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Coefficient bounds for certain subclasses of starlike functions

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Abstract

The conjecture proposed by Raina and Sokòł [Hacet. J. Math. Stat. 44(6):1427–1433 (2015)] for a sharp upper bound on the fourth coefficient has been settled in this manuscript. An example is constructed to show that their conjectures for the bound on the fifth coefficient and the bound related to the second Hankel determinant are false. However, the correct bound for the latter is stated and proved. Further, a sharp bound on the initial coefficients for normalized analytic function f such that $zf'(z)/f(z) \prec \sqrt{1+\lambda z}$, $\lambda \in (0,1]$, have also been obtained, which contain many existing results.

MSC: Primary 30C80; secondary 30C50; 30C45

Keywords: Starlike functions; Coefficient bounds; Hankel determinant

1 Introduction

The class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathcal{A}$ and its subclass containing univalent functions is denoted by $\mathcal{S}$. Among the many subclasses of $\mathcal{S}$ the classes of starlike and convex functions are the most studied classes. We recall that a domain D in the complex plane $\mathbb{C}$ is called starlike with respect to $w_0 \in D$ if each line joining w_0 to other points of D lies entirely in D. A domain which is starlike with respect to all its points is called a convex domain. Using the concept of subordination, in 1994, Ma and Minda [12] introduced general form of starlike and convex functions as follows: $\mathcal{S}^*(\varphi) := \{f \in \mathcal{A} : zf'(z)/f(z) \prec \varphi(z)\}$ and $\mathcal{K}(\varphi) := \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec \varphi(z)\}$, where the symbol '$\prec$' denotes the subordination and φ is an analytic function with positive real part in the unit disk $\mathbb{D}$ and mapping $\mathbb{D}$ onto a domain starlike with respect to 1, $\varphi'(0) > 0$ which is symmetric about the real axis.

For various choices of the function φ, the class $\mathcal{S}^*(\varphi)$ gives to several well-known/new classes. The class $\mathcal{S}^*_l := \mathcal{S}^*(\sqrt{1+z})$ was introduced by Sokół and Stankiewicz [25]. In 2009, Sokół [24] derived the sharp upper bound for first four coefficients for the class $\mathcal{S}^*_l$ and conjectured that $|a_{n+1}| \leq 1/2n$. In 2015, Ravichandran and Verma [21] verified this conjecture for the fifth coefficient. In 1998, Sokół generalized this class by introducing a more general class $\mathcal{S}^*_{l_\lambda} := \mathcal{S}^*(\sqrt{1+\lambda z})$, $\lambda \in (0,1]$ and obtained structural formula, growth theorem and also derived the sharp radius of convexity for this class. The functions in this class are strongly starlike of order $\arcsin(\lambda/\pi)$ and hence are univalent. Actuated by these classes, Mendiratta *et al.* [13] put before us a subclass of starlike functions associated with left-half of the shifted lemniscate of Bernoulli and discussed the geometric properties, coefficient estimates and the radius of starlikeness. Inspired by their work, Naveen *et al.* [23] considered the class starlike functions associated with cardioid discussed various properties of this class. In 2015, Raina and Sokół [19] introduced the interesting class $\mathcal{S}^*_q := \mathcal{S}^*(q)$, $q(z) = \sqrt{1+z^2} + z$ and proved that the class $\mathcal{S}^*_q$ is a subclass of the class consisting of functions $f \in \mathcal{A}$ such that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 2\left|\frac{zf'(z)}{f(z)}\right|$$

and discussed several other properties of the class $\mathcal{S}^*_q$. They derived bound on the coefficients. They proved the bounds (a) $|a_2| \leq 1$, (b) $|a_3| \leq 3/4$, (c) $|a_4| \leq 1/2$, (d) $|a_3 - \lambda a_2^2| \leq \max\{1/2, |\lambda - 3/4|\}$, $\lambda \in \mathbb{C}$ and (e) $|a_2a_4 - a_3^2| \leq 39/48$. The bounds (a), (b) and (d) were proven to be sharp. Further they conjectured that $|a_4| \leq 5/12$, $|a_5| \leq 2/9$ and $|a_2a_4 - a_3^2| \leq 7/48$. Recently, Gandhi and Ravichandran [6] discussed radius problems for this class.

