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A new approach to multivalued nonlinear weakly Picard operators

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Abstract

The notion of nonlinear $(\mathcal{F}_s, \mathcal{L})$-contractive multivalued operators is initiated and some related fixed point results are considered. We also give an example to show the validity of obtained theoretical results. Our results generalize many existing ones in the literature.

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1 Introduction

A multivalued weakly Picard operator (in short, MWP) has been introduced as a connection with the successive approximation method and the data dependence problem in fixed point theory for multivalued operators by Rus et al. [25].

Given a metric space (X, d). Let $P(X)$ be the class of nonempty subsets of X. Denote by $C(X)$ (resp. $CB(X)$) the class of nonempty closed (resp. all nonempty bounded and closed) subsets of X. For $A, B \in CB(X)$, consider the Pompeiu–Hausdorff functional

$$H(A,B) := \max\Big\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\Big\}.$$

For $\eta \in X$, define $D(\eta, B) = \inf_{\mu\in B} d(\eta, \mu)$.

Lemma 1.1 ([22]) *Given a metric space (X, d). Let $B \subseteq X$ and $\alpha > 1$. For $\eta \in X$, there is $\xi \in B$ such that $d(\eta, \xi) \le \alpha D(\eta, B)$.*

Berinde [14] introduced the following notion which was later named from 'weak contraction' to 'almost contraction' by Berinde [15].

Definition 1.1 Given a metric space (X, d). A mapping $F : X \to X$ is said to be an almost contraction or an (δ, L)-contraction if there are $\delta \in (0, 1)$ and $L \ge 0$ such that, for $\zeta, \theta \in X$,

$$d(F\zeta, F\theta) \le \delta d(\zeta, \theta) + Ld(\theta, F\zeta). \tag{1}$$

Nadler [22] used the notion of the Pompeiu–Hausdorff metric to ensure the existence of fixed points for multivalued contraction mappings.

M. Berinde and V. Berinde [13] initiated the notion of multivalued almost contractions as follows:

A mapping $F : X \to CB(X)$ is an almost contraction if there are $\delta \in (0, 1)$ and $L \geq 0$ such that, for $\zeta, \theta \in X$, the following inequality holds:

$$H(F\zeta, F\theta) \leq \delta d(\zeta, \theta) + LD(\theta, F\zeta). \tag{2}$$

Berinde [13] established the Nadler fixed point theorem in [22].

Theorem 1.1 *Let $F : X \to CB(X)$ be an almost contraction mapping on a complete metric space. Then F has a fixed point.*

Definition 1.2 ([25]) A mapping $T : X \to CB(X)$ is called an MWP operator if, for all $\zeta \in X$ and $\theta \in T\zeta$, there is $\{\zeta_n\}$ in X such that the following statements hold:

(i) $\zeta_0 = \zeta$ and $\zeta_1 = \theta$;
(ii) $\zeta_{n+1} \in T\zeta_n$ for all $n \geq 0$;
(iii) $\{\zeta_n\}$ converges to a fixed point of T.

Popescu [27] introduced the concept of (s, r)-contractive multivalued operators and obtained some (strict) fixed point results.

Definition 1.3 ([27]) Let $T : X \to CB(X)$ be a multivalued operator on a complete metric space (X, d). Such T is an (s, r)-contraction if $r \in [0, 1)$, $s \geq r$, and $\zeta, \theta \in X$

$$D(\theta, T\zeta) \leq sd(\theta, \zeta) \quad \text{implies} \quad H(T\zeta, T\theta) \leq rP(\zeta, \theta), \tag{3}$$

where

$$P(\zeta, \theta) = \max\left\{d(\zeta, \theta), D(\zeta, T\zeta), D(\theta, T\theta), \frac{D(\zeta, T\theta) + D(\theta, T\zeta)}{2}\right\}.$$

Theorem 1.2 ([27]) *Let $T : X \to CB(X)$ be an (s, r)-contractive multivalued operator (with $s > r$) on a complete metric space. Then T is an MWP operator.*

Theorem 1.3 ([27]) *Let $T : X \to CB(X)$ be an (s, r)-contractive multivalued operator on a complete metric space. Then T has a fixed point. Moreover, if $s \geq 1$, such a fixed point is unique.*

Kamran [17] improved the results of Popescu [27] to weakly (s, r)-contractive multivalued operators.

