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Some new fractional integral inequalities for exponentially *m*-convex functions via extended generalized Mittag-Leffler function

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Abstract
In the article, we establish some new general fractional integral inequalities for exponentially *m*-convex functions involving an extended Mittag-Leffler function, provide several kinds of fractional integral operator inequalities and give certain special cases for our obtained results.

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1 Introduction

Convex functions and their variant forms are being used to study a wide class of problems which arises in various branches of pure and applied mathematics [1–24]. This theory provides us a natural, unified and general framework to study a wide class of unrelated problems. Many applications, generalizations and other aspects of convex functions and their variant forms can be found in the recent literature [25–62].

An important class of convex functions, which is called the class of exponential convex functions, was introduced and studied by Antczak [63] and Dragomir et al. [64]. Alirezai and Mathar [65] investigated their mathematical properties along with their potential applications in statistics and information theory. Due to its significance, Awan et al. [66], and Jakšetić and Pečarić [67] defined another kind of exponential convex functions, which have shown that the class of exponential convex functions unifies various concepts in different manners.

In [68], Toader defined the m-convexity as an intermediate between the usual convexity and star shaped property. If we set $m = 0$, then we have the concept of star shaped functions on $[a, b]$. We recall that $f : [a, b] \to \mathbb{R}$ is said to be star shaped if $f(tx) \leq tf(x)$ for all $t \in [0, 1]$ and $x \in [a, b]$.

We would like to emphasize that exponentially convex functions and m-convex functions are two distinct classes of convex functions. It is natural to introduce a new class of convex functions to unify these concepts. For this purpose, we need to recall some basic concepts as follows.

Definition 1.1 (See [68]) Let $m \in [0,1]$. Then the real number set $I \subseteq \mathbb{R}$ is said to be m-convex if

$$(1-t)a + mtb \in I$$

for all $a, b \in I$ and $t \in [0,1]$.

From Definition 1.1 we clearly see that the m-convex set I contains the line segment between points a and mb for every pair of points a and b of I.

Definition 1.2 (See [68]) Let $m \in [0,1]$ and $I \subseteq \mathbb{R}$ be a m-convex set. Then A real-valued function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be a m-convex if

$$f\big[(1-t)a + mtb\big] \le (1-t)f(a) + mtf(b)$$

for all $a, b \in I$ and $t \in [0,1]$.

Remark 1.3 From Definition 1.2 we clearly see that the 1-convex function is a convex function in the ordinary sense and the 0-convex function is the star shaped function. If we take $m = 1$, then we recapture the concept of convex functions. If we take $t = 1$, then we get

$$f(mb) \le mf(b)$$

for all $a, b \in I$, which implies that the function f is sub-homogeneous.

Definition 1.4 (See [64]) A real-valued function $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be an exponentially convex on K if

$$e^{f[(1-t)a+tb]} \le (1-t)e^{f(a)} + te^{f(b)}$$

for all $a, b \in K$ and $t \in [0,1]$.

Definition 1.5 Let $m \in [0,1]$ and $K \subseteq \mathbb{R}$ be a m-convex set. Then a real-valued function $f : K \to \mathbb{R}$ is said to be exponentially m-convex if

$$e^{f[(1-t)a+mtb]} \le (1-t)e^{f(a)} + mte^{f(b)}$$

for all $a, b \in K$ and $t \in [0,1]$.

Example 1.6 Let $f(x) = 2\log x$. Then $f(x)$ is exponentially m-convex on $(0,\infty)$ for any $m \in [0,1]$. Indeed,

$$\begin{aligned}
&e^{f[(1-t)a+mtb]} = \big[(1-t)a + mtb\big]^2, \\
&(1-t)e^{f(a)} + mte^{f(b)} = (1-t)a^2 + mtb^2, \\
&(1-t)e^{f(a)} + mte^{f(b)} - e^{f[(1-t)a+mtb]} \\
&\quad = t\big[(1-t)a^2 + m(1-mt)b^2 - 2m(1-t)ab\big]
\end{aligned}$$

$$\geq 2t\left[\sqrt{m(1-t)(1-mt)} - m(1-t)\right]ab$$
$$= 2t\sqrt{m(1-t)}\left[\sqrt{1-mt} - \sqrt{m(1-t)}\right]ab \geq 0$$

for all $a, b \in (0, \infty)$ and $m, t \in [0, 1]$.

Fractional analysis can be regarded as an expansion of classical analysis. Fractional analysis has been studied by many scientists and they have expressed the fractional derivative and integral in different ways with different notations. Although the expressions between these different definitions can be transformed into each other, but these different definitions and expressions have different physical meanings. It is well known that the first fractional integral operator is the Riemann–Liouville fractional integral operator. Recently, some new definitions of the fractional derivative were given by many mathematicians, which are the natural extensions of the classical derivative. These new definitions drew attention with their variability to classical derivative.

Let $b > a \geq 0$, $u > 0$ and $f \in L_1[a, b]$. Then the left and right sided Riemann–Liouville fractional integrals of order u are defined by

$$I^u_{a^+} f(t) = \frac{1}{\Gamma(u)} \int_a^t (t-\xi)^{u-1} f(\xi)\, d\xi, \quad t > a,$$

and

$$I^u_{b^-} f(t) = \frac{1}{\Gamma(u)} \int_t^b (\xi - t)^{u-1} f(\xi)\, d\xi, \quad t > a,$$

where $\Gamma(u) = \int_0^\infty t^{u-1} e^{-t}\, dt$ denotes the Gamma function [69–71].