Finding the upper bound for coefficients have been one of the central topic of research in geometric function theory as it gives several properties of functions. In particular, bound for the second coefficient gives growth and distortion theorems for functions in the class $\mathcal{S}$. Similarly, using the Hankel determinants (which also deals with the bound on coefficients), Cantor [1] proved that the "if ratio of two bounded analytic functions in $\mathbb{D}$, then the function is rational". For given natural numbers n, q, the Hankel determinant $H_{q,n}(f)$ of a function $f \in \mathcal{A}$ is defined by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

with $a_1 = 1$.

Note that $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete–Szegö functional. The second Hankel determinant is given by $H_{2,2}(f) = a_2a_4 - a_3^2$. The Hankel determinant $H_{q,n}(f)$ for the class of univalent functions was investigated by Pommerenke [15] and Hayman [7]. For successive developments in this direction till 2013, refer to [9]. In 2013, Sarfraz and Malik [22] obtained the upper bound on the third Hankel determinant for functions in the class $\mathcal{S}^*_l$. For more results and recent development in this direction, see [4, 5, 15, 16].

Motivated by the above work, in this manuscript, the conjecture $|a_4| \leq 5/12$ posed by Raina and Sokòł [18] for functions in the class $\mathcal{S}^*_q$ has been settled. However, an example is given to show that their conjecture $|a_2a_4 - a_3^2| \leq 7/48$ is false and a sharp upper bound for this functional is shown to be 1/4, that is, $|a_2a_4 - a_3^2| \leq 1/4$. The same example also shows that their conjecture $|a_5| \leq 2/9$ is also false. In addition to that, for functions in the class $\mathcal{S}^*_q$ a sharp upper bound on the functional $|a_2a_3 - a_4|$ is also derived. Furthermore, all the results proved by Sarfraz and Malik [22] have been generalized by proving sharp upper bound on the initial coefficients and bounds on $|a_2a_4 - a_3^2|$ and $|a_2a_3 - a_4|$ for functions in the class $\mathcal{S}^*_{l_\lambda}$. There were several mistakes/typos in their paper which have also been corrected.

Throughout this manuscript, let $\mathcal{P}$ denote the class of Carathéodory [2, 3] functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

The following results related to the class $\mathcal{P}$ are required for the discussion of the result in this manuscript.

Lemma 1.1 ([10, 11, Libera and Zlotkiewicz]) *If $p \in \mathcal{P}$ has the form given by* (2) *with $p_1 \geq 0$, then*

$$2p_2 = p_1^2 + x\bigl(4 - p_1^2\bigr) \tag{3}$$

and

$$4p_3 = p_1^3 + 2p_1\bigl(4 - p_1^2\bigr)x - p_1\bigl(4 - p_1^2\bigr)x^2 + 2\bigl(4 - p_1^2\bigr)\bigl(1 - |x|^2\bigr)y \tag{4}$$

for some x and y such that $|x| \leq 1$ and $|y| \leq 1$.

Lemma 1.2 ([21, Ravichandran and Verma]) *Let α, β, γ and a satisfy the inequalities $0 < \alpha < 1$, $0 < a < 1$ and*

$$8a(1-a)\bigl[(\alpha\beta - 2\gamma)^2 + \bigl(\alpha(a+\alpha) - \beta\bigr)^2\bigr] + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \leq 4a\alpha^2(1-\alpha)^2(1-a).$$

If $p \in \mathcal{P}$ has the form given by (2), *then*

$$\bigl|\gamma p_1^4 + ap_2^2 + 2\alpha p_1p_3 - (3/2)\beta p_1^2p_2 - p_4\bigr| \leq 2.$$

Let $\mathcal{B}$ be the class of analytic functions w of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{5}$$

and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. And let us consider a functional $\Psi(w) = |c_3 + \mu c_1c_2 + \nu c_1^3|$ for $w \in \mathcal{B}$ and $\mu, \nu \in \mathbb{R}$. Now we define sets A and B by

$$A = \left\{(\mu,\nu) \in \mathbb{R}^2 : 2 \leq |\mu| \leq 4,\ \nu \geq \frac{1}{12}\bigl(\mu^2 + 8\bigr)\right\}$$

and

$$B = \left\{(\mu,\nu) \in \mathbb{R}^2 : \frac{1}{2} \leq |\mu| \leq 2,\ -\frac{2}{3}\bigl(|\mu|+1\bigr) \leq \nu \leq \frac{4}{27}\bigl(|\mu|+1\bigr)^3 - \bigl(|\mu|+1\bigr)\right\},$$

respectively.