Definition 1.4 ([17]) Let $T : X \to CB(X)$ be a multivalued operator on a metric space (X, d). Such T is a weakly (s, r)-contraction if there are $r \in [0, 1)$, $s \geq r$, and $L \geq 0$ such that

$$D(\theta, T\zeta) \leq sd(\theta, \zeta) \quad \text{implies} \quad H(T\zeta, T\theta) \leq rN(\zeta, \theta),$$

where

$$N(\zeta,\theta) = \max\left\{d(\zeta,\theta), D(\zeta,T\zeta), D(\theta,T\theta), \frac{D(\zeta,T\theta)+D(\theta,T\zeta)}{2}\right\} + L\min\{d(\zeta,\theta), D(\theta,T\zeta)\}.$$

Theorem 1.4 ([17]) *Let $T: X \to CB(X)$ be a weakly (s,r)-contraction (with $s > r$ and $L \geq 0$) on a complete metric space. Then T is an MWP operator.*

On the other hand, Wardowski [34] introduced a generalized version of contraction mappings, called $\mathcal{F}$-contractions, i.e., a mapping $T: X \to X$ satisfying

$$\tau + \mathcal{F}\bigl(d(T\zeta,T\theta)\bigr) \leq \mathcal{F}\bigl(d(\zeta,\theta)\bigr)$$

for all $\zeta,\theta \in X$ with $T\zeta \neq T\theta$, where $\tau > 0$ and $\mathcal{F}: (0,\infty) \to \mathbb{R}$ is a function verifying the following conditions:

($\mathcal{F}_1$) $\mathcal{F}$ is strictly increasing;

($\mathcal{F}_2$) for each $\{a_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} \mathcal{F}(a_n) = -\infty$;

($\mathcal{F}_3$) there is $0 < k < 1$ such that $\lim_{a\to 0^+} a^k\mathcal{F}(a) = 0$.

It was proved that every $\mathcal{F}$-contraction on a complete metric space possesses a unique fixed point.

In 2014, Piri and Kumam [26] combined the notion of $\mathcal{F}$-contractions with a Suzuki type contraction as follows:

$$\frac{1}{2}d(\zeta,T\zeta) < d(\zeta,\theta) \quad \text{implies} \quad \tau + \mathcal{F}\bigl(d(T\zeta,T\theta)\bigr) \leq \mathcal{F}\bigl(d(\zeta,\theta)\bigr).$$

Recently, Turinici in [33] relaxed condition ($\mathcal{F}_2$) by

($\mathcal{F}_2'$) for each $\{a_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} \mathcal{F}(a_n) = -\infty$.

Then the following

($\mathcal{F}_2''$) $\mathcal{F}(a_n) \to -\infty$ implies $a_n \to 0$

can be derived from ($\mathcal{F}_1$).

Recently, Wardowski [35] considered the class of $\mathcal{F}$-contractions in a generalized way by replacing τ by a function $\varphi: (0,\infty) \to (0,\infty)$ and defined $(\varphi,\mathcal{F})$-contractions (nonlinear contractions) on a metric space (X,d) so that

($\mathcal{H}_1$) $\mathcal{F}$ verifies ($\mathcal{F}_1$) and ($\mathcal{F}_2'$);

($\mathcal{H}_2$) $\liminf_{q\to p^+} \varphi(q) > 0$ for $p \geq 0$;

($\mathcal{H}_3$) $\varphi(d(\zeta,\theta)) + \mathcal{F}(d(T\zeta,T\theta)) \leq \mathcal{F}(d(\zeta,\theta))$ for all $\zeta,\theta \in X$ so that $T\zeta \neq T\theta$.

Wardowski [35] proved a fixed point result for such nonlinear contractions by omitting ($\mathcal{F}_3$).

Altun et al. [6] used an extra condition on $\mathcal{F}$:

($\mathcal{F}_4$) $\mathcal{F}(\inf(P)) = \inf\mathcal{F}(P)$ for $P \subset (0,\infty)$ such that $\inf(P) > 0$.

Define:

($\mathcal{H}_1'$) $\mathcal{F}$ satisfies ($\mathcal{F}_1$), ($\mathcal{F}_2'$), and ($\mathcal{F}_4$).

