis of nonlinear systems with bounded gain error

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Abstract

Considering technology limitation or device restriction in practical application, we formulate new nonlinear systems with bounded gain error, which contain switched control and impulsive control. We then investigate the exponential stability of the considered systems. Finally, the effectiveness of the proposed criteria is confirmed via an example based on Chua's oscillator.

MSC: 37N35; 49N25

Keywords: Exponential stability; Nonlinear systems; Bounded gain error; Impulsive control; Switched control

1 Introduction

Nonlinear systems have been paid considerable attention from different areas since they have been successfully used in many practical applications including robotics, information science, artificial intelligence, automatic control systems, and so forth [1, 2]. Due to various effects, the states of systems will become oscillations and instability. Thus, it is significant to discuss stability of nonlinear systems. There are many methods to stabilize the nonlinear systems, for example, adaptive control [3], fuzzy control [4], sliding mode control [5], feedback control [6], impulsive control [7], switched control [8], etc. In view of engineering applications, the control cost of continuous control is expensive. By intermittent control, control cost and the amount of the transmitted information can be reduced drastically. It should be noticed that both impulsive control and switched control are discontinuous control methods.

Impulsive control of nonlinear systems has been one of the focal points in many research and application fields, such as complex networks, orbital transfer of satellite, dosage supply in pharmacokinetics, ecosystems management, synchronization in chaotic secure communication systems [9–15], etc. Impulsive control can stabilize nonlinear systems by using it only at some isolated points.

Switched control system is a hybrid system that is composed of several subsystems and a switching rule that orchestrates the switching among subsystems. In the real world, many biological, physical, engineering, and economical systems can be presented by switched systems. Compared with ordinary differential dynamic systems, not only should we focus on each subsystem, but the switching rule as well. Switching among different subsystems can cause chaos and instability. It is well known that a switched system might be stable even if each subsystem is unstable and also might be unstable even if all subsystems are stable for specified switching rules.

In this paper, switched control and impulsive control are combined together. We construct a control system, in which some of the inputs are continuous and some are impulsive.

The remainder of this paper is organized as follows. The considered model of general nonlinear systems with bounded gain error is given in Sect. 2. Some necessary notations and lemmas are also presented in this section. In Sect. 3, we establish an exponential stability criterion. Then, in Sect. 4, an example is presented to show the effectiveness of our result. Finally, we conclude the paper.

2 Problem formulation and preliminaries

A class of nonlinear systems can be described as

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + w(t), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

where $x(t) \in R^n$ is the state vector, $A \in R^{n\times n}$ is a constant matrix, $f : R^n \to R^n$ is a continuous nonlinear function satisfying $f(0) = 0$, and $\|f(x)\| \le l\|x\|$, $l \ge 0$ is a constant. $w(t)$ is control input. Without loss of generally, let $t_0 = 0$, $x_0 \in R^n$ is a given vector.

In order to stabilize system (2.1) at the origin, we set three kinds of control, i.e., in the first period of continuous time, we set $w(t) = B_1x(t)$, where $B_1 \in R^{n\times n}$ is a known matrix; in the second period of continuous time, we set $w(t) = B_2x(t)$, where $B_2 \in R^{n\times n}$ is a known matrix; at the same time, where the system is changed from the first control to the second control, we impose an impulse.

So system (2.1) is rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + B_1x(t), & kT \le t < kT + \tau, \\ x(t) = (Q + \Delta Q)x(t^-), & t = kT + \tau, \\ \dot{x}(t) = Ax(t) + f(x(t)) + B_2x(t), & kT + \tau < t < (k + 1)T, \end{cases} \tag{2.2}$$

where $T > 0$ represents control period, $\tau \in (0, T)$ is a constant. $Q \in R^{n\times n}$ is the impulsive control gain matrix, $\Delta Q \in R^{n\times n}$ denotes impulsive gain error caused by technology limitation or device restriction. In general, let

$$\Delta Q = mG(t)Q,$$

where m is a positive constant. The uncertain matrix $G(t) \in R^{n\times n}$ satisfies

$$G^T(t)G(t) \le I.$$

Remark 1 Recently, periodic control has been extensively investigated, intermittent and alternate cases have been separately studied. To unify them, the generalized control protocol (2.2) is presented in this paper, which contains the traditional periodically intermittent control [16], periodically alternate control [17, 18], and impulsive control.