Lemma 1.3 ([17, Prokhorov and Szynal]) *If $w \in \mathcal{B}$, then for any real numbers μ and ν the following sharp estimate $\Psi(w) \leq \Phi(\mu,\nu)$ holds*:

$$\Phi(\mu,\nu) = \begin{cases} |\nu|, & \text{if } (\mu,\nu) \in A, \\ \frac{2}{3}(|\mu|+1)(\frac{|\mu|+1}{3(|\mu|+1+\nu)})^{1/2}, & \text{if } (\mu,\nu) \in B. \end{cases}$$

Lemma 1.4 ([14, Ohno and Sugawa]) *For any real numbers a, b and c, let the quantity $Y(a,b,c)$ be given by*

$$Y(a,b,c) = \max_{z\in\overline{\mathbb{D}}} \left\{ \left| a + bz + cz^2 \right| + 1 - |z|^2 \right\},$$

where $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. If $ac \geq 0$, then

$$Y(a,b,c) = \begin{cases} |a| + |b| + |c|, & \text{if } |b| \geq 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1-|c|)}, & \text{if } |b| < 2(1-|c|). \end{cases}$$

Furthermore, if $ac < 0$, then

$$Y(a,b,c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1-|c|)}, & \text{if } -4ac(c^{-2}-1) \leq b^2 \text{ and } |b| < 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1+|c|)}, & \text{if } b^2 < \min\{4(1+|c|)^2, -4ac(c^{-2}-1)\}, \\ R(a,b,c), & \text{otherwise}, \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a| + |b| - |c|, & \text{if } |c|(|b| + 4|a|) \leq |ab|, \\ -|a| + |b| + |c|, & \text{if } |ab| \leq |c|(|b| - 4|a|), \\ (|c| + |a|)\sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise}. \end{cases}$$

2 Main results

Raina and Sokół [18], for functions in the class $\mathcal{S}_q^*$, proved that $|a_4| \leq 1/2$ and $|a_2a_4 - a_3^2| \leq 39/48$ and conjectured that $|a_4| \leq 5/12$, $|a_5| \leq 2/9$ and $|a_2a_4 - a_3^2| \leq 7/48$. In the following proposition, the conjecture for $|a_4|$ has been settled. However, their conjectures $|a_2a_4 - a_3^2| \leq 7/48$ and $|a_5| \leq 2/9$ are shown to be false. To this aim, consider the Schwarz function $w(z) = z(\sqrt{6} - 3z)/(3 - \sqrt{6}z)$ such that $zf'(z)/f(z) = (w(z) + \sqrt{1 + w(z)^2})$. The solution of this equation is

$$f_1(z) := z + \sqrt{\frac{2}{3}}z^2 + \frac{z^3}{3} - \frac{1}{9}\sqrt{\frac{2}{3}}z^4 - \frac{13}{54}z^5 + \cdots. \tag{6}$$

Here we see that $|a_5| = 13/54 \approx 0.240 > 2/9 \approx 0.222$ and $|a_2a_3 - a_4| = 4\sqrt{6}/27 \approx 0.362887 > 7/48 \approx 0.145833$. We shall provide two proofs, both of which give sharp bounds on $|a_2a_3 - a_4|$ and $|a_2a_4 - a_3^2|$. The following proposition gives sharp bounds on $|a_4|$, $|a_2a_3 - a_4|$ and $|a_2a_4 - a_3^2|$.

Theorem 2.1 *Let $f \in \mathcal{S}_q^*$ with the form given by* (1). *Then the following inequalities hold*:

(1) $|a_4| \le 5/12$;
(2) $|a_2a_3 - a_4| \le 4\sqrt{6}/27$ *and* $|a_2a_4 - a_3^2| \le 1/4$.
The inequalities are sharp.