($\mathcal{H}_3'$) There are $s \geq 0$ and $\mathcal{L} \geq 0$ such that, for $\zeta,\theta \in X$ with $H(T\zeta,T\theta) > 0$, we have

$$D(\theta,T\zeta) \leq sd(\theta,\zeta) \quad \text{implies} \quad \varphi\bigl(d(\zeta,\theta)\bigr) + \mathcal{F}\bigl(H(T\zeta,T\theta)\bigr) \leq \mathcal{F}\bigl(M(\zeta,\theta)\bigr), \tag{4}$$

where

$$M(\zeta,\theta) = \max\left\{d(\zeta,\theta), D(\zeta,T\zeta), D(\theta,T\theta), \frac{D(\zeta,T\theta)+D(\theta,T\zeta)}{2}\right\} + \mathcal{L}\min\{d(\zeta,\theta), D(\theta,T\zeta), D(\zeta,T\zeta)\}.$$

($\mathcal{H}_3''$) There are $r,s \in [0,1)$ with $r < s$ such that, for $\zeta,\theta \in X$ with $H(T\zeta,T\theta) > 0$,

$$\frac{1}{1+r}D(\zeta,T\zeta) \leq d(\zeta,\theta) \leq \frac{1}{1-s}D(\zeta,T\zeta)$$

implies

$$\varphi(d(\zeta,\theta)) + \mathcal{F}(H(T\zeta,T\theta)) \leq \mathcal{F}(M(\zeta,\theta)).$$

For more works concerning $\mathcal{F}$-contractions, we refer to [1–3, 5, 7–12, 18–21, 23, 24, 28, 32] and the references therein.

The graph of $T : X \to 2^X$ is given as

$$\mathrm{Gr}(T) = \{(\mu,\nu) \in X^2, \nu \in T\mu\}.$$

The mapping T is said to be upper semi-continuous if the inverse image of closed sets is closed.

Here, we introduce the concept of $(\mathcal{F}_s,\mathcal{L})$-contractive multivalued operators. We will extend the results of Kamran [17] and Popescu [27]. For more details, see [4, 16, 29–31]. An example is given to show the validity of our results.

2 Main results

We begin with the following definition.

Definition 2.1 Let (X,d) be a metric space. The multivalued operator $T : X \to CB(X)$ is an $(\mathcal{F}_s,\mathcal{L})$-contraction if conditions ($\mathcal{H}_1'$), ($\mathcal{H}_2$), and ($\mathcal{H}_3'$) are satisfied.

Our first result is as follows.

Theorem 2.1 *Let $T : X \to CB(X)$ be an $(\mathcal{F}_s,\mathcal{L})$-contractive multivalued operator on a complete metric space. Assume that* Gr(T) *is a closed subset of X^2. Then T is an MWP operator.*

Proof Let $\zeta_0 \in X$ and $\zeta_1 \in T\zeta_0$, then $D(\zeta_1,T\zeta_0) = 0$. In the case that $\zeta_0 = \zeta_1$, then ζ_1 is a fixed point of T, and so the proof is done.

Assume that $\zeta_0 \neq \zeta_1$. If $\zeta_1 \in T\zeta_1$, the proof is completed. Otherwise, if $\zeta_1 \notin T\zeta_1$, then since $T\zeta_1$ is closed, we have $D(\zeta_1,T\zeta_1) > 0$. Therefore, $H(T\zeta_0,T\zeta_1) \geq D(\zeta_1,T\zeta_1) > 0$, we also have $D(\zeta_1,T\zeta_0) \leq sd(\zeta_1,\zeta_0)$. Since T is an $(\mathcal{F}_s,\mathcal{L})$-contractive multivalued operator, we have

$$\varphi(d(\zeta_0,\zeta_1)) + F(H(T\zeta_0,T\zeta_1)) \leq F(M(\zeta_0,\zeta_1)),$$

where

$$\begin{aligned} M(\zeta_0,\zeta_1) &= \max\left\{d(\zeta_0,\zeta_1), D(\zeta_0,T\zeta_0), D(\zeta_1,T\zeta_1), \frac{D(\zeta_0,T\zeta_1)+D(\zeta_1,T\zeta_0)}{2}\right\} \\ &\quad + L\min\bigl\{d(\zeta_0,\zeta_1), D(\zeta_1,T\zeta_0), D(\zeta_0,T\zeta_0)\bigr\} \\ &\leq \max\left\{d(\zeta_0,\zeta_1), D(\zeta_1,T\zeta_1), \frac{d(\zeta_0,\zeta_1)+D(\zeta_1,T\zeta_1)}{2}\right\} \\ &= d(\zeta_0,\zeta_1). \end{aligned}$$