Definition 1.7 Let $\mu, \alpha, \jmath, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(\jmath) > 0$, $\Re(\mathfrak{c}) > \Re(\gamma) > 0$, $p \geq 0$, $\delta > 0$ and $0 < \kappa \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}(t;p)$ is defined by

$$E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + n\kappa, c - \gamma)(c)_{n\kappa}}{\beta(\gamma, c-\gamma)\Gamma(\mu n + \alpha)} \frac{t^n}{(\jmath)_{n\delta}},$$

where β_p is the generalized beta function defined by

$$\beta_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt$$

and $(c)_{n\kappa}$ is the Pochhammer symbol [72–74] defined as $(c)_{n\kappa} = \Gamma(c + n\kappa)/\Gamma(c)$.

In [75], several properties of the generalized Mittag-Leffler function are discussed, and it has been proved that $E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}(t;p)$ is absolutely convergent for $k < \delta + \mu$. If S is the sum of the series of absolute terms of the Mittag-Leffler function $E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}(t;p)$, then we have $|E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}(\cdot;p)| \leq S$. This property will be used to prove our main results.

The corresponding left and right sided extended generalized fractional integral operators are defined by Theorem 1.8 as follows.

Theorem 1.8 (See [75]) *Let $\mu, \alpha, \jmath, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(\jmath) > 0$, $\Re(c) > \Re(\gamma) > 0$, $p \geq 0$, $\delta > 0$, $0 < \kappa \leq \delta + \Re(\mu)$, $f \in L_1[a,b]$ and $x \in [a,b]$. Then the extended generalized fractional integral operators $\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,a^+} f$ and $\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,b^-} f$ can be defined by*

$$\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,a^+} f(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\big(w(x-t)^{\mu};p\big) f(t)\,dt$$

and

$$\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,b^-} f(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\big(w(t-x)^{\mu};p\big) f(t)\,dt.$$

From the extended generalized fractional integral operators, we have

$$\begin{aligned}
\big(\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,a^+} 1\big)(x;p) &= \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\big(w(x-t)^{\mu};p\big)\,dt \\
&= \int_a^x (x-t)^{\alpha-1} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+n\kappa, c-\gamma)(c)_{n\kappa}}{\beta(\gamma, c-\gamma)\Gamma(\mu n+\alpha)} \frac{w^n (x-t)^{\mu n}}{(\jmath)_{n\delta}}\,dt \\
&= \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+n\kappa, c-\gamma)(c)_{n\kappa}}{\beta(\gamma, c-\gamma)\Gamma(\mu n+\alpha)} \frac{w^n}{(\jmath)_{n\delta}} \int_a^x (x-t)^{\mu n+\alpha-1}\,dt \\
&= (x-a)^{\alpha} \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+n\kappa, c-\gamma)(c)_{n\kappa}}{\beta(\gamma, c-\gamma)\Gamma(\mu n+\alpha)} \frac{w^n}{(\jmath)_{n\delta}} (x-a)^{\mu n} \frac{1}{\mu n+\alpha}.
\end{aligned}$$

Hence

$$\big(\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,a^+} 1\big)(x;p) = (x-a)^{\alpha} E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\big(w(x-a)^{\mu};p\big),$$

and similarly

$$\big(\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,b^-} 1\big)(x;p) = (x-a)^{\alpha} E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\big(w(b-x)^{\mu};p\big).$$

We will use the following notations in the article:

$$\zeta_{\alpha,a^+}(x;p) = \big(\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,a^+} 1\big)(x;p)$$

and

$$\zeta_{\alpha,b^-}(x;p) = \big(\epsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w,b^-} 1\big)(x;p).$$

More information related to the Mittag-Leffler functions and the corresponding fractional integral operators can be found in the literature [76–78].

The main purpose of the article is to establish a Hadamard type inequality and several general fractional integral inequalities for the exponentially m-convex functions involving an extended Mittag-Leffler function, and deduce some new results which are quite general.

2 Main results

In order to establish our main results we need a lemma, which we present in this section.

Lemma 2.1 *Let $0 \leq a < mb$ and $f : [a, mb] \to \mathbb{R}$ be a differentiable function such that $(e^f)' \in L_1[a, mb]$. Then one has*

$$\begin{aligned}
&\left(\int_a^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} \left[e^{f(a)} + e^{f(mb)}\right] \\
&\quad - \alpha \int_a^{mb} \left(\int_a^t w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha-1} w(t) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^{\mu}; p) e^{f(t)}\, dt \\
&\quad - \alpha \int_a^{mb} \left(\int_t^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha-1} w(t) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^{\mu}; p) e^{f(t)}\, dt \\
&= \int_a^{mb} \left(\int_a^t w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} e^{f(t)} f'(t)\, dt \\
&\quad - \int_a^{mb} \left(\int_t^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} e^{f(t)} f'(t)\, dt. \qquad (2.1)
\end{aligned}$$

Proof Integrating by parts gives

$$\begin{aligned}
&\int_a^{mb} \left(\int_a^t w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} e^{f(t)} f'(t)\, dt \\
&= \left(\int_a^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} e^{f(mb)} \\
&\quad - \alpha \int_a^{mb} \left(\int_a^t w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha-1} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p) e^{f(t)}\, dt \qquad (2.2)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^{mb} \left(\int_t^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} e^{f(t)} f'(t)\, dt \\
&= -\left(\int_a^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} e^{f(a)} \\
&\quad + \alpha \int_a^{mb} \left(\int_t^{mb} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha-1} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p) e^{f(t)}\, dt. \qquad (2.3)
\end{aligned}$$

Therefore, identity (2.1) follows from (2.2) and (2.3). □

Let $m = 1$. Then Lemma 2.1 leads to Corollary 2.2 immediately.