Proof Since $f \in \mathcal{S}_q^*$, it follows that there exists a Schwarz function $w \in \mathcal{B}$, with the form given by (5), such that

$$\frac{zf'(z)}{f(z)} = w(z) + \sqrt{1 + w(z)^2}. \tag{7}$$

Thus, we have

$$a_2 = c_1, \qquad a_3 = \frac{1}{2}\left(c_2 + \frac{3}{2}c_1^2\right) \quad \text{and} \quad a_4 = \frac{1}{3}\left(\frac{5}{4}c_1^3 + \frac{5}{2}c_1c_2 + c_3\right). \tag{8}$$

(1) Setting $\mu = 5/2$ and $\nu = 5/4$ in (8), we have

$$|a_4| = \frac{1}{3}\left|\nu c_1^3 + \mu c_1c_2 + c_3\right|.$$

We now use Lemma 1.3 for $\mu = 5/2$ and $\nu = 5/4$. In this case, we see that $|\nu c_1^3 + \mu c_1c_2 + c_3| \le |\nu| = 5/4$ as $(\mu, \nu) = (5/2, 5/4) \in A$. Thus, we conclude that $|a_4| \le 5/12$. The result is sharp as equality in the result holds for the function

$$\begin{aligned} f_2(z) &:= \frac{2(\sqrt{1+z^2} - 1)}{z} \exp\left\{z + \sqrt{1+z^2} - 1\right\} \\ &= z + z^2 + \frac{3}{4}z^3 + \frac{5}{12}z^4 + \frac{1}{8}z^5 + \cdots. \end{aligned}$$

(2) *First proof*: We now find sharp upper bound for functional $|a_2a_3 - a_4|$. To this aim, from (8), we have

$$|a_2a_3 - a_4| = \frac{1}{3}\left|c_3 + c_1c_2 - c_1^3\right|. \tag{9}$$

Setting $\mu = 1$ and $\nu = -1$ and using Lemma 1.3, we see that $(\mu, \nu) = (1, -1) \in B$ and $|\nu c_1^3 + \mu c_1c_2 + c_3| \le 4\sqrt{6}/9$. Thus, we conclude from (9) that $|a_4| \le 4\sqrt{6}/27$. The result is sharp as equality occurs in the case of the function f satisfying (7) with the Schwarz function is defined by $w(z) = z(u_0 - 2z)/(2 - u_0z)$, where $u_0 = 2\sqrt{6}/3$, that is, the equality occurs in the case of the function f_1 given by (6).

Now it remains to find sharp upper bound for $|a_2a_4 - a_3^2|$. To find bound on this functional, we shall use the relation between Carathéodory and Schwarz's functions. Setting $w(z) = (p(z) - 1)/(p(z) + 1)$ with $p \in \mathcal{P}$ of the form given by (2) in (7) and equating the coefficients, we have

$$a_2 = \frac{p_1}{2}, \qquad a_3 = \frac{1}{16}\left(p_1^2 + 4p_2\right) \quad \text{and} \quad a_4 = \frac{1}{96}\left(16p_3 + 4p_1p_2 - p_1^3\right).$$

A computation gives

$$a_2a_4 - a_3^2 = \frac{1}{768}\left(-7p_1^4 - 8p_1^2p_2 - 48p_2^2 + 64p_1p_3\right). \tag{10}$$

We substitute expression for p_2 and p_3 from (3) and (4) in (10). Since $|x| \leq 1$, $|y| \leq 1$ for some x and y and the class $\mathcal{S}_q^*$ is invariant under rotation, without loss of any generality we can assume that $p_1 = |p_1| =: s \in [0,2]$ and $|x| =: t \in [0,1]$, we get

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{768}F_1(s,t),$$

where

$$F_1(s,t) := 7s^4 + 32\left(4-s^2\right) + 4\left(4-s^2\right)\left(s^2+4\right)t^2 + 4s^2\left(4-s^2\right)t$$

with $s \in [0,2]$ and $t \in [0,1]$.

