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Some Opial-type integral inequalities via (p,q)-calculus

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Abstract

In this paper, we introduce a new Opial-type inequality by using (p,q)-calculus and establish some integral inequalities. We find a (p,q)-generalization of a Steffensens-type integral inequality and some other inequalities.

MSC: 81P68; 26D10; 26D15

Keywords: (p,q)-integers; (p,q)-derivatives; (p,q)-integral; Opial inequality; q-Opial inequality

1 Introduction and preliminaries

The applications of q-calculus play an important role in mathematics and the field of natural sciences, such as physics and chemistry. It has many applications in orthogonal polynomials, number theory, and quantum theory etc. The q-integer for integer n is denoted by $[n]_q$ and defined by $[n]_q = \frac{1-q^n}{1-q}$. In recent years we have also a generalization of q-calculus with one or more parameters such as (p,q)-calculus known as two parameter quantum calculus or post quantum calculus. The (p,q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q} & (p \neq q \neq 1), \\ \frac{1-q^n}{1-q} & (p = 1), \\ n & (p = q = 1). \end{cases} \tag{1.1}$$

In 1960, Opial [10] established some important integral inequalities. In this article our purpose is to obtain Opial-type classical and some recent integral inequalities in the quantum calculus of two parameters, i.e., in (p,q)-analog, which generalize the results of [2, 4]. In particular, we will find a new generalization of Steffensen's and some other new inequalities.

We recall here some basic definitions and elementary results concerning the (p,q)-derivative, the (p,q)-Jacson integral and Opial-type integral inequalities. For $0 < q < p \leq 1$, the (p,q)-derivative of the function $f(x)$ is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0. \tag{1.2}$$

The $(p;q)$-integral of f is defined as

$$\int f(x)\,\mathrm{d}_{p,q}x = (p-q)x\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}f\left(\frac{q^i}{p^{i+1}}x\right). \tag{1.3}$$

The (p,q)-integrals from 0 to α are defined by

$$\int_0^{\alpha} f(x)\,\mathrm{d}_{p,q}x = (p-q)\alpha\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}f\left(\frac{q^i}{p^{i+1}}\alpha\right) \quad \text{if } \left|\frac{p}{q}\right| > 1 \tag{1.4}$$

and

$$\int_0^{\alpha} f(x)\,\mathrm{d}_{p,q}x = (q-p)\alpha\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}f\left(\frac{q^i}{p^{i+1}}\alpha\right) \quad \text{if } \left|\frac{p}{q}\right| < 1. \tag{1.5}$$

Also for two nonnegative numbers such that $\alpha < \beta$, we have

$$\int_{\alpha}^{\beta} f(x)\,\mathrm{d}_{p,q}x = \int_0^{\beta} f(x)\,\mathrm{d}_{p,q}x - \int_0^{\alpha} f(x)\,\mathrm{d}_{p,q}x. \tag{1.6}$$

If $f(t)\in C^1$ on $0\le t\le h$ such that $f(0)=f(h)=0, f(t)>0$ on $(0,h)$. Then the Opial integral inequality is given by

$$\int_0^{h}\left|f(x)f'(x)\right|\mathrm{d}t \le \frac{h}{4}\int_0^{h}\left(f'(x)\right)^2\mathrm{d}x. \tag{1.7}$$

For $f(t)$ an absolutely continuous function with $f(t)=0$, we have

$$\int_{\alpha}^{\beta}\left|f(t)f'(t)\right|\mathrm{d}t \le \frac{(\beta-\alpha)^2}{4}\int_{\alpha}^{\beta}\left(f'(t)\right)^2\mathrm{d}x. \tag{1.8}$$

The purpose of this article is to obtain some Opial-type integral inequalities in (p,q)-analog. We will find a (p,q)-generalization of Steffensen's inequality as well as some other inequalities which give a modification of [2, 4]. In particular, we use the nodes ϕ defined as

$$\phi_i = \frac{q^i}{p^{i+1}} \quad \text{for} \quad i\in\mathbb{N}\cup\{0\}, \quad 0<q<p\le 1. \tag{1.9}$$

Take $\alpha=\beta\phi_n=\psi_n$(suppose), then $\frac{\alpha}{\beta}=\frac{q^n}{p^{n+1}}$. In the case of $p=1$, the (p,q)-integral is reduced to the q-Jackson integral and further for $q=1$ it is reduced to the usual Riemann integral on the interval $[\alpha,\beta]$.

As a natural phenomenon we order the nodes ψ_n, such as $\psi_n\le\psi_{n+1}$ for $n\in\mathbb{N}\cup\{0\}$ and define the (p,q)-decreasing function as well as the (p,q)-increasing function, respectively, by

$$f(\psi_n)\ge f(\psi_{n+1})$$

and

$$f(\psi_n) \leq f(\psi_{n+1}).$$

Recently, Mursaleen et al. [5] applied (p,q)-calculus in approximation theory and introduced first (p,q)-analogue of Bernstein operators. Recent results on (p,q)-calculus are obtained in [3, 6, 8, 9, 11, 12].