Corollary 2.2 *Let $0 \leq a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable exponential function such that $(e^f)' \in L_1[a, b]$. Then the identity for the extended generalized fractional integral operators*

$$\begin{aligned}
&\left(\int_a^{b} w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha} \left[e^{f(a)} + e^{f(b)}\right] \\
&\quad - \alpha \int_a^{b} \left(\int_a^t w(s) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^{\mu}; p)\right)^{\alpha-1} w(t) E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^{\mu}; p) e^{f(t)}\, dt
\end{aligned}$$

$$\begin{aligned}
&-\alpha\int_a^b\left(\int_t^b w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt\\
&=\int_a^b\left(\int_a^t w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha} e^{f(t)}f'(t)\,dt\\
&\quad-\int_a^b\left(\int_t^b w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha} e^{f(t)}f'(t)\,dt
\end{aligned}\tag{2.4}$$

holds.

Theorem 2.3 *Let $0 \le a < mb$ and $f : [a, mb] \to \mathbb{R}$ be a differentiable function such that $(e^f)' \in L_1[a, mb]$. Then the inequality*

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha}\left[e^{f(a)}+e^{f(mb)}\right]\\
&\quad-\alpha\int_a^{mb}\left(\int_a^t w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt\\
&\quad-\alpha\int_a^{mb}\left(\int_t^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt\Bigg|\\
&\le\frac{(mb-a)^{\alpha+1}\|w\|_\infty^\alpha S^\alpha}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&\quad\times\Big(\big(\alpha^2+3\alpha\big)\big[\big|e^{f(a)}f'(a)\big|+m^2\big|e^{f(b)}f'(b)\big|\big]+2m(\alpha+1)\Delta(a,b)\Big)
\end{aligned}\tag{2.5}$$

for the extended generalized fractional integral operators holds if $|f'|$ is an exponentially m-convex function on $[a, mb]$, where

$$\Delta(a,b)=\left\{\left|e^{f(a)}f'(b)\right|+\left|e^{f(b)}f'(a)\right|\right\}.$$

Proof It follows from Lemma 2.1 that

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha}\left[e^{f(a)}+e^{f(mb)}\right]\\
&\quad-\alpha\int_a^{mb}\left(\int_a^t w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt\\
&\quad-\alpha\int_a^{mb}\left(\int_t^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt\Bigg|\\
&\le\int_a^{mb}\left|\int_a^t w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right|^{\alpha}\left|e^{f(t)}f'(t)\right|dt\\
&\quad+\int_a^{mb}\left|\int_t^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right|^{\alpha}\left|e^{f(t)}f'(t)\right|dt.
\end{aligned}\tag{2.6}$$

By using absolute convergence property of the Mittag-Leffler function and $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$, we have

$$\begin{aligned}
&\bigg|\bigg(\int_a^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta s^\mu;p\big)\bigg)^\alpha\big[e^{f(a)}+e^{f(mb)}\big]\\
&\quad-\alpha\int_a^{mb}\bigg(\int_a^t w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta s^\mu;p\big)\bigg)^{\alpha-1}w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta t^\mu;p\big)e^{f(t)}\,dt\\
&\quad-\alpha\int_a^{mb}\bigg(\int_t^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta s^\mu;p\big)\bigg)^{\alpha-1}w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta t^\mu;p\big)e^{f(t)}\,dt\bigg|\\
&\leq\|w\|_\infty^\alpha S^\alpha\bigg(\int_a^{mb}(t-a)^\alpha\big|e^{f(t)}f'(t)\big|\,dt+\int_a^{mb}(mb-t)^\alpha\big|e^{f(t)}f'(t)\big|\,dt\bigg).
\end{aligned}\tag{2.7}$$

Since $|e^{f(t)}f'(t)|$ is an exponentially m-convex function, we get

$$\begin{aligned}
&\big|e^{f(t)}f'(t)\big|\\
&\leq\bigg[\frac{mb-t}{mb-a}\big|e^{f(a)}\big|+m\frac{t-a}{mb-a}\big|e^{f(b)}\big|\bigg]\bigg[\frac{mb-t}{mb-a}\big|f'(a)\big|+m\frac{t-a}{mb-a}\big|f(b)\big|\bigg]\\
&=\bigg(\frac{mb-t}{mb-a}\bigg)^2\big|e^{f(a)}f'(a)\big|+m^2\bigg(\frac{t-a}{mb-a}\bigg)^2\big|e^{f(b)}f'(b)\big|\\
&\quad+m\bigg(\frac{mb-t}{mb-a}\bigg)\bigg(\frac{t-a}{mb-a}\bigg)\big\{\big|e^{f(a)}f'(b)\big|+\big|e^{f(b)}f'(a)\big|\big\}\\
&=\bigg(\frac{mb-t}{mb-a}\bigg)^2\big|e^{f(a)}f'(a)\big|+m^2\bigg(\frac{t-a}{mb-a}\bigg)^2\big|e^{f(b)}f'(b)\big|\\
&\quad+m\bigg(\frac{mb-t}{mb-a}\bigg)\bigg(\frac{t-a}{mb-a}\bigg)\Delta(a,b).
\end{aligned}\tag{2.8}$$