So

$$\varphi\bigl(d(\zeta_0,\zeta_1)\bigr) + F\bigl(H(T\zeta_0,T\zeta_1)\bigr) \leq F\bigl(d(\zeta_0,\zeta_1)\bigr). \tag{5}$$

Since $D(\zeta_1,T\zeta_1) \leq H(T\zeta_0,T\zeta_1)$, from $(\mathcal{F}_1)$ and (5), we have

$$F\bigl(D(\zeta_1,T\zeta_1)\bigr) \leq F\bigl(H(T\zeta_0,T\zeta_1)\bigr) \leq F\bigl(d(\zeta_0,\zeta_1)\bigr) - \varphi\bigl(d(\zeta_0,\zeta_1)\bigr). \tag{6}$$

Recall that $D(\zeta_1,T\zeta_1) > 0$, so from $(\mathcal{F}_4)$ we obtain

$$F\bigl(D(\zeta_1,T\zeta_1)\bigr) = \inf_{y\in T\zeta_1} F\bigl(d(\zeta_1,y)\bigr).$$

By (6), we have

$$\inf_{y\in T\zeta_1} F\bigl(d(\zeta_1,y)\bigr) \leq F\bigl(d(\zeta_0,\zeta_1)\bigr) - \varphi\bigl(d(\zeta_0,\zeta_1)\bigr). \tag{7}$$

There is $\zeta_2 \in T\zeta_1$ such that

$$F\bigl(d(\zeta_1,\zeta_2)\bigr) \leq F\bigl(d(\zeta_0,\zeta_1)\bigr) - \varphi\bigl(d(\zeta_0,\zeta_1)\bigr).$$

Continuing in this manner, we get $\{\zeta_n\}$ such that $\zeta_{n+1} \in T\zeta_n$ and

$$F\bigl(d(\zeta_n,\zeta_{n+1})\bigr) \leq F\bigl(d(\zeta_{n-1},\zeta_n)\bigr) - \varphi\bigl(d(\zeta_{n-1},\zeta_n)\bigr) \tag{8}$$

for all $n \geq 1$. Let $\alpha_n = d(\zeta_{n-1},\zeta_n)$ for all $n \geq 0$. We suppose that $\alpha_n > 0$ for each $n \in \mathbb{N}$. From (8), there is $c > 0$ such that

$$\mathcal{F}(\alpha_{n+1}) \leq \mathcal{F}(\alpha_n) - \varphi(\alpha_n) \quad \text{for each } n \in \mathbb{N}.$$

By $(\mathcal{F}_1)$, (α_n) is decreasing, and so $\alpha_n \searrow t \geq 0$. By $(\mathcal{H}_2)$ there are $c > 0$ and $n_0 \in \mathbb{N}$ such that $\varphi(\alpha_n) > 0$ for each $n \geq n_0$. Thus,

$$\begin{aligned} \mathcal{F}(\alpha_n) &\leq \mathcal{F}(\alpha_{n-1}) - \varphi(\alpha_{n-1}) \leq \cdots \leq \mathcal{F}(\alpha_1) - \sum_{i=1}^{n-1}\varphi(\alpha_i) \\ &= \mathcal{F}(\alpha_1) - \sum_{i=1}^{n_0-1}\varphi(\alpha_i) - \sum_{i=n_0}^{n-1}\varphi(\alpha_i) < \mathcal{F}(\alpha_1) - (n-n_0)c, \quad n > n_0. \end{aligned}$$

Taking $n \to \infty$, $\mathcal{F}(\alpha_n) \to -\infty$, so using $(\mathcal{F}_2'')$, $\alpha_n \to 0$.