Lemma 1.1 *Let $0 < q < p \leq 1$, $\beta > 0$ and $n \in \mathbb{N} \cup \{0\}$. Then for an arbitrary function $f(x)$ the restricted (p,q)-integral is defined by*

$$\begin{aligned}\int_{\alpha}^{\beta} f(x)\,\mathrm{d}_{p,q}x = \int_{\beta\phi_n=\psi_n}^{\beta} f(x)\,\mathrm{d}_{p,q}x &= (p-q)\beta\sum_{i=0}^{n-1}\frac{q^i}{p^{i+1}}f\left(\beta\frac{q^i}{p^{i+1}}\right)\\ &= (p-q)\sum_{i=0}^{n-1}\psi_i f(\psi_i).\end{aligned} \tag{1.10}$$

2 Main results

The purpose of this paper is to find (p,q)-analogues of some classical integral inequalities. In particular, we find (p,q)-generalizations of the inequalities of [2], as well as some new inequalities involving Taylor's remainder [1].

Theorem 2.1 *Suppose $0 < q < p \leq 1$, $\beta > 0$, $n \in \mathbb{N}$ and F, G to be two functions defined by $F, G : [\alpha, \beta] \to \mathbb{R}$ with $\alpha = \beta\phi_n$. Let on $[\alpha, \beta]$ F be (p,q)-decreasing and $0 \leq G \leq 1$. Moreover, assume that we have any numbers $k, \ell \in \{0, 1, 2, \dots\}$ and the functions F and G are such that*

$$\begin{aligned}p\psi_k - \alpha \leq \int_{\alpha}^{\beta} G(x)\,\mathrm{d}_{p,q}x \leq \beta - p\psi_\ell \quad &\text{if } F \geq 0 \text{ on } [\alpha,\beta],\\ \beta - p\psi_\ell \leq \int_{\alpha}^{\beta} G(x)\,\mathrm{d}_{p,q}x \leq p\psi_k - \alpha \quad &\text{if } F \leq 0 \text{ on } [\alpha,\beta].\end{aligned}$$

Then

$$\int_{\psi_\ell}^{\beta} F(x)\,\mathrm{d}_{p,q}x \leq \int_{\alpha}^{\beta} F(x)G(x)\,\mathrm{d}_{p,q}x \leq \int_{\alpha}^{\psi_k} F(x)\,\mathrm{d}_{p,q}x. \tag{2.1}$$

Proof We prove the case for F a (p,q)-decreasing function when $F \geq 0$ as well as $F \leq 0$. For $F \geq 0$, we prove only the left inequality (2.1). We have $\ell \in \mathbb{N} \cup \{0\}$. Take $j = 0, 1, \dots, \ell - 1$. In the case of $F \geq 0$, we have $F(\psi_\ell) \leq F(\psi_j)$ for $\psi_j \leq \psi_\ell$ $(j = 0, 1, \dots, n - \ell)$ and $F(\psi_\ell) \geq F(\psi_{\ell+j})$ for $\psi_\ell \leq \psi_{\ell+j}$. Similarly, in the case $F \leq 0$, we have $F(\psi_\ell) \geq F(\psi_j)$ for $\psi_j \geq \psi_\ell$ and $F(\psi_\ell) \leq F(\psi_{\ell+j})$ for $\psi_\ell \geq \psi_{\ell+j}$. The proof is straightforward, so we omit it. Now

$$\begin{aligned}&\int_{\alpha}^{\beta} F(x)G(x)\,\mathrm{d}_{p,q}x - \int_{\psi_\ell}^{\beta} F(x)\,\mathrm{d}_{p,q}x\\ &\quad= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x + \int_{\psi_\ell}^{\beta} F(x)G(x)\,\mathrm{d}_{p,q}x - \int_{\psi_\ell}^{\beta} F(x)\,\mathrm{d}_{p,q}x\end{aligned}$$