A computation reveals that the function F_1 has no critical point inside $(0,2) \times (0,1)$. Now we shall check the boundary of the rectangular domain $(0,2) \times (0,1)$ for maxima.

(i) $F_1(0,t) = (2+t^2)/12 \leq 1/4$, $t \in [0,1]$;

(ii) $F_1(2,t) = 5/42 < 1/4$, $t \in [0,1]$;

(iii) $F_1(s,0) = (7s^4 + 32(4-s^2))/768 \leq 5/42$, $s \in [0,2]$;

(iv) $F_1(s,1) = (192 - 16s^2 - s^4)/768 \leq 1/4$, $s \in [0,2]$.

It is clear, therefore, that $F_1(s,t) \leq 1/4$ for all $(s,t) \in [0,2] \times [0,1]$. Thus, $|a_2a_4 - a_3^2| \leq 1/4$. Equality holds in the case of the function

$$f_3(z) := z\exp\left(\int_0^z \frac{\sqrt{1+\zeta^4}+\zeta^2-1}{\zeta}\,\mathrm{d}\zeta\right) = z + \frac{z^3}{2} + \frac{z^5}{4} + \cdots. \tag{11}$$

Hence the result is sharp.

(2) *Second proof (estimate on* $|a_2a_3 - a_4|$*)*: From (7) with the relation $w(z) = (p(z) - 1)/(p(z)+1)$, where p is a Carathéodory function with the form given by (2). From (9), we have

$$|a_2a_3 - a_4| = \frac{1}{24}\left(p_1^3 + 2p_1p_2 - 4p_3\right).$$

Applying Lemma 1.1 and the invariant property for the class $\mathcal{S}_q^*$ under rotation, we have

$$|a_2a_3 - a_4| = \frac{1}{24}\left[s^3 - s\left(4-s^2\right)x + s\left(4-s^2\right)x^2 - 2\left(4-s^2\right)\left(1-|x|^2\right)y\right], \tag{12}$$

where $s := p_1 \in [0,2]$, $|x| \leq 1$ and $|y| \leq 1$. We note that, for $s = 0$ and $s = 2$

$$|a_2a_3 - a_4| \leq 1/3. \tag{13}$$

Now assume that $s \in (0,2)$. Then from (12) we obtain

$$|a_2a_3 - a_4| \leq \frac{1}{12}\left(4-s^2\right)F_2(s,x),$$

where

$$F_2(s,x) := \left|a + bx + cx^2\right| + 1 - |x|^2$$

with

$$a = \frac{s^3}{2(4-s^2)}, \qquad b = -\frac{1}{2}s \quad \text{and} \quad c = \frac{1}{2}s.$$

Here it is easy to verify that $ac > 0$. Here we have two cases now:

(i) When $s \in [4/3, 2)$, we obtain $|b| \geq 2(1-|c|)$. Therefore, by Lemma 1.4, we have

$$|a_2a_3 - a_4| \leq \frac{1}{12}(4-s^2)F_2(s,x) \leq \frac{1}{12}(4-s^2)\big(|a|+|b|+|c|\big) = \frac{1}{12}g(s),$$

where $g : [4/3, 2) \to \mathbb{R}$ is a function defined by $g(s) = (8s - s^3)/2$. Since g has its maximum at $s = s_1 := \sqrt{8/3}$, we have

$$|a_2a_3 - a_4| \leq \frac{1}{12}g(s_1) = \frac{4}{27}\sqrt{6}.$$

(ii) When $s \in (0, 4/3)$, we obtain $|b| < 2(1-|c|)$. Therefore, by Lemma 1.4, we have

$$|a_2a_3 - a_4| \leq \frac{1}{12}(4-s^2)F_2(s,x) \leq \frac{1}{12}(4-s^2)\left(1 + |a| + \frac{b^2}{4(1-|c|)}\right) = \frac{1}{12}h(s),$$

where $h : (0, 4/3) \to \mathbb{R}$ is a function defined by $h(s) = (32 - 6s^2 + 5s^3)/8$. Since $h'(s) = 0$ occurs only at $s = s_2 := 4/5$ in $(0, 4/3)$ and $h''(s_2) > 0$, h has no maximum in $(0, 4/3)$ and

$$h(s) \leq h\left(\frac{4}{3}\right) = \frac{112}{27} < \frac{4}{27}\sqrt{6}, \quad s \in (0, 4/3).$$