By taking into account the inequalities (2.7) and (2.8), we deduce

$$\begin{aligned}
&\bigg|\bigg(\int_a^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta s^\mu;p\big)\bigg)^\alpha\big[e^{f(a)}+e^{f(mb)}\big]\\
&\quad-\alpha\int_a^{mb}\bigg(\int_a^t w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta s^\mu;p\big)\bigg)^{\alpha-1}w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta t^\mu;p\big)e^{f(t)}\,dt\\
&\quad-\alpha\int_a^{mb}\bigg(\int_t^{mb} w(s)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta s^\mu;p\big)\bigg)^{\alpha-1}w(t)E_{\mu,\alpha,J}^{\gamma,\delta,\kappa,c}\big(\eta t^\mu;p\big)e^{f(t)}\,dt\bigg|\\
&\leq\|w\|_\infty^\alpha S^\alpha\bigg(\int_a^{mb}(t-a)^\alpha\bigg[\bigg(\frac{mb-t}{mb-a}\bigg)^2\big|e^{f(a)}f'(a)\big|+m^2\bigg(\frac{t-a}{mb-a}\bigg)^2\big|e^{f(b)}f'(b)\big|\\
&\quad+m\bigg(\frac{mb-t}{mb-a}\bigg)\bigg(\frac{t-a}{mb-a}\bigg)\Delta(a,b)\bigg]\,dt+\int_a^{mb}(mb-t)^\alpha\bigg[\bigg(\frac{mb-t}{mb-a}\bigg)^2\big|e^{f(a)}f'(a)\big|\\
&\quad+m^2\bigg(\frac{t-a}{mb-a}\bigg)^2\big|e^{f(b)}f'(b)\big|+m\bigg(\frac{mb-t}{mb-a}\bigg)\bigg(\frac{t-a}{mb-a}\bigg)\Delta(a,b)\bigg]\,dt\bigg)\\
&=\frac{(mb-a)^{\alpha+1}\|w\|_\infty^\alpha S^\alpha}{(\alpha+1)(\alpha+2)(\alpha+3)}\big(\big(\alpha^2+3\alpha\big)\big[\big|e^{f(a)}f'(a)\big|+m^2\big|e^{f(b)}f'(b)\big|\big]+2m(\alpha+1)\Delta(a,b)\big).
\end{aligned}$$

This completes the proof. □

Let $m = 1$. Then Theorem 2.3 leads to Corollary 2.4 immediately.

Corollary 2.4 *Let $0 \leq a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $(e^f)' \in L_1[a,b]$. Then the inequality*

$$\begin{aligned}
&\biggl|\biggl(\int_a^b w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta s^\mu;p\bigr)\biggr)^\alpha \bigl[e^{f(a)} + e^{f(b)}\bigr] \\
&\quad - \alpha \int_a^b \biggl(\int_a^t w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta s^\mu;p\bigr)\biggr)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta t^\mu;p\bigr)e^{f(t)}\,dt \\
&\quad - \alpha \int_a^{mb} \biggl(\int_t^b w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta s^\mu;p\bigr)\biggr)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta t^\mu;p\bigr)e^{f(t)}\,dt\biggr| \\
&\quad \leq \frac{(b-a)^{\alpha+1}\|w\|^\alpha_\infty S^\alpha}{(\alpha+1)(\alpha+2)(\alpha+3)}\bigl(\bigl(\alpha^2+3\alpha\bigr)\bigl[\bigl|e^{f(a)}f'(a)\bigr| + \bigl|e^{f(b)}f'(b)\bigr|\bigr] + 2(\alpha+1)\Delta(a,b)\bigr)
\end{aligned}$$

for extended generalized fractional integral operators holds if $|f'|$ is a convex function on $[a,b]$ and $k < \delta + \Re(\mu)$, where $\|w\|^\alpha_\infty = \sup_{t\in[a,b]}|w(t)|$ and $\Delta(a,b)$ are given in Theorem 2.3.

Corollary 2.5 *If $p = 0$ and all the assumptions of Theorem* 2.3 *are satisfied, then one has*

$$\begin{aligned}
&\biggl|\biggl(\int_a^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta s^\mu\bigr)\biggr)^\alpha \bigl[e^{f(a)} + e^{f(mb)}\bigr] \\
&\quad - \alpha \int_a^{mb} \biggl(\int_a^t w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta s^\mu\bigr)\biggr)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta t^\mu\bigr)e^{f(t)}\,dt \\
&\quad \alpha \int_a^{mb} \biggl(\int_t^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta s^\mu\bigr)\biggr)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath}\bigl(\eta t^\mu\bigr)e^{f(t)}\,dt\biggr| \\
&\quad \leq \frac{(mb-a)^{\alpha+1}\|w\|^\alpha_\infty S^\alpha}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
&\qquad \times \bigl(\bigl(\alpha^2+3\alpha\bigr)\bigl[\bigl|e^{f(a)}f'(a)\bigr| + m^2\bigl|e^{f(b)}f'(b)\bigr|\bigr] + 2m(\alpha+1)\Delta(a,b)\bigr)
\end{aligned}$$

for $k < \delta + \Re(\mu)$, where $\|w\|^\alpha_\infty = \sup_{t\in[a,b]}|w(t)|$ and $\Delta(a,b)$ is given in Theorem 2.3.