Lemma 1 ([19]) *Let $x, y \in R^n$, then*

$$\left|x^T y\right| \le \|x\| \|y\|.$$

Lemma 2 ([20]) *Let $x, y \in R^n$ and $\varepsilon > 0$, then*

$$2x^T y \le \varepsilon x^T x + \frac{1}{\varepsilon} y^T y.$$

Lemma 3 ([19]) *Suppose that $A \in R^{n\times n}$ is a symmetric matrix. Then, for all $x \in R^n$,*

$$\lambda_{\min}(A)x^T x \le x^T Ax \le \lambda_{\max}(A)x^T x.$$

As customary, R^n is an n-dimensional real Euclidean space with norm $\|\cdot\|$. $R^{m\times n}$ denotes the set of all $m \times n$-dimensional real matrices. $\lambda_{\min}(A)$, $\lambda_{\max}(A)$, and A^T are the minimum, the maximum eigenvalue, and the transpose of matrix A, respectively. $A > 0$ implies that A is a positive definite matrix. I is an identity matrix of proper dimension. $f(x(t_0^-))$ is defined by $f(x(t_0^-)) = \lim_{t\to t_0^-} f(x(t))$.

3 Stability analysis

In this section, we aim at proposing the exponential stability criterion of system (2.2).

Theorem 1 *Let $0 < P \in R^{n\times n}$ such that the following two conditions are satisfied:*

(1) $h_1 < 0$,

(2) $h_1\tau + h_2(T - \tau) + \ln\eta < 0$,

where $\beta_1 = \lambda_{\max}(P^{-1}(PA + A^T P + PB_1 + B_1^T P))$, $\beta_2 = \lambda_{\max}(P)$, $\beta_3 = \lambda_{\min}(P)$, $\beta_4 = \lambda_{\max}(P^{-1}(Q^T Q))$, $\beta_5 = \lambda_{\max}(P^{-1}(PA + A^T P + PB_2 + B_2^T P))$, $h_1 = \beta_1 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$, $\eta = \beta_2\beta_4((1 + \varepsilon) + m^2(1 + \frac{1}{\varepsilon}))$, $h_2 = \beta_5 + 2l\sqrt{\frac{\beta_2}{\beta_3}}$. *Then system (2.2) is exponentially stable at origin.*

Proof Define

$$V\big(x(t)\big) = x^T(t)Px(t).$$

Let $t \in [kT, kT + \tau)$, from Lemmas 1 and 3, we obtain

$$\begin{aligned}
D^+\big(V\big(x(t)\big)\big) &= 2x^T(t)P\big(Ax(t) + f\big(x(t)\big) + B_1x(t)\big) \\
&= 2x^T(t)PAx(t) + 2x^T(t)Pf\big(x(t)\big) + 2x^T(t)PB_1x(t) \\
&= x^T(t)\big(PA + A^T P + PB_1 + B_1^T P\big)x(t) + 2x^T(t)P^{\frac{1}{2}}P^{\frac{1}{2}}f\big(x(t)\big) \\
&\le \beta_1x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)f^T\big(x(t)\big)Pf\big(x(t)\big)} \\
&\le \beta_1x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)\beta_2f^T\big(x(t)\big)f\big(x(t)\big)} \\
&\le \beta_1x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)\beta_2l^2x^T(t)x(t)} \\
&\le \beta_1x^T(t)Px(t) + 2l\sqrt{x^T(t)Px(t)\frac{\beta_2}{\beta_3}x^T(t)Px(t)}
\end{aligned}$$

$$= h_1 V(x(t)),$$

which means

$$V(x(t)) \leq V(x(kT)^-) e^{h_1(t-kT)}. \tag{3.1}$$

If $t = kT + \tau$, then from Lemmas 2 and 3 we have

$$\begin{aligned}
V(x(t)) &= ((Q + \Delta Q)x(t^-))^T P(Q + \Delta Q)x(t^-) \\
&= x^T(t^-)(Q^T PQ + Q^T P\Delta Q + \Delta Q^T PQ + \Delta Q^T P\Delta Q)x(t^-) \\
&\leq x^T(t^-)\left((1+\varepsilon)Q^T PQ + \left(1 + \frac{1}{\varepsilon}\right)\Delta Q^T P\Delta Q\right)x(t^-) \\
&\leq \beta_2 x^T(t^-)\left((1+\varepsilon)Q^T Q + \left(1 + \frac{1}{\varepsilon}\right)\Delta Q^T \Delta Q\right)x(t^-) \\
&= \beta_2 x^T(t^-)\left((1+\varepsilon)Q^T Q + m^2\left(1 + \frac{1}{\varepsilon}\right)Q^T G^T(t)G(t)Q\right)x(t^-) \\
&\leq \beta_2 x^T(t^-)\left((1+\varepsilon)Q^T Q + m^2\left(1 + \frac{1}{\varepsilon}\right)Q^T Q\right)x(t^-) \\
&\leq \beta_2\beta_4\left((1+\varepsilon) + m^2\left(1 + \frac{1}{\varepsilon}\right)\right)V(x(t^-)) \\
&= \eta V(x(t^-)).
\end{aligned} \tag{3.2}$$