Therefore, by (13) and as discussed in the cases (i) and (ii), we have $|a_2a_3 - a_4| \leq 4\sqrt{6}/27$. To show sharpness of this bound, we note that equality holds when $p_1 = s_1 = \sqrt{8/3}$, $x = -1$ and $|z| = 1$. In this condition, it follows from Lemma 1.1 that $p_2 = 2/3$ and $p_3 = -2\sqrt{6}/9$. We can easily check that the function p defined by $p(z) = (1-z^2)/(1-u_0z+z^2)$ with $u_0 = 2\sqrt{6}/3$ satisfies them. The relation $w(z) = (p(z)-1)/(p(z)+1)$ shows that for the function f_1, given by (6), the resulting equality holds.

Estimate on $|a_2a_4 - a_3^2|$: From (10) with Lemma 1.1, we have

$$\begin{aligned} &a_2a_4 - a_3^2 \\ &\quad = \frac{1}{768}\big[-7s^4 + 4s^2(4-s^2)x - 4(4-s^2)(12+s^2)x^2 + 32s(4-s^2)(1-|x|^2)y\big], \end{aligned} \tag{14}$$

where $s := p_1 \in [0, 2]$, $|x| \leq 1$ and $|y| \leq 1$. We have the following two cases now:

(I) For $s = 0$ and $s = 2$, we get the bound $1/4$ and $7/48$, respectively, for $|a_2a_4 - a_3^2|$.

(II) Now assume that $s \in (0, 2)$. Then from (14), we have

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{24}s(4-s^2)F_3(s,x),$$

where

$$F_3(s,x) := \left|a + bx + cx^2\right| + 1 - |x|^2$$

with

$$a=\frac{-7s^3}{32(4-s^2)},\qquad b=\frac{1}{8}s\quad\text{and}\quad c=-\frac{12+s^2}{8s}.$$

We note that $ac>0$ and $|b|\ge 2(1-|c|)$ for all $s\in(0,2)$. Therefore, by Lemma 1.4, we have

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &\le \frac{1}{24}s\left(4-s^2\right)\left(|a|+|b|+|c|\right)\\ &=\frac{1}{24}\left(6-\frac{1}{2}s^2-\frac{1}{32}s^4\right)<\frac{1}{4},\quad s\in(0,2).\end{aligned}$$

Therefore, we have $|a_2a_4-a_3^2|\le 1/4$. To find the extremal function, we note that the maximum of the bound for $|a_2a_4-a_3^2|$ occurs when $p_1=s=0$ and $x=1$ and by applying Lemma 1.1 again, we get $p_1=0$ and $p_2=2$ and $p_3=0$. Thus, we get the function $p\in\mathcal{P}$ defined by $p(z)=(1+z^2)/(1-z^2)$ and the corresponding function for which equality holds in the result is f_3, given by (11). □

The function (6) suggests the following conjecture.

Conjecture 2.2 *Let $f\in\mathcal{S}_q^*$. Then $|a_5|\le 13/54$.*

3 Coefficient bounds for the class $\mathcal{S}_{l_\lambda}^*$

In this section, the work of Sarfraz and Malik [22] has been generalized for the class $\mathcal{S}_{l_\lambda}^*$. In addition to that a sharp upper bound for $|a_5|$ is also obtained.

Theorem 3.1 *Let $f\in\mathcal{S}_{l_\lambda}^*$, $\lambda\in(0,1]$ with the form given by (1). Then the following inequalities hold*:

(1) $|a_2|\le\lambda/2$, $|a_3|\le\lambda/4$, $|a_4|\le\lambda/6$, $|a_5|\le\lambda/8$; *and for any complex number μ*

$$\left|a_3-\mu a_2^2\right|\le\frac{\lambda}{4}\max\left\{1;\frac{|4\mu-1|}{4}\right\};$$

(2) $|a_2a_4-a_3^2|\le\lambda^2/16$ *and* $|a_2a_3-a_4|\le\lambda/6$.