Corollary 2.6 *Let $\jmath = p = 0$, $m = 1$ and all the assumptions of Theorem* 2.3 *are satisfied, then we get*

$$\begin{aligned}
&\biggl|\biggl(\int_a^b w(s)\,ds\biggr)^\alpha \bigl[e^{f(a)} + e^{f(b)}\bigr] - \alpha \int_a^b \biggl(\int_a^t w(s)\,ds\biggr)^{\alpha-1} w(t)e^{f(t)}\,dt \\
&\quad - \alpha \int_a^b \biggl(\int_a^t w(s)\,ds\biggr)^{\alpha-1} w(t)e^{f(t)}\,dt\biggr| \\
&\quad \leq \frac{(b-a)^{\alpha+1}\|w\|^\alpha_\infty}{(\alpha+1)(\alpha+2)(\alpha+3)}\bigl(\bigl(\alpha^2+3\alpha\bigr)\bigl[\bigl|e^{f(a)}f'(a)\bigr| + \bigl|e^{f(b)}f'(b)\bigr|\bigr] + 2(\alpha+1)\Delta(a,b)\bigr),
\end{aligned}$$

where $\alpha > 0$, $\|w\|^\alpha_\infty = \sup_{t\in[a,b]}|w(t)|$ and $\Delta(a,b)$ is given in Theorem 2.3.

Corollary 2.7 *If we choose $\jmath = p = 0$, $m = 1$, $\alpha = \mu/k$ and $w(s) = 1$, then we have the new result*

$$\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{\Gamma_k(\mu + k)}{2(b-a)^{\frac{\mu}{k}}}\right|\left[I_{a^+}^{\mu,k} e^{f(b)} + I_{b^-}^{\mu,k} e^{f(a)}\right]$$
$$\leq \frac{(b-a)}{(\frac{\mu}{k}+1)(\frac{\mu}{k}+2)(\frac{\mu}{k}+3)}\left(\left(\frac{\mu}{k}\right)^2 + 3\frac{\mu}{k}\right)\left[\left|e^{f(a)}f'(a)\right| + \left|e^{f(b)}f'(b)\right|\right]$$
$$+ 2\left(\frac{\mu}{k}+1\right)\Delta(a,b))$$

under the assumptions of Theorem 2.3.

Corollary 2.8 *If $\jmath = p = 0$, $m = 1$, $\alpha = \frac{\mu}{k}$, $w(s) = 1$ and $\alpha = \mu$, then the inequality*

$$\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{\Gamma(\mu + 1)}{2(b-a)^{\mu}}\right|\left[I_{a^+}^{\mu} e^{f(b)} + I_{b^-}^{\mu} e^{f(a)}\right]$$
$$\leq \frac{(b-a)}{(\mu+1)(\frac{\mu}{k}+2)(\mu+3)}\left((\mu)^2 + 3\mu\right)\left[\left|e^{f(a)}f'(a)\right| + \left|e^{f(b)}f'(b)\right|\right] + 2(\mu+1)\Delta(a,b))$$

holds under the assumption of Theorem 2.3.

Theorem 2.9 *Let $0 \leq a < mb$, $q, r > 1$ such that $1/q + 1/r = 1$, and $f : [a, mb] \to \mathbb{R}$ be a differentiable function such that $(e^f)' \in L_1[a, mb]$. Then the inequality*

$$\left|\left(\int_a^{mb} w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^{\mu};p)\right)^{\alpha}\left[e^{f(a)} + e^{f(mb)}\right]\right.$$
$$- \alpha\int_a^{mb}\left(\int_a^t w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^{\mu};p)e^{f(t)}\,dt$$
$$\left.- \alpha\int_a^{mb}\left(\int_t^{mb} w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^{\mu};p)e^{f(t)}\,dt\right|$$
$$\leq \frac{2(mb-a)^{\alpha+1}\|w\|_{\infty}^{\alpha}S^{\alpha}}{(\alpha r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q + m^2|e^{f(b)}f'(b)|^q\} + m\Delta_1(a,b)}{6}\right)^{\frac{1}{q}} \tag{2.9}$$

for extended generalized fractional integral operators holds if $|f'|^q$ is an exponentially m-convex function on $[a, mb]$ and $k < \delta + \Re(\mu)$, where $\|w\|_{\infty} = \sup_{t\in[a,mb]}|w(t)|$ and

$$\Delta_1(a,b) = \left|e^{f(a)}f'(b)\right|^q + \left|e^{f(b)}f'(a)\right|^q.$$

Proof From Lemma 2.1 and the Hölder inequality we get

$$\left|\left(\int_a^{mb} w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^{\mu};p)\right)^{\alpha}\left[e^{f(a)} + e^{f(mb)}\right]\right.$$
$$- \alpha\int_a^{mb}\left(\int_a^t w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^{\mu};p)e^{f(t)}\,dt$$
$$\left.- \alpha\int_a^{mb}\left(\int_t^{mb} w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^{\mu};p)e^{f(t)}\,dt\right|$$

$$\leq \left(\int_a^{mb}\left|\int_a^t w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\,ds\right|^{\alpha r}\right)^{\frac{1}{r}}\left(\int_a^{mb}\left|e^{f(t)}f'(t)\right|dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^{mb}\left|\int_t^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\,ds\right|^{\alpha r}\right)^{\frac{1}{r}}\left(\int_a^{mb}\left|e^{f(t)}f'(t)\right|dt\right)^{\frac{1}{q}}.$$