Suppose that (ζ_n) is not a Cauchy sequence. Using $(\mathcal{F}_1)$, the set ∇ of all discontinuity elements of $\mathcal{F}$ is at most countable. There is $\gamma > 0$, $\gamma \notin \nabla$ in order that for each $k \geq 0$ there are $m_k, n_k \in \mathbb{N}$ such that

$$k \leq m_k < n_k \quad \text{and} \quad d(\zeta_{m_k}, \zeta_{n_k}) > \gamma, \qquad d(\zeta_{m_k}, \zeta_{n_k-1}) < \gamma, \qquad d(\zeta_{n_k}, \zeta_{m_k+1}) < \gamma. \tag{9}$$

Denote by $\bar{m}_k$ the least of m_k satisfying (9) and by $\bar{n}_k$ the least of n_k so that $\bar{m}_k < n_k$ and $d(\zeta_{\bar{m}_k}, \zeta_{n_k}) > \gamma$. Naturally, one writes that

$$d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k}) > \gamma, \qquad d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k-1}) < \gamma, \qquad d(\zeta_{\bar{n}_k}, \zeta_{\bar{m}_k+1}) < \gamma. \tag{10}$$

Taking $k_0 \in \mathbb{N}$ such that for $\alpha_k < \gamma$ for each $k \geq k_0$, we have

$$\gamma < d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k}) \leq d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k-1}) + d(\zeta_{\bar{n}_k-1}, \zeta_{\bar{n}_k}) \leq \gamma + \alpha_{\bar{n}_k} \quad \text{for each } k \geq k_0.$$

Therefore,

$$\lim_{k\to\infty} d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k}) = \gamma. \tag{11}$$

Thus, we conclude that

$$D(\zeta_{\bar{n}_k}, T\zeta_{\bar{m}_k}) \leq d(\zeta_{\bar{n}_k}, \zeta_{\bar{m}_k+1}) < \gamma < d(\zeta_{\bar{n}_k}, \zeta_{\bar{m}_k}) \leq sd(\zeta_{\bar{n}_k}, \zeta_{\bar{m}_k}). \tag{12}$$

From $(\mathcal{H}_3')$, we get

$$\varphi\big(d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k})\big) \leq \mathcal{F}\big(d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k})\big) - \mathcal{F}\big(d(\zeta_{\bar{m}_k+1}, \zeta_{\bar{n}_k+1})\big), \tag{13}$$

$k \geq 0$. Now, using (10)–(13) and by the continuity of $\mathcal{F}$ at γ, we get

$$\begin{aligned}\liminf_{s\to\gamma^+} \varphi(s) &\leq \liminf_{k\to\infty} \varphi\big(d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k})\big) \\ &\leq \lim_{k\to\infty} \Big(\mathcal{F}\big(d(\zeta_{\bar{m}_k}, \zeta_{\bar{n}_k})\big) - \mathcal{F}\big(d(\zeta_{\bar{m}_k+1}, \zeta_{\bar{n}_k+1})\big)\Big) = 0,\end{aligned}$$

which is a contradiction to $(\mathcal{H}_2)$. Therefore (ζ_n) is a Cauchy sequence. Hence, $\zeta_n \to z \in X$ as $n \to \infty$.

Since $\mathrm{Gr}(T)$ is closed, at the limit $n \to \infty$, $(\zeta_n, \zeta_{n+1}) \to (z, z)$ with $(z, z) \in \mathrm{Gr}(T)$. Thus, $z \in Tz$, i.e., z is a fixed point of T. □

The upper semi-continuity condition is stronger than the closedness of $\mathrm{Gr}(T)$. Consequently, we have the following result.

Theorem 2.2 *Let $T : X \to CB(X)$ be an $(\mathcal{F}_s, \mathcal{L})$-contractive multivalued operator on a complete metric space. Assume that T is upper semi-continuous. Then T is an MWP operator.*

Remark 2.1 Taking $T : X \to K(X)$ in Theorem 2.1 and $s \geq 1$, we may omit condition $(\mathcal{F}_4)$. In fact, let $\zeta_0 \in X$ and $\zeta_1 \in T\zeta_0$. If $\zeta_1 \in T\zeta_1$, then the proof is complete. Let $\zeta_1 \notin T\zeta_1$. Then, as $T\zeta_1$ is closed, $D(\zeta_1, T\zeta_1) > 0$. On the other hand, as $D(\zeta_1, T\zeta_1) \leq H(T\zeta_0, T\zeta_1)$, from $(\mathcal{F}_1)$ we have

$$\mathcal{F}\big(D(\zeta_1, T\zeta_1)\big) \leq \mathcal{F}\big(H(T\zeta_0, T\zeta_1)\big).$$