The inequalities are sharp.

Proof Since $f\in\mathcal{S}_{l_\lambda}^*$, there exists a Schwarz function $w\in\mathcal{B}$, with the form given by (5), such that

$$\frac{zf'(z)}{f(z)}=\sqrt{1+\lambda w(z)}.\tag{15}$$

The function w is related with the Carathéodory [2, 3] function p with the form given by (2) as follows:

$$w(z)=\frac{p(z)-1}{p(z)+1}.$$

Thus, from (15), we have

$$\begin{aligned} a_2 &= \frac{\lambda}{4}p_1, \qquad a_3 = \frac{\lambda}{8}\left(p_2 + \frac{\lambda-4}{8}p_1^2\right), \\ a_4 &= \frac{\lambda}{12}p_3 + \frac{\lambda(\lambda-8)}{96}p_1p_2 + \frac{\lambda(\lambda^2-4\lambda+16)}{768}p_1^3, \end{aligned} \tag{16}$$

and

$$a_5 = -\frac{\lambda}{16}\left(\frac{\lambda^3+8\lambda^2-8\lambda+48}{384}p_1^4 - \frac{\lambda^2-2\lambda+18}{24}p_1^2p_2 - \frac{\lambda-12}{12}p_1p_3 + \frac{1}{2}p_2^2 - p_4\right).$$

(1) Upper bounds on $|a_2|$, $|a_3|$ and $|a_3 - \mu a_2^2|$ are readily obtained by just an application of the well-known results: $|p_n| \leq 2$ ($n \in \mathbb{N}$); and for any complex number ν, $|p_2 - \nu p_1^2| \leq 2\max\{1; |2\nu - 1|\}$ (see [8, 20]).

Now to find upper bound on $|a_4|$, we write

$$a_4 = \frac{\lambda}{768}\left[(\lambda^2-4\lambda+16)p_1^3 + 8(\lambda-8)p_1p_2 + 64p_3\right]. \tag{17}$$

Substituting expression for p_2 and p_3 from (3) and (4) in (17) and simplifying, we get

$$a_4 = \frac{\lambda}{768}\left[\lambda^2p_1^3 + 4\lambda(4-p_1^2)p_1x - 16(4-p_1^2)p_1x^2 + 32(4-p_1^2)(1-|x|^2)y\right].$$

Since $|x| \leq 1$, $|y| \leq 1$, for some x and y and the class $\mathcal{S}^*_{l_\lambda}$ is invariant under rotation, without loss of any generality we can assume that $p_1 = |p_1| =: s \in [0,2]$ and $|x| =: t \in [0,1]$. Thus, we can write

$$\begin{aligned} |a_4| &\leq \frac{\lambda}{768}\left[\lambda^2s^3 + 4\lambda(4-s^2)st + 16(4-s^2)st^2 + 32(4-s^2)(1-t^2)\right] \\ &= \frac{\lambda}{768}\left[\lambda^2s^3 + 4(4-s^2)\left(4(s-2)t^2 + \lambda st + 8\right)\right]. \end{aligned}$$

Let us denote

$$G_1(s,t) := \lambda^2s^3 + 4(4-s^2)\left(4(s-2)t^2 + \lambda st + 8\right).$$

Now we need to find the least upper bound of G_1 on $[0,2] \times [0,1]$. For this consider the function G_1 defined on the interior to the rectangular domain $[0,2] \times [0,1]$. A computation shows that the function G_1 has no critical point in $(0,2) \times (0,1)$. To this aim we note that G_1 has a unique critical point (s_1, t_1), where $s_1 := (256 - 4\lambda^2)/(15\lambda^2)$ and $t_1 = \lambda(\lambda^2 - 64)/(512 - 68\lambda^2)$ which possible lies in $(0,2) \times (0,1)$. It follows from $t_1 < 0$ for all $\lambda \in (0,1]$ that G_1 has no critical point in $(0,2) \times (0,1)$. Now we check the boundary of $(0,2) \times (0,1)$ for maxima of G_1. On the boundary of the rectangular domain $(0,2) \times (0,1)$, we have

(i) $G_1(0,t) = 128(1-t^2) \leq 128$, $t \in [0,1]$;
(ii) $G_1(2,t) = 8\lambda^2 \leq 8$, $t \in [0,1]$;
(iii) $G_1(s,0) = 128 - s^2(32 - \lambda^2 s) \leq 128$, $s \in [0,2]$;
(iv) $G_1(s,1) = (\lambda^2 - 4\lambda - 16)s^3 + 16(4+\lambda)s =: H_1(s)$, $s \in [0,2]$.