$$
\begin{aligned}
&= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - \int_{\psi_\ell}^{\beta} F(x)\bigl(1-G(x)\bigr)\,\mathrm{d}_{p,q}x \\
&= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - (p-q)\beta\sum_{j=0}^{\ell-1}\frac{q^j}{p^{j+1}}F(\psi_j)\bigl(1-G(\psi_j)\bigr) \\
&\geq \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - (p-q)\beta\sum_{j=0}^{\ell-1}\frac{q^j}{p^{j+1}}F(\psi_\ell)\bigl(1-G(\psi_j)\bigr) \\
&= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - (p-q)\beta F(\psi_\ell)\sum_{j=0}^{\ell-1}\frac{q^j}{p^{j+1}} + (p-q)\beta F(\psi_\ell)\sum_{j=0}^{\ell-1}\frac{q^j}{p^{j+1}}G(\psi_j) \\
&= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - F(\psi_\ell)(\beta-p\psi_\ell) + F(\psi_\ell)\int_{\psi_\ell}^{\beta} G(x)\,\mathrm{d}_{p,q}x \\
&\geq \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - F(\psi_\ell)\int_{\alpha}^{\beta} G(x)\,\mathrm{d}_{p,q}x + F(\psi_\ell)\int_{\psi_\ell}^{\beta} G(x)\,\mathrm{d}_{p,q}x \\
&= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - F(\psi_\ell)\biggl[\int_{\alpha}^{\beta} G(x)\,\mathrm{d}_{p,q}x - \int_{\psi_\ell}^{\beta} G(x)\,\mathrm{d}_{p,q}x\biggr] \\
&= \int_{\alpha}^{\psi_\ell} F(x)G(x)\,\mathrm{d}_{p,q}x - F(\psi_\ell)\int_{\alpha}^{\psi_\ell} G(x)\,\mathrm{d}_{p,q}x \\
&= \int_{\alpha}^{\psi_\ell} \bigl(F(x)-F(\psi_\ell)\bigr)G(x)\,\mathrm{d}_{p,q}x \\
&= (p-q)\psi_\ell\sum_{j=0}^{n-\ell-1}\frac{q^j}{p^{j+1}}\biggl[F\biggl(\psi_\ell\frac{q^j}{p^{j+1}}\biggr)-F(\psi_\ell)\biggr]G\biggl(\psi_\ell\frac{q^j}{p^{j+1}}\biggr) \\
&= (p-q)\psi_\ell\sum_{j=0}^{n-\ell-1}\frac{q^j}{p^{j+1}}\bigl[F(\psi_j)-F(\psi_\ell)\bigr]G(\psi_j) \geq 0.
\end{aligned}
$$

□

Theorem 2.2 *For $0 < q < p \leq 1$, $n \in \mathbb{N}$, we have the following integral identities*:

$$
\begin{aligned}
&\int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu+1}f\bigr)(x)\frac{(\gamma_k-qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}\,\mathrm{d}_{p,q}x \\
&\quad= \int_{\alpha}^{\gamma_k} R_{(p,q),\mu,f}(\alpha,x)\,\mathrm{d}_{p,q}x + \int_{\gamma_k}^{\beta} R_{(p,q),\mu,f}(\beta,x)\,\mathrm{d}_{p,q}x,
\end{aligned}
\tag{2.2}
$$

where μ is any integer such that $\mu \geq -1$, and $R_{(p,q),-1,f}(\alpha,x) = f(x)$.

Moreover,

$$
\int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu+1}f\bigr)(x)\frac{(\nu-qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} = \int_{\alpha}^{\beta} R_{(p,q),\mu,f}(\alpha,x), \tag{2.3}
$$

$$
\int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu+1}f\bigr)(x)\frac{(u-qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} = \int_{\alpha}^{\beta} R_{(p,q),\mu,f}(\beta,x). \tag{2.4}
$$

Proof We prove it by induction. We prove it for $\mu = -1$, that is, $\int_{\alpha}^{\beta} f(x)\,\mathrm{d}_{p,q}x = \int_{\alpha}^{\gamma_k} f(x)\,\mathrm{d}_{p,q}x + \int_{\gamma_k}^{\beta} f(x)\,\mathrm{d}_{p,q}x$. Suppose it is true for μ. Then we prove it for $\mu + 1$.

By integration of by parts, we have

$$\begin{aligned}
&\int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu} f\bigr)(x) \frac{(\gamma_k - qx)_{p,q}^{\mu}}{[\mu]_{p,q}!} \,\mathrm{d}_{p,q}x \\
&\quad = -\frac{1}{[\mu+1]_{p,q}!} \int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu} f\bigr)(px) D_{p,q}(\gamma_k - qx)_{p,q}^{r+1} \,\mathrm{d}_{p,q}x \\
&\quad = -\frac{1}{[\mu+1]_{p,q}!} \biggl[\int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu} f\bigr)(\nu)(\gamma_k - \nu)_{p,q}^{\mu+1} - \bigl(D_{p,q}^{\mu} f\bigr)(u)(\gamma_k - u)_{p,q}^{\mu+1} \\
&\qquad - \int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu+1} f\bigr)(x)(\gamma_k - qx)_{p,q}^{\mu+1} \,\mathrm{d}_{p,q}x\biggr].
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\int_{\alpha}^{\beta} (D_{p,q}^{\mu+1} f)(x) \frac{(\gamma_k - qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} \,\mathrm{d}_{p,q}x \\
&\quad = \bigl(D_{p,q}^{\mu} f\bigr)(\beta) \frac{(\gamma_k - \nu)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} - \bigl(D_{p,q}^{\mu} f\bigr)(\alpha) \frac{(\gamma_k - u)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} \\
&\qquad + \int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu} f\bigr)(x) \frac{(\gamma_k - qx)_{p,q}^{\mu}}{[\mu]_{p,q}!} \,\mathrm{d}_{p,q}x.
\end{aligned} \tag{2.5}$$