It follows from the absolute convergence property of the Mittag-Leffler function and $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$ that

$$\left|\left(\int_a^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\right)^{\alpha}\left[e^{f(a)}+e^{f(mb)}\right]\right.$$
$$-\alpha\int_a^{mb}\left(\int_a^{t} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta t^{\mu};p)e^{f(t)}\,dt$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta t^{\mu};p)e^{f(t)}\,dt\right|$$
$$\leq \|w\|_\infty^{\alpha}S^{\alpha}\left(\int_a^{mb}|t-a|^{\alpha r}\,dt\right)^{\frac{1}{p}}$$
$$+\int_a^{mb}|mb-t|^{\alpha r}\,dt)^{\frac{1}{p}}\left(\int_a^{mb}\left|e^{f(t)}f'(t)\right|^{q}dt\right)^{\frac{1}{q}}. \tag{2.10}$$

Since $|e^{f(t)}f'(t)|^q$ is an exponentially m-convex function, we obtain

$$\left|e^{f(t)}f'(t)\right|^q$$
$$\leq\left[\frac{mb-t}{mb-a}\left|e^{f(a)}\right|^q+m\frac{t-a}{mb-a}\left|e^{f(b)}\right|^q\right]\left[\frac{mb-t}{mb-a}\left|f'(a)\right|^q+m\frac{t-a}{mb-a}\left|f(b)\right|^q\right]$$
$$=\left(\frac{mb-t}{mb-a}\right)^2\left|e^{f(a)}f'(a)\right|^q+m^2\left(\frac{t-a}{mb-a}\right)^2\left|e^{f(b)}f'(b)\right|^q$$
$$+m\left(\frac{mb-t}{mb-a}\right)\left(\frac{t-a}{mb-a}\right)\left\{\left|e^{f(a)}f'(b)\right|^q+\left|e^{f(b)}f'(a)\right|^q\right\}$$
$$=\left(\frac{mb-t}{mb-a}\right)^2\left|e^{f(a)}f'(a)\right|^q+m^2\left(\frac{t-a}{mb-a}\right)^2\left|e^{f(b)}f'(b)\right|^q$$
$$+m\left(\frac{mb-t}{mb-a}\right)\left(\frac{t-a}{mb-a}\right)\Delta(a,b). \tag{2.11}$$

Inequalities (2.10) and (2.11) lead to

$$\left|\left(\int_a^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\right)^{\alpha}\left[e^{f(a)}+e^{f(mb)}\right]\right.$$
$$-\alpha\int_a^{mb}\left(\int_a^{t} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta t^{\mu};p)e^{f(t)}\,dt$$
$$\left.-\alpha\int_a^{mb}\left(\int_t^{mb} w(s)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta s^{\mu};p)\right)^{\alpha-1} w(t)E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(\eta t^{\mu};p)e^{f(t)}\,dt\right|$$
$$\leq \|w\|_\infty^{\alpha}S^{\alpha}\left[\left(\int_a^{mb}|t-a|^{\alpha r}\,dt\right)^{\frac{1}{r}}+\left(\int_a^{mb}|mb-t|^{\alpha r}\,dt\right)^{\frac{1}{r}}\right]$$

$$\times \left(\int_a^{mb}\left\{\left(\frac{mb-t}{mb-a}\right)^2\left|e^{f(a)}f'(a)\right|^q\right.\right.$$

$$\left.\left.+m^2\left(\frac{t-a}{mb-a}\right)^2\left|e^{f(b)}f'(b)\right|^q+m\left(\frac{mb-t}{mb-a}\right)\left(\frac{t-a}{mb-a}\right)\Delta(a,b)\right\}dt\right)^{\frac{1}{q}}$$

$$=\frac{2(mb-a)^{\alpha+1}\|w\|_\infty^\alpha S^\alpha}{(\alpha r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+m^2|e^{f(b)}f'(b)|^q\}+m\Delta_1(a,b)}{6}\right)^{\frac{1}{q}},$$

which is the required result. □

If we take $m = 1$ in (2.9), then we get the following result for an exponentially convex function.

Corollary 2.10 *Let $0 \le a < b$, $p,q > 1$ such that $1/p + 1/q = 1$, and $f : [a,b] \to \mathbb{R}$ be a differentiable exponentially convex function such that $f' \in L_1[a,b]$. Then*

$$\left|\left(\int_a^b w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^\alpha\left[e^{f(a)}+e^{f(b)}\right]\right.$$

$$-\alpha\int_a^b\left(\int_a^t w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1}w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt$$

$$\left.-\alpha\int_a^b\left(\int_t^b w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^\mu;p)\right)^{\alpha-1}w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt\right|$$

$$\le\frac{2(b-a)^{\alpha+1}\|w\|_\infty^\alpha S^\alpha}{(\alpha r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\Delta_1(a,b)}{6}\right)^{\frac{1}{q}}$$

for $k < \delta + \Re(\mu)$ if $|f'|^q$ is a convex function on $[a,b]$, where $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Corollary 2.11 *If we set $p = 0$, then we get the inequality*

$$\left|\left(\int_a^b w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^\mu)\right)^\alpha\left[e^{f(a)}+e^{f(b)}\right]\right.$$

$$-\alpha\int_a^b\left(\int_a^t w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^\mu)\right)^{\alpha-1}w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^\mu;p)e^{f(t)}\,dt$$