In the same way, let $t \in (kT + \tau, (k+1)T)$, we also obtain

$$\begin{aligned}
D^+(V(x(t))) &= 2x^T(t)P(Ax(t) + f(x(t)) + B_2 x(t)) \\
&= x^T(t)(PA + A^T P + PB_2 + B_2^T P)x(t) + 2x^T(t)Pf(x(t)) \\
&\leq \beta_5 x^T(t)Px(t) + 2\sqrt{x^T(t)Px(t)f^T(x(t))Pf(x(t))} \\
&\leq h_2 V(x(t)),
\end{aligned}$$

which together with (3.2) infers that

$$V(x(t)) \leq \eta V(x(kT + \tau)^-) e^{h_2(t-kT-\tau)}, \tag{3.3}$$

where $t \in [kT + \tau, (k+1)T)$.

When $k = 0$, let $t \in [0, \tau)$, from (3.1) we can obtain

$$V(x(t)) \leq V(x(0)) e^{h_1 t},$$

hence

$$V(x(\tau^-)) \leq V(x(0)) e^{h_1 \tau}. \tag{3.4}$$

Let $t \in [\tau, T)$, applying (3.3) and (3.4), we get

$$\begin{aligned} V\big(x(t)\big) &\le \eta V\big(x\big(\tau^-\big)\big)e^{h_2(t-\tau)} \\ &\le \eta V\big(x(0)\big)e^{h_1\tau + h_2(t-\tau)}, \end{aligned}$$

hence

$$V\big(x\big(T^-\big)\big) \le \eta V\big(x(0)\big)e^{h_1\tau + h_2(T-\tau)}. \tag{3.5}$$

When $k = 1$, let $t \in [T, T+\tau)$, applying (3.1) and (3.5), we get

$$\begin{aligned} V\big(x(t)\big) &\le V\big(x\big(T^-\big)\big)e^{h_1(t-T)} \\ &\le \eta V\big(x(0)\big)e^{h_1\tau + h_2(T-\tau)+h_1(t-T)}, \end{aligned}$$

hence

$$V\big(x\big((T+\tau)^-\big)\big) \le \eta V\big(x(0)\big)e^{2h_1\tau + h_2(T-\tau)}. \tag{3.6}$$

Let $t \in [T+\tau, 2T)$, applying (3.3) and (3.6), we get

$$\begin{aligned} V\big(x(t)\big) &\le \eta V\big(x\big((T+\tau)^-\big)\big)e^{h_2(t-T-\tau)} \\ &\le \eta^2 V\big(x(0)\big)e^{2h_1\tau + h_2(T-\tau)+h_2(t-T-\tau)}. \end{aligned}$$

By induction, when $k = m,\ m = 0, 1, \dots$, let $t \in [mT, mT+\tau)$, we get

$$V\big(x(t)\big) \le \eta^m V\big(x(0)\big)e^{mh_1\tau + mh_2(T-\tau)+h_1(t-mT)}, \tag{3.7}$$

hence

$$V\big(x\big((mT+\tau)^-\big)\big) \le \eta^m V\big(x(0)\big)e^{(m+1)h_1\tau + mh_2(T-\tau)}. \tag{3.8}$$

Let $t \in [mT+\tau, (m+1)T)$, applying (3.3) and (3.8), we obtain

$$\begin{aligned} V\big(x(t)\big) &\le \eta V\big(x\big((mT+\tau)^-\big)\big)e^{h_2(t-mT-\tau)} \\ &\le \eta^{m+1} V\big(x(0)\big)e^{(m+1)h_1\tau + mh_2(T-\tau)+h_2(t-mT-\tau)}. \end{aligned} \tag{3.9}$$

Applying (3.7), we get

$$\begin{aligned} V\big(x(t)\big) &\le \eta^m V\big(x(0)\big)e^{mh_1\tau + mh_2(T-\tau)} \\ &= V\big(x(0)\big)e^{m(h_1\tau + h_2(T-\tau)+\ln\eta)} \\ &< V\big(x(0)\big)e^{\frac{t-\tau}{T}(h_1\tau + h_2(T-\tau)+\ln\eta)} \\ &< V\big(x(0)\big)e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau)+\ln\eta)}, \end{aligned} \tag{3.10}$$

where $t \in [mT, mT+\tau)$.

Let $t \in [mT + \tau, (m+1)T)$, applying (3.9), we get the following.