From (2.2), we have

$$\begin{aligned}
&\int_{\alpha}^{\beta} (D_{p,q}^{\mu} f)(x) \frac{(\gamma_k - qx)_{p,q}^{\mu}}{[\mu]_{p,q}!} \,\mathrm{d}_{p,q}x \\
&\quad = \int_{\alpha}^{\gamma_k} R_{(p,q),\mu-1,f}(\alpha, x) \,\mathrm{d}_{p,q}x + \int_{\gamma_k}^{\beta} R_{(p,q),\mu-1,f}(\beta, x) \,\mathrm{d}_{p,q}x.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}
&\int_{\alpha}^{\beta} (D_{p,q}^{\mu+1} f)(x) \frac{(\gamma_k - qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} \,\mathrm{d}_{p,q}x \\
&\quad = (D_{p,q}^{\mu} f)(\nu) \frac{(\gamma_k - \beta)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} - (D_{p,q}^{\mu} f)(\alpha) \frac{(\gamma_k - \alpha)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} \\
&\qquad + \int_{\alpha}^{\gamma_k} R_{(p,q),\mu-1,f}(\alpha, x) \,\mathrm{d}_{p,q}x + \int_{\gamma_k}^{\mathrm{beta}} R_{(p,q),\mu-1,f}(\beta, x) \,\mathrm{d}_{p,q}x.
\end{aligned} \tag{2.7}$$

And we know that for the (p, q)-integral

$$\int_{\alpha}^{\gamma_k} \frac{(x-\alpha)_{p,q}^{\mu}}{[\mu]_{p,q}!} = \frac{(\gamma_k - \alpha)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}, \qquad \int_{\gamma_k}^{\beta} \frac{(x-\beta)_{p,q}^{\mu}}{[\mu]_{p,q}!} = -\frac{(\gamma_k - \beta)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}. \tag{2.8}$$

Hence (2.7) implies that

$$\begin{aligned}
&\int_{\alpha}^{\beta} \bigl(D_{p,q}^{\mu+1} f\bigr)(x) \frac{(\gamma_k - qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!} \,\mathrm{d}_{p,q}x \\
&\quad = \int_{\alpha}^{\gamma_k} \biggl[R_{(p,q),\mu-1,f}(\alpha, x) - \bigl(D_{p,q}^{\mu} f\bigr)(\alpha) \frac{(x-\alpha)_{p,q}^{\mu}}{[\mu]_{p,q}!}\biggr] \,\mathrm{d}_{p,q}x
\end{aligned}$$

$$+\int_{\gamma_k}^{\beta}\left[R_{(p,q),\mu-1,f}(\beta,x)-\left(D_{p,q}^{\mu}f\right)(\beta)\frac{(x-\beta)_{p,q}^{\mu}}{[\mu]_{p,q}!}\right]\mathrm{d}_{p,q}x$$
$$=\int_{\alpha}^{\gamma_k}R_{(p,q),\mu,f}(\alpha,x)\,\mathrm{d}_{p,q}x+\int_{\gamma_k}^{\beta}R_{(p,q),\mu,f}(\beta,x)\,\mathrm{d}_{p,q}x,$$

where

$$\int_{\alpha}^{\beta}\left(D_{p,q}^{\mu+1}f\right)(x)\frac{(\beta-qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}=\int_{\alpha}^{\beta}R_{(p,q),\mu,f}(\alpha,x),\tag{2.9}$$

$$\int_{\alpha}^{\beta}\left(D_{p,q}^{\mu+1}f\right)(x)\frac{(\alpha-qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}=\int_{\alpha}^{\beta}R_{(p,q),\mu,f}(\beta,x).\tag{2.10}$$

This completes the proof. □

Theorem 2.3 *Suppose* $0<q<p\leq 1$, $\beta>0$, $n\in\mathbb{N}$ *and* f *be the function defined by* $f:[\alpha,\beta]\to\mathbb{R}$ *with* $\alpha=\beta\phi_n$. *Let, on* $[\alpha,\beta]$, $D_{p,q}^{\mu}f$ *be either* (p,q)*-decreasing or* (p,q)*-increasing on* $[\alpha,\beta]$. *Moreover, assume that the numbers* $k,\ell\in\{0,1,2,\ldots\}$ *are such that*

$$p\psi_k-\alpha\leq\frac{\beta-\alpha}{[\mu+2]_{p,q}}\leq\beta-p\psi_\ell\quad\text{if } D_{p,q}^{\mu}f \text{ is } (p,q)\text{-decreasing},$$
$$\beta-p\psi_\ell\leq\frac{\beta-\alpha}{[\mu+2]_{p,q}}\leq p\psi_k-\alpha\quad\text{if } D_{p,q}^{\mu}f \text{ is } (p,q)\text{-increasing}.$$