$$\left.-\alpha\int_a^b\left(\int_t^b w(s)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta s^\mu)\right)^{\alpha-1}w(t)E_{\mu,\alpha,\jmath}^{\gamma,\delta,\kappa,c}(\eta t^\mu)e^{f(t)}\,dt\right|$$

$$\le\frac{2(b-a)^{\alpha+1}\|w\|_\infty^\alpha S^\alpha}{(\alpha r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\Delta_1(a,b)}{6}\right)^{\frac{1}{q}}$$

for $k < \delta + \Re(\mu)$, where $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Corollary 2.12 *If we set $\jmath = p = 0$ and $m = 1$, then one has*

$$\left|\left(\int_a^b w(s)\,ds\right)^\alpha\left[e^{f(a)}+e^{f(b)}\right]-2\alpha\int_a^b\left(\int_a^t w(s)\,ds\right)^{\alpha-1}w(t)e^{f(t)}\,dt\right|$$

$$\le\frac{2(b-a)^{\alpha+1}\|w\|_\infty^\alpha}{(\alpha r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\Delta_1(a,b)}{6}\right)^{\frac{1}{q}}$$

for $k < \delta + \Re(\mu)$, where $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Corollary 2.13 *If we set* $\jmath = p = 0$, $m = 1$ *and* $\alpha = \frac{\mu}{k}$, *then we have*

$$\begin{aligned}&\left|\left(\int_a^b w(s)\,ds\right)^{\frac{\mu}{k}}\left[e^{f(a)}+e^{f(b)}\right]-\frac{\mu}{k}\int_a^b\left(\int_a^t w(s)\,ds\right)^{\frac{\mu}{k}-1}w(t)e^{f(t)}\,dt\right|\\&\quad\le\frac{2(b-a)^{\frac{\mu}{k}+1}\|w\|_\infty^{\frac{\mu}{k}}}{(\frac{\mu}{k}r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\Delta_1(a,b)}{6}\right)^{\frac{1}{q}}\end{aligned}$$

for $k < \delta + \Re(\mu)$, *where* $\|w\|_\infty = \sup_{t\in[a,b]}|w(t)|$.

Corollary 2.14 *If we set* $\jmath = p = 0$, $m = 1$ *and* $w(s) = 1$, *then*

$$\begin{aligned}&\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{1}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k}e^{f(b)}+I_{b^-}^{\alpha,k}e^{f(a)}\right]\right|\\&\quad\le\frac{(b-a)}{(\frac{\mu}{k}r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\Delta_1(a,b)}{6}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.15 *If we set* $\jmath = p = 0$, $m = 1$, $w(s) = 1$ *and* $\alpha = 1$, *then*

$$\begin{aligned}&\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{1}{b-a}\int_a^b e^{f(x)}\,dx\right|\\&\quad\le\frac{(b-a)}{(r+1)^{\frac{1}{r}}}\left(\frac{2\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\Delta_1(a,b)}{6}\right)^{\frac{1}{q}}.\end{aligned}$$

Next, we establish the Hermite–Hadamard type inequalities for exponentially m-convex functions via an extended Mittag-Leffler function.

Theorem 2.16 *Let* $0 \le a < mb, f : [a, mb] \longrightarrow \mathbb{R}$ *be an exponentially* m*-convex such that* $f \in L_1[a, mb]$. *Then the inequalities for extended generalized fractional integral operators*

$$\begin{aligned}&2e^{f(\frac{a+mb}{2})}\zeta_{\alpha,(\frac{a+mb}{2})^+}(mb;p)\\&\quad\le\left(\varepsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w',(\frac{a+mb}{2})^+}e^f\right)(mb;p)+m^{\alpha+1}\left(\varepsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,\jmath,w',(\frac{a+mb}{2m})^+}e^f\right)\left(\frac{a}{m};p\right)\\&\quad\le\frac{a}{mb-a}\left(e^{f(a)}-m^2e^{(\frac{a}{m^2})}\right)\zeta_{\alpha+1,(\frac{a+mb}{2})^+}(mb;p)+m^{\alpha+1}\left(e^{f(b)}+me^{f\frac{a}{m^2}}\right)\zeta_{\alpha,(\frac{a+mb}{2m})^-}\end{aligned}\tag{2.12}$$

hold, where $w' = \frac{2^\mu w}{(mb-u)^\mu}$.

Proof Since f is an exponentially m-convex function, we get

$$2e^{f(\frac{a+mb}{2})}\le e^{f(\frac{t}{2}a+\frac{2-t}{2}mb)}+me^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}.\tag{2.13}$$

It follows from the definition of exponentially m-convexity that

$$\begin{aligned}&e^{f(\frac{t}{2}a+\frac{2-t}{2}mb)}+me^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}\\&\quad\le\frac{t}{2}\left(e^{f(a)}-m^2e^{\frac{a}{m^2}}\right)+m\left(e^{f(b)}+me^{f(\frac{a}{m^2})}\right).\end{aligned}\tag{2.14}$$

Multiplying (2.13) by $t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(wt^{\mu};p)$ on both sides and then integrating over $[0,1]$, we get

$$\begin{aligned}&2e^{f(\frac{a+mb}{2})}\int_0^1 t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)\,dt\\&\quad\leq t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)e^{f(\frac{t}{2}a+\frac{2-t}{2}mb)}\,dt+mt^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}\,dt.\end{aligned}\tag{2.15}$$

Let $u=\frac{t}{2}a+\frac{2-t}{2}mb$ and $v=\frac{2-t}{2m}a+\frac{t}{2}b$. Then (2.15) gives

$$\begin{aligned}&2e^{f(\frac{a+mb}{2})}\int_a^{mb}(mb-u)^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(w(mb-u)^{\mu};p\bigr)\,du\\&\leq (mb-u)^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(w(mb-u)^{\mu};p\bigr)e^{f(u)}\,du\\&\quad+m^{\alpha+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\biggl(v-\frac{a}{m}\biggr)^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\biggl(w\biggl(v-\frac{a}{m}\biggr)^{\mu};p\biggr)e^{f(v)}\,dv.\end{aligned}\tag{2.16}$$

By using (2.13), (2.15) and (2.16), we get the first inequality of (2.12).