Case 1. When $h_2 > 0$, we get

$$\begin{aligned} V\bigl(x(t)\bigr) &< \eta^{m+1} V\bigl(x(0)\bigr) e^{(m+1)h_1\tau + (m+1)h_2(T-\tau)} \\ &< V\bigl(x(0)\bigr) e^{\frac{t}{T}(h_1\tau + h_2(T-\tau) + \ln\eta)} \\ &< V\bigl(x(0)\bigr) e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau) + \ln\eta)}. \end{aligned} \tag{3.11}$$

Case 2. When $h_2 \leq 0$, we get

$$\begin{aligned} V\bigl(x(t)\bigr) &\leq \eta^{m+1} V\bigl(x(0)\bigr) e^{(m+1)h_1\tau + mh_2(T-\tau)} \\ &< \eta^{m+1} V\bigl(x(0)\bigr) e^{mh_1\tau + mh_2(T-\tau)} \\ &< \eta V\bigl(x(0)\bigr) e^{\frac{t-T}{T}(h_1\tau + h_2(T-\tau) + \ln\eta)}. \end{aligned} \tag{3.12}$$

For all $t > 0$, by (3.10), (3.11), and (3.12), we can conclude that system (2.2) is exponentially stable at origin.

This completes the proof. □

4 A numerical example

For verifying the effectiveness of Theorem 1, a numerical example is presented in this section.

Example 1 Chua's oscillator [21] is given by

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1 - g(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\gamma x_2, \end{cases} \tag{4.1}$$

where α and γ are parameters,

$$g(x_1) = bx_1 + 0.5(a-b)\bigl(|x_1 + 1| - |x_1 - 1|\bigr),$$

where $a < b < 0$ are two given constants.

For using the above result, system (4.1) is rewritten as

$$\dot{x}(t) = Ax + f(x),$$

where

$$A = \begin{pmatrix} -\alpha(1+b) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\gamma & 0 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} -0.5\alpha(a-b)(|x_1+1| - |x_1-1|) \\ 0 \\ 0 \end{pmatrix}.$$

Figure 1 The chaotic phenomenon of (4.1) with the initial condition $x(0) = (5, 1, -3)^T$

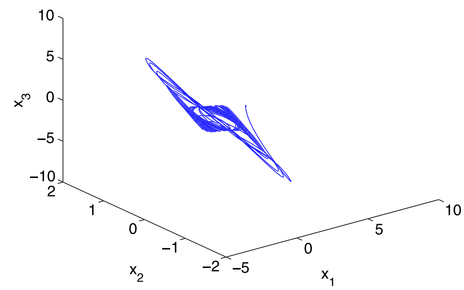
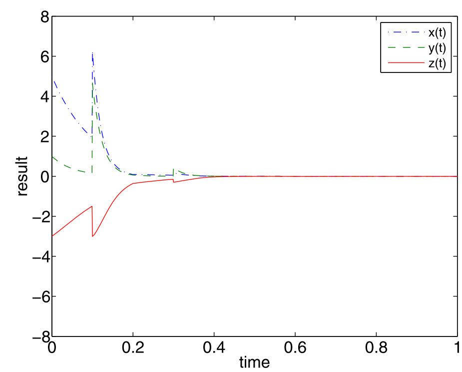


Figure 2 Time response curves of (4.1) with the initial condition $x(0) = (5, 1, -3)^T$



In the initial condition $x(0) = (5, 1, -3)^T$, system (4.1) has chaotic phenomenon when

$$\alpha = 9.2156, \qquad \gamma = 15.9946, \qquad a = -1.24905, \qquad b = -0.75735,$$

as shown in Fig. 1. In this case, we can get $l = 4.5313$.

In order to simplify the calculation, let $P = I$, $\tau = 0.1$, $T = 0.2$, $\varepsilon = 0.5$, $m = 1$, $B_1 = \operatorname{diag}(-10, -20, -10)$, $B_2 = \operatorname{diag}(-50, -40, -30)$, and

$$Q = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}.$$

Through simple computation, we get $\beta_1 = -10.5898$, $\beta_2 = \beta_3 = 1$, $\beta_4 = 8.21$, $\beta_5 = -51.9794$, $h_1 = -1.5272 < 0$, $h_2 = -42.9168$, $\eta = 36.945$, and $h_1\tau + h_2(T - \tau) + \ln\eta = -0.7630 < 0$. So, system (4.1) is exponentially stable by Theorem 1 with the initial condition $x(0) = (5, 1, -3)^T$, as shown in Fig. 2.

5 Conclusions

In this paper, we investigate the exponential stability of nonlinear systems with bounded gain error. A generalized control model, which contains switched control and impulsive control, is introduced. Compared with the traditional periodic control [16–18], system (2.2) is more general and more practical. Finally, a numerical example is given to show the effectiveness of the proposed method.

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Authors' contributions
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