Then

$$\left(D_{p,q}^{\mu}f\right)(p\psi_k)-\left(D_{p,q}^{\mu}f\right)(\alpha)\leq\frac{[\mu+1]_{p,q}!}{(p\beta-q\alpha)_{p,q}^{\mu+1}}\int_{\alpha}^{\beta}R_{(p,q),\mu,f}(\alpha,x)$$
$$\leq\left(D_{p,q}^{\mu}f\right)(\beta)-\left(D_{p,q}^{\mu}f\right)(p\psi_\ell).\tag{2.11}$$

Proof We prove the result for $D_{p,q}^{\mu}f$ a (p,q)-decreasing function and in a similar way we can prove it for the case if $D_{p,q}^{\mu}f$ is (p,q)-increasing function. Let $F(x)=-(D_{p,q}^{\mu+1}f)(x)$. Then $F(x)$ is a (p,q)-decreasing function. Since $D_{p,q}^{\mu}f$ is (p,q)-decreasing, $D_{p,q}^{\mu+1}f\leq 0$. Therefore $F(x)\geq 0$ and will be a (p,q)-decreasing. Suppose that

$$G(x)=\frac{(\beta-qx)_{p,q}^{\mu+1}}{(p\beta-q\alpha)_{p,q}^{\mu+1}}.$$

Then

$$G(x)=\frac{(\beta-qx)(p\beta-q^2x)\cdots(p^{\mu}\beta-q^{\mu+1}x)}{(p\beta-q\alpha)(p^2\beta-q^2\alpha)\cdots(p^{\mu+1}\beta-q^{\mu+1}\alpha)},$$

$$\int_{\alpha}^{\beta}G(x)\,\mathrm{d}_{p,q}x=\frac{1}{(\beta-q\alpha)_{p,q}^{\mu+1}}\int_{\alpha}^{\beta}(\beta-qx)_{p,q}^{\mu+1}\,\mathrm{d}_{p,q}x$$
$$=-\frac{1}{[\mu+2]_{p,q}}\frac{1}{(p\beta-q\alpha)_{p,q}^{\mu+1}}\int_{\alpha}^{\nu}D_{p,q}(\beta-x)_{p,q}^{\mu+2}\,\mathrm{d}_{p,q}x$$

$$
\begin{aligned}
&= \frac{1}{[\mu+2]_{p,q}} \frac{1}{(p\beta-q\alpha)_{p,q}^{\mu+1}} (\beta-\alpha)_{p,q}^{\mu+2} \\
&= \frac{1}{[\mu+2]_{p,q}} \frac{(\beta-\alpha)(p\beta-q\alpha)\cdots(p^{\mu+1}\beta-q^{\mu+1}\alpha)}{(p\beta-q\alpha)(p^2\beta-q^2\alpha)\cdots(p^{\mu+1}\beta-q^{\mu+1}\alpha)} \\
&= \frac{\beta-\alpha}{[\mu+2]_{p,q}}.
\end{aligned}
$$

If

$$
p\psi_k - \alpha \le \frac{\beta-\alpha}{[\mu+2]_{p,q}} \le \beta - p\psi_\ell,
$$

then

$$
\begin{aligned}
-\int_{p\psi_\ell}^{\beta} \big(D_{p,q}^{\mu+1} f\big)(x)\, \mathrm{d}_{p,q}x &\le -\int_{\alpha}^{\beta} \big(D_{p,q}^{\mu+1} f\big)(x) \frac{(\beta-qx)_{p,q}^{\mu+1}}{(p\beta-q\alpha)_{p,q}^{\mu+1}}\, \mathrm{d}_{p,q}x \\
&\le -\int_{\alpha}^{p\psi_k} \big(D_{p,q}^{\mu+1} f\big)(x)\, \mathrm{d}_{p,q}x.
\end{aligned}
$$

From (2.9), we can write

$$
\big(D_{p,q}^{\mu} f\big)(x)\big|_{x=\alpha}^{x=p\psi_k} \le \frac{[\mu+1]_{p,q}!}{(p\beta-q\alpha)_{p,q}^{\mu+1}} \int_{\alpha}^{\beta} R_{(p,q),\mu,f}(\alpha,x) \le \big(D_{p,q}^{\mu} f\big)(x)\big|_{x=p\psi_\ell}^{x=\beta}. \qquad \square
$$