Now multiplying (2.14) by $t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}(wt^{\mu};p)$ on both sides and then integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)e^{f(\frac{t}{2}a+\frac{2-t}{2}mb)}\,dt\\&\quad+m\int_0^1 t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)e^{f(\frac{2-t}{2m}a+\frac{t}{2}b)}\,dt\\&\leq\int_0^1 t^{\alpha}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)\,dt\frac{t}{2}\bigl(e^{f(a)}-m^2e^{\frac{a}{m^2}}\bigr)\,dt\\&\quad+m\int_0^1 t^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(wt^{\mu};p\bigr)\bigl(e^{f(b)}+me^{f(\frac{a}{m^2})}\bigr)\,dt.\end{aligned}\tag{2.17}$$

By changing of the variables $u=\frac{t}{2}a+\frac{2-t}{2}mb$ and $v=\frac{2-t}{2m}a+\frac{t}{2}b$ in (2.17), we get

$$\begin{aligned}&\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(w(mb-u)^{\mu};p\bigr)e^{f(u)}\,du\\&\quad+m\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\biggl(v-\frac{a}{m}\biggr)^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\biggl(w\biggl(v-\frac{a}{m}\biggr)^{\mu};p\biggr)e^{f(v)}\,dv\\&\leq\frac{1}{2}\bigl(e^{f(a)}-m^2e^{\frac{a}{m^2}}\bigr)\int_{\frac{a+mb}{2m}}^{mb}(mb-u)^{\alpha}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\bigl(w(mb-u)^{\mu};p\bigr)\,dt\frac{t}{2}\,du\\&\quad+m^{\alpha+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\biggl(v-\frac{a}{m}\biggr)^{\alpha-1}E^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J}\biggl(m^{\mu}w\biggl(v-\frac{a}{m}\biggr)^{\mu};p\biggr))\,dv.\end{aligned}\tag{2.18}$$

By using (2.14), (2.17) and (2.18), we obtain the second inequality of (2.12). □

Let $m=1$. Then (2.12) leads to the Hermite–Hadamard type inequality for exponentially convex function.

Corollary 2.17 *Let $0 \leq a < b$ and $f : [a,b] \to \mathbb{R}$ be an exponentially convex function such that $f \in L_1[a,b]$. Then the inequalities for extended generalized fractional integral operators*

$$\begin{aligned}
&2e^{f(\frac{a+b}{2})}\zeta_{\alpha,(\frac{a+b}{2})^+}(b;p) \\
&\leq \big(\varepsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J,w',(\frac{a+b}{2})^+}e^f\big)(b;p) + \big(\varepsilon^{\gamma,\delta,\kappa,c}_{\mu,\alpha,J,w',(\frac{a+b}{2})^+}e^f\big)(a;p) \\
&\leq \frac{e^{f(a)}+e^{f(b)}}{2}\zeta_{\alpha,(\frac{a+b}{2})^-}(a;p)
\end{aligned}$$

hold, where $w' = \frac{2^\mu w}{(mb-u)^\mu}$.

Corollary 2.18 *If we set $p = 0$, then we have the following inequalities*:

$$\begin{aligned}
&2e^{f(\frac{a+mb}{2})}\big(\varepsilon^{\gamma,\delta,\kappa}_{\mu,\alpha,J,w',(\frac{a+mb}{2})^+}1\big)(mb) \\
&\leq \big(\varepsilon^{\gamma,\delta,\kappa}_{\mu,\alpha,J,w',(\frac{a+mb}{2})^+}e^f\big)(mb) + m^{\alpha+1}\big(\varepsilon^{\gamma,\delta,\kappa}_{\mu,\alpha,J,w',(\frac{a+mb}{2m})^-}e^f\big)\left(\frac{a}{m}\right) \\
&\leq \frac{1}{mb-a}\big(e^{f(a)} - me^{f(\frac{a}{m^2})}\big)\big(\varepsilon^{\gamma,\delta,\kappa}_{\mu,\alpha+1,J,w',(\frac{a+mb}{2})^+}1\big)(mb) \\
&\quad + m^{\alpha+1}\big(e^{f(b)} + me^{f(\frac{a}{m^2})}\big)\big(\varepsilon^{\gamma,\delta,\kappa}_{\mu,\alpha,J,w',(\frac{a+mb}{2m})^+}1\big)\left(\frac{a}{m}\right),
\end{aligned}$$

where $w' = \frac{2^\mu w}{(mb-u)^\mu}$.

3 Conclusion

We have investigated more general fractional integral inequalities. By selecting specific values of parameters quite interesting results can be obtained. The idea can be extended for more diversified classes for convex and exponentially convex functions.

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