We also have $D(\zeta_1, T\zeta_0) \leq sd(\zeta_1, \zeta_0)$. Using $(\mathcal{H}_3')$, we have

$$\begin{aligned}
\mathcal{F}\big(D(\zeta_1, T\zeta_1)\big) &\leq \mathcal{F}\big(H(T\zeta_0, T\zeta_1)\big) \\
&\leq \mathcal{F}\big(M(\zeta_0, \zeta_1)\big) - \varphi\big(d(\zeta_0, \zeta_1)\big) \\
&\leq \mathcal{F}\big(d(\zeta_0, \zeta_1)\big) - \varphi\big(d(\zeta_0, \zeta_1)\big). \qquad (14)
\end{aligned}$$

Since $T\zeta_1$ is compact, there exists $\zeta_2 \in T\zeta_1$ such that $d(\zeta_1, \zeta_2) = D(\zeta_1, T\zeta_1)$. Then from (14) we have

$$\mathcal{F}\big(d(\zeta_1, \zeta_2)\big) \leq \mathcal{F}\big(d(\zeta_0, \zeta_1)\big) - \varphi\big(d(\zeta_0, \zeta_1)\big).$$

The rest of the proof is similar to that of the proof of Theorem 2.1.

Our second result is as follows.

Theorem 2.3 *Let $T : X \to X$ be an $(\mathcal{F}_s, \mathcal{L})$-contractive single-valued operator on a complete metric space. Assume that* $\mathrm{Gr}(T)$ *is a closed subset of X^2. Then T has a fixed point. Moreover, if $s \geq 1$, then such a fixed point is unique.*

Proof Similar to the proof of Theorem 2.1, T has a fixed point. Let $s \geq 1$ and ζ, θ be two distinct fixed points of T. Then

$$d(\theta, T\zeta) = d(\theta, \zeta) \leq sd(\theta, \zeta)$$

implies

$$\varphi\big(d(\zeta, \theta)\big) + F\big(d(T\zeta, T\theta)\big) \leq F\big(d(\zeta, \theta)\big)$$

or

$$\varphi\big(d(\zeta, \theta)\big) + F\big(d(\zeta, \theta)\big) \leq F\big(d(\zeta, \theta)\big),$$

hence $\zeta = \theta$. □

Definition 2.2 Let $T : X \to CB(X)$ be a multivalued operator on a metric space (X, d). Such T is an $(\mathcal{F}_{r,s}, \mathcal{L})$-contraction if conditions $(\mathcal{H}_1')$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3'')$ are satisfied.

Our third main result is as follows.

Theorem 2.4 *Let $T : X \to CB(X)$ be an $(\mathcal{F}_{r,s}, \mathcal{L})$-contraction on a complete metric space. Assume that* $\mathrm{Gr}(T)$ *is a closed subset of X^2. Then T is a multivalued weakly Picard operator.*

Proof Consider $t < 1$ so that $0 \le r < t < s$. Since $\frac{1-t}{1-s} > 1$, by Lemma 1.1, $\zeta_1 \in X$, and so there is $\zeta_2 \in T\zeta_1$ such that

$$d(\zeta_1, \zeta_2) \le \frac{1-t}{1-s} D(\zeta_1, T\zeta_1),$$

then

$$\frac{1}{1+r} D(\zeta_1, T\zeta_1) \le D(\zeta_1, T\zeta_1) \le d(\zeta_1, \zeta_2) \le \frac{1}{1-s} D(\zeta_1, T\zeta_1).$$

Since T is an $(\mathcal{F}_{r,s}, \mathcal{L})$-contraction, we have

$$\varphi\bigl(d(\zeta_1, \zeta_2)\bigr) + F\bigl(H(T\zeta_1, T\zeta_2)\bigr) \le F\bigl(M(\zeta_1, \zeta_2)\bigr), \tag{15}$$

where

$$\begin{aligned} M(\zeta_1, \zeta_2) = \max&\left\{ d(\zeta_1, \zeta_2), D(\zeta_1, T\zeta_1), D(\zeta_2, T\zeta_2), \frac{D(\zeta_1, T\zeta_2) + D(\zeta_2, T\zeta_1)}{2} \right\} \\ &+ L \min\bigl\{ d(\zeta_1, \zeta_2), D(\zeta_2, T\zeta_1), D(\zeta_1, T\zeta_1) \bigr\} \\ \le{}& d(\zeta_1, \zeta_2). \end{aligned}$$