Theorem 2.4 *Suppose* $0 < q < p \le 1$, $\beta > 0$, $n \in \mathbb{N}$ *and* f *be the function defined by* $f : [\alpha,\beta] \to \mathbb{R}$ *with* $\alpha = \beta\phi_n$. *Let, on* $[\alpha,\beta]$, $D_{p,q}^2 f \ge 0$ *(*f *is* (p,q)*-convex). Assume that we have any numbers* $k,\ell \in \{0,1,2,\ldots\}$ *such that*

$$
\begin{aligned}
p\psi_\ell &\le \frac{(p-1)\beta+q\beta+\alpha}{p+q} \quad \text{and} \\
p\psi_k &\le \frac{(p-1)\alpha+q\alpha+\beta}{p+q}, \quad \text{for } f \text{ is } (p,q)\text{-decreasing}, \\
p\psi_\ell &\ge \frac{(p-1)\beta+q\beta+\alpha}{p+q} \quad \text{and} \\
p\psi_k &\ge \frac{(p-1)\alpha+q\alpha+\beta}{p+q}, \quad \text{for } f \text{ is } (p,q)\text{-increasing}.
\end{aligned}
$$

Then

$$
\begin{aligned}
f(p\psi_k) + \frac{f(\alpha)}{p\beta-q\alpha}\big((1-p)\beta-(1-q)\alpha\big) &\le \frac{1}{p\beta-q\alpha}\int_{\alpha}^{\beta} f(x)\, \mathrm{d}_{p,q}x \\
&\le \frac{(\beta-\alpha)f(\alpha)}{p\beta-q\alpha} + f(\beta) - f(p\psi_\ell).
\end{aligned}
$$

Proof We prove it for f being a (p,q)-decreasing function, and for f a (p,q)-increasing function the proof is similar. For $D_{p,q}^2 f \ge 0$ on $[\alpha,\beta]$, $f(x)$ is a (p,q)-convex function and clearly for all x, p^2x, q^2x and $p,q,x \in [\alpha,\beta]$, we have $qf(p^2x) - (p+q)f(pqx) + pf(q^2x) \ge 0$. If $f(x)$ is convex on $[\alpha,\beta]$, then it is also q-convex on $[\alpha,\beta]$ and hence it is also a (p,q)-convex on $[\alpha,\beta]$. Take $\mu = 0$, then $[\mu+2]_{p,q} = p+q$. Assume that f is (p,q)-decreasing, then we have $p\psi_k - \alpha \le \frac{\beta-\alpha}{[\mu+2]_{p,q}} \le \beta - p\psi_\ell$ which implies that, for $\mu = 0$, $p\psi_k \le \alpha + \frac{\beta-\alpha}{p+q} \le \beta - p\psi_\ell$ and hence $p\psi_\ell \le \frac{(p-1)\beta+q\beta+\alpha}{p+q}$, as well as $p\psi_k \le \frac{(p-1)\alpha+\alpha+\beta}{p+q}$. Since $(D^0_{p,q}f)(x) = f(x)$, $R_{(p,q),-1,f}(\alpha,x) = f(x)$ from Theorem 2.2 and

$$\int_\alpha^{\gamma_k} R_{(p,q),\mu,f}(\alpha,x)\,\mathrm{d}_{p,q}x = \int_\alpha^{\gamma_k}\left[R_{(p,q),\mu-1,f}(\alpha,x) - \left(D^\mu_{p,q}f\right)(\alpha)\frac{(x-\alpha)^\mu_{p,q}}{[\mu]_{p,q}!}\right]\mathrm{d}_{p,q}x.$$

Therefore,

$$\int_\alpha^{\gamma_k} R_{(p,q),0,f}(\alpha,x)\,\mathrm{d}_{p,q}x = \int_\alpha^{\gamma_k}\left[R_{(p,q),-1,f}(\alpha,x) - f(\alpha)\right]\mathrm{d}_{p,q}x.$$

This implies that $R_{(p,q),0,f}(\alpha,x) = f(x) - f(\alpha)$. Hence from (2.11) of Theorem 2.3, we have

$$f(p\psi_k) - f(\alpha) \le \frac{1}{p\beta - q\alpha}\int_\alpha^\beta\left[f(x) - f(\alpha)\right]\mathrm{d}_{p,q}x \le f(\beta) - f(p\psi_\ell).$$

This completes the proof. □

Theorem 2.5 *Suppose* $0 < q < p \le 1$, $\beta > 0$, $n \in \mathbb{N}$ *and* f *be the function defined by* $f : [\alpha,\beta] \to \mathbb{R}$ *with* $\alpha = \beta\phi_n$. *Suppose, on* $[\alpha,\beta]$, $D^2_{p,q}f \ge 0$ *(*f *is* (p,q)*-convex). Moreover, assume that we have any numbers* $k,\ell \in \{0,1,2,\dots\}$ *such that*

$$p\psi_\ell \ge \frac{(\alpha+\beta)p}{p+q} \quad \textit{and} \quad p\psi_k \ge \frac{(\alpha+\beta)q}{p+q}, \quad \textit{for } f \textit{ is } (p,q)\textit{-increasing},$$
$$p\psi_\ell \le \frac{(\alpha+\beta)p}{p+q}, \quad \textit{and} \quad p\psi_k \le \frac{(\alpha+\beta)q}{p+q}, \quad \textit{for } f \textit{ is } (p,q)\textit{-decreasing}.$$

Then

$$f(\psi_k) \le \frac{1}{\beta-\alpha}\int_\alpha^\beta f(qx)\,\mathrm{d}_{p,q}x \le f(\alpha) + f(\beta) - f(\psi_\ell).$$

Proof We prove it for f being (p,q) decreasing, and for f (p,q)-increasing, the proof is similar. Suppose f is (p,q)-convex on $[\alpha,\beta]$, $D^2_{p,q}f \ge 0$ on $[\alpha,\beta]$ and $D_{p,q}f$ is (p,q)-increasing on $[\alpha,\beta]$. Suppose $F(x) = -D_{p,q}f$, $G(x) = \frac{\beta-x}{\beta-\alpha}$, then $F(x)$ is (p,q)-decreasing on $[\alpha,\beta]$. Similarly, f is (p,q)-decreasing on $[\alpha,\beta]$, $D_{p,q}f \le 0$. Hence $F(x) \ge 0$ and (p,q)-decreasing on $[\alpha,\beta]$. Therefore, F and G satisfy the hypothesis of Theorem 2.1. From Theorem 2.3, we have

$$p\psi_k - \alpha \le \frac{\beta-\alpha}{[\mu+2]_{p,q}} \le \beta - p\psi_\ell,$$

which implies that

$$p\psi_k - \alpha \le \int_\alpha^\beta G(x)\,\mathrm{d}_{p,q}x \le \beta - p\psi_\ell,$$
$$p\psi_k - \alpha \le \frac{q\beta - p\alpha}{p+q} \le \beta - p\psi_\ell,$$

which takes the form $pc_\ell \le \frac{(\alpha+\beta)p}{p+q}$, $p\psi_k \le \frac{(\alpha+\beta)q}{p+q}$.

We find $\int_\alpha^\beta F(x)G(x)\,\mathrm{d}_{p,q}x$, by using integration by parts and $(1+qx)_{p,q}^{n-1} = \frac{1}{[n]_{p,q}} D_{p,q}(1+x)_{p,q}^n$ from [7], we have

$$\int_\alpha^\beta G(x)\,\mathrm{d}_{p,q}x = \frac{1}{\beta-\alpha}\int_\alpha^\beta (\beta - x)\,\mathrm{d}_{p,q}x = \frac{(\alpha-\beta)_{p,q}^2}{(\beta-\alpha)(p+q)} = \frac{q\beta - p\alpha}{p+q},$$

$$\int_\alpha^\beta F(x)G(x)\,\mathrm{d}_{p,q}x = \frac{1}{\beta-\alpha}\int_\alpha^\beta (x-\beta)(D_{p,q}f)\,\mathrm{d}_{p,q}x = \frac{-1}{\beta-\alpha}\int_\alpha^\beta f(qx)\,\mathrm{d}_{p,q}x + f(\alpha).$$

From (2.1) we have

$$\int_{\psi_\ell}^\beta F(x)\,\mathrm{d}_{p,q}x \le \int_\alpha^\beta F(x)G(x)\,\mathrm{d}_{p,q}x \le \int_\alpha^{\psi_k} F(x)\,\mathrm{d}_{p,q}x.$$

Therefore,

$$-\int_{\psi_\ell}^\nu (D_{p,q}f)\,\mathrm{d}_{p,q}x \le \frac{-1}{\beta-\alpha}\int_\alpha^\beta f(qx)\,\mathrm{d}_{p,q}x + f(\alpha) \le -\int_\alpha^{\psi_k} (D_{p,q}f)\,\mathrm{d}_{p,q}x.$$

This implies that

$$-f(x)|_{\psi_\ell}^\beta \le \frac{-1}{\beta-\alpha}\int_\alpha^\beta f(qx)\,\mathrm{d}_{p,q}x + f(\alpha) \le -f(x)|_u^{\psi_k}.$$

Hence

$$f(x)|_\alpha^{\psi_k} \le \frac{1}{\beta-\alpha}\int_\alpha^\beta f(qx)\,\mathrm{d}_{p,q}x - f(\alpha) \le f(x)|_{\psi_\ell}^\beta.$$

This completes the proof. □

Theorem 2.6 *Let $f:[\alpha,\beta]\to\mathbb{R}$ and $m \le D_{p,q}^{\mu+1} \le M$, where $m < M$. Assume that we have any numbers $k,\ell \in \{0,1,2,\ldots\}$ such that*

$$p\psi_k - \alpha \le \frac{1}{M-m}\big[\big(D_{p,q}^\mu f\big) - \big(D_{p,q}^\mu f\big)(\alpha) - m(\beta-\alpha)\big] \le \beta - p\psi_\ell.$$