So (15) becomes

$$F\bigl(H(T\zeta_1, T\zeta_2)\bigr) \le F\bigl(d(\zeta_1, \zeta_2)\bigr) - \varphi\bigl(d(\zeta_1, \zeta_2)\bigr). \tag{16}$$

Since $D(\zeta_2, T\zeta_2) \le H(T\zeta_1, T\zeta_2)$, from $(\mathcal{F}_1)$ and (16), we have

$$F\bigl(D(\zeta_2, T\zeta_2)\bigr) \le F\bigl(H(T\zeta_1, T\zeta_2)\bigr) \le F\bigl(d(\zeta_1, \zeta_2)\bigr) - \varphi\bigl(d(\zeta_1, \zeta_2)\bigr). \tag{17}$$

As $T\zeta_2$ is closed, $D(\zeta_2, T\zeta_2) > 0$, and from $(\mathcal{F}_4)$

$$F\bigl(D(\zeta_2, T\zeta_2)\bigr) = \inf_{y \in T\zeta_2} F\bigl(d(\zeta_2, y)\bigr).$$

By (17), we have

$$\inf_{y \in T\zeta_2} F\bigl(d(\zeta_2, y)\bigr) \le F\bigl(d(\zeta_1, \zeta_2)\bigr) - \varphi\bigl(d(\zeta_1, \zeta_2)\bigr). \tag{18}$$

There is $\zeta_3 \in T\zeta_2$ such that

$$F\bigl(d(\zeta_2, \zeta_3)\bigr) \le F\bigl(d(\zeta_1, \zeta_2)\bigr) - \varphi\bigl(d(\zeta_1, \zeta_2)\bigr).$$

Continuing in this manner, we construct a sequence $\{\zeta_n\}$ such that $\zeta_{n+1} \in T\zeta_n$, and the following inequality holds:

$$F\bigl(d(\zeta_n, \zeta_{n+1})\bigr) \le F\bigl(d(\zeta_{n-1}, \zeta_n)\bigr) - \varphi\bigl(d(\zeta_{n-1}, \zeta_n)\bigr) \tag{19}$$

for each $n \ge 1$.

As in the proof of Theorem 2.1, (ζ_n) is a Cauchy sequence, and so $\zeta_n \to z \in X$ as $n \to \infty$. By the arguments similar to those given in Theorem 2.1, we have that $D(z, Tz) = 0$. □

The following example is in support of Theorem 2.1.

Example 2.1 Let $X = \{0, 1, 2, 3\}$ and take $d(\zeta, \theta) = |\zeta - \theta|$. Consider $T : X \to CB(X)$ as

$$T\eta = \begin{cases} \{1,3\} & \text{if } \eta = 3, \\ \{2\} & \text{if not.} \end{cases}$$

Then, for $(\zeta, \theta) \in \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,3)\}$,

$$H(T\zeta, T\Theta) = 0,$$

and for $(\zeta, \theta) \in \{(0,3), (1,3), (2,3), (3,0), (3,1), (3,2)\}$,

$$H(T\zeta, T\theta) = 1.$$

Choosing $s = 0.5$ and $(\zeta, \theta) \in \{(2,3), (3,2)\}$, we have

$$D(\theta, T\zeta) = 1 = d(\theta, \zeta),$$

which gives

$$D(\theta, T\zeta) > sd(\theta, \zeta).$$

Now, for $(\zeta, \theta) \in \{(0,3), (1,3), (3,0), (3,1)\}$, we have

$$D(\theta, T\zeta) \le sd(\theta, \zeta).$$

Hence, for any $\mathcal{L} \ge 0$, choosing $\varphi(t) = \frac{1}{t}$ and $\mathcal{F}(t) = t + \ln(t)$, we have

$$\varphi\bigl(d(\zeta, \theta)\bigr) + \mathcal{F}\bigl(H(T\zeta, T\theta)\bigr) < \mathcal{F}\bigl(M(\zeta, \theta)\bigr).$$

That is, T is an $(\mathcal{F}_s, \mathcal{L})$-contraction. Also, $\mathrm{Gr}(T)$ is a closed subset of X^2. By Theorem 2.1, T has 2 and 3 as fixed points.

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Authors' contributions
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