Then

$$m + (M-m)\frac{(\beta-\psi_\ell)_{p,q}^{\mu+2}}{(\beta-\alpha)_{p,q}^{\mu+2}} \le \frac{[\mu+2]_{p,q}!}{(\beta-\alpha)_{p,q}^{\mu+2}}\int_\alpha^\beta R_{(p,q),\mu,f}(\alpha,x)\,\mathrm{d}_{p,q}x$$
$$\le M - (M-m)\frac{(\beta-\psi_k)_{p,q}^{\mu+2}}{(\beta-\alpha)_{p,q}^{\mu+2}}.$$

Proof Suppose $F(x)$ is (p,q)-decreasing function and $F(x)\ge 0$. We take $F(x) = \frac{(\beta-qx)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}$ and $G(x) = D_{p,q}^{\mu+1}g$, where $g(x) = \frac{1}{M-m}[f(x) - m\frac{(x-\alpha)_{p,q}^{\mu+1}}{[\mu+1]_{p,q}!}]$ and hence $G(x) = \frac{1}{M-m}(D_{p,q}^{\mu+1}f - m)$. Therefore,

$$\int_\alpha^\beta G(x)\,\mathrm{d}_{p,q}x = \frac{1}{M-m}\int_\alpha^\beta D_{p,q}^{\mu+1}f\,\mathrm{d}_{p,q}x - \frac{m(\beta-\alpha)}{M-m}$$
$$= \frac{1}{M-m}\big[\big(D_{p,q}^\mu f\big)(\beta) - \big(D_{p,q}^\mu f\big)(\alpha) - m(\beta-\alpha)\big].$$

This implies that

$$p\psi_k - \alpha \le \frac{1}{M-m}\big[\big(D^{\mu}_{p,q}f\big)(\beta) - \big(D^{\mu}_{p,q}f\big)(\alpha) - m(\beta-\alpha)\big] \le \beta - p\psi_\ell.$$

And

$$\begin{aligned}\int_{\psi_\ell}^{\beta} F(x)\,\mathrm{d}_{p,q}x = \int_{\psi_\ell}^{\beta} \frac{(\beta-qx)^{\mu+1}_{p,q}}{[\mu+1]_{p,q}!}\,\mathrm{d}_{p,q}x &= -\frac{1}{[\mu+2]_{p,q}!}\int_{\psi_\ell}^{\beta} D_{p,q}(\beta-x)^{\mu+2}_{p,q}\,\mathrm{d}_{p,q}x \\ &= \frac{(\beta-\psi_\ell)^{\mu+2}_{p,q}}{[\mu+2]_{p,q}!}.\end{aligned}$$

Similarly

$$\int_{\alpha}^{\psi_k} F(x)\,\mathrm{d}_{p,q}x = \int_{\alpha}^{\psi_k} \frac{(\beta-qx)^{\mu+1}_{p,q}}{[\mu+1]_{p,q}!}\,\mathrm{d}_{p,q}x = \frac{1}{[\mu+2]_{p,q}!}\big[(\beta-\alpha)^{\mu+2}_{p,q} - (\beta-\psi_k)^{\mu+2}_{p,q}\big],$$

$$\begin{aligned}\int_{\alpha}^{\beta} F(x)G(x)\,\mathrm{d}_{p,q}x &= \int_{\alpha}^{\beta} \frac{(\beta-qx)^{\mu+1}_{p,q}}{[\mu+1]_{p,q}!} D^{\mu+2}_{p,q}g\,\mathrm{d}_{p,q}x \\ &= \int_{\alpha}^{\beta} R_{(p,q),\mu,g}(\alpha,x)\,\mathrm{d}_{p,q}x \\ &= \int_{\alpha}^{\beta}\left[\frac{1}{M-m}R_{(p,q),\mu,f}(\alpha,x) - \frac{m}{M-m}\frac{(x-\alpha)^{\mu+1}_{p,q}}{[\mu+1]_{p,q}!}\right]\mathrm{d}_{p,q}x \\ &= \frac{1}{M-m}\int_{\alpha}^{\beta} R_{(p,q),\mu,f}(\alpha,x)\,\mathrm{d}_{p,q}x - \frac{m}{M-m}\frac{(\beta-\alpha)^{\mu+2}_{p,q}}{[\mu+2]_{p,q}!}.\end{aligned}$$

By substituting the integrals $\int_{\psi_\ell}^{\beta} F(x)\,\mathrm{d}_{p,q}x$, $\int_{\alpha}^{\psi_k} F(x)\,\mathrm{d}_{p,q}x$ and $\int_{\alpha}^{\beta} F(x)G(x)\,\mathrm{d}_{p,q}x$ in (2.1) of Theorem 2.1, we get the desired result. □

3 Conclusion

These types of integral inequalities have a great impetus and large interest in their own right as well as have an important applications in the theory of different mathematical areas, such as ordinary differential equations, hyper-geometric functions, combinatorics, number theory, mechanics, theory of relativity etc. We have studied here the Opial-type inequalities defined by (p,q)-integers which are more general than the classical Opial and q-Opial integral inequalities.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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