We now find the maximum of the function $H_1(s)$, $s \in [0,2]$. To this aim we note that $H_1'(s) = 0$ if and only if $s = s_2 := \sqrt{16(4+\lambda)/(3(16+4\lambda-\lambda^2))}$ and $H_1(s_2) = (32(4+\lambda)s_2)/3 \le 320/3 \le 128$, as $s_2 < 2$. Thus, we conclude that

$$|a_4| \le \max_{(s,t)\in[0,2]\times[0,1]} F_1(s,t) = \frac{\lambda}{6}.$$

To find the upper bound for $|a_5|$, we use Lemma 1.2 with

$$a = \frac{1}{2}, \qquad \alpha = -\frac{\lambda-12}{24}, \qquad \beta = \frac{\lambda^2-2\lambda+18}{36} \quad \text{and} \quad \gamma = \frac{\lambda^3+8\lambda^2-8\lambda+48}{384},$$

in

$$|a_5| = \frac{\lambda}{16}\left|\frac{\lambda^3+8\lambda^2-8\lambda+48}{384}p_1^4 - \frac{\lambda^2-2\lambda+18}{24}p_1^2p_2 - \frac{\lambda-12}{12}p_1p_3 + \frac{1}{2}p_2^2 - p_4\right|. \tag{18}$$

Then we see that all the conditions of Lemma 1.2 are satisfied. Indeed, we have

$$\begin{aligned}
&8a(1-a)\bigl[(\alpha\beta-2\gamma)^2) + \bigl(\alpha(a+\alpha)-\beta\bigr)^2\bigr] + \alpha(1-\alpha)(\beta-2a\alpha)^2 - 4a\alpha^2(1-\alpha)^2(1-a)\\
&\quad= \frac{1}{1{,}492{,}992}\bigl(-93{,}312 + 1656\lambda^2 + 1848\lambda^3 + 4508\lambda^4 + 970\lambda^5 + 119\lambda^6\bigr)\\
&\quad\le -\frac{84{,}211}{1{,}492{,}992} < 0
\end{aligned}$$

for all $\lambda \in (0,1]$. Thus,

$$\left|\frac{\lambda^3+8\lambda^2-8\lambda+48}{384}p_1^4 - \frac{\lambda^2-2\lambda+18}{24}p_1^2p_2 - \frac{\lambda-12}{12}p_1p_3 + \frac{1}{2}p_2^2 - p_4\right| \le 2$$

and, therefore, the result follows at once from (18).

Bounds on $|a_n|$ ($n = 2,3,4,5$) are sharp as equality holds in the results in the case of the function $g_{n,\lambda}$ defined by

$$g_{n,\lambda}(z) := z\exp\left(\int_0^z \frac{\sqrt{1+\lambda\zeta^{n-1}}-1}{\zeta}\,\mathrm{d}\zeta\right) = z + \frac{\lambda}{2n-2}z^n + \cdots, \tag{19}$$

respectively. Here we note that

$$g_{2,\lambda}(z) = \frac{4z\exp(2\sqrt{1+\lambda z}-2)}{(\sqrt{1+\lambda z}+1)^2} = z + \frac{\lambda}{2}z^2 + \cdots.$$

The extremal function of the functional $|a_3 - \mu a_2^2|$ is $g_{2,\lambda}$ when $|1-4\mu| \le 4$ and $g_{2,\sqrt{\lambda}}$ when $|1-4\mu| \ge 4$, respectively.

(2) From (16), we have

$$12{,}288\bigl(a_2a_4 - a_3^2\bigr) = \lambda^2\bigl[(4+\lambda)^2p_1^4 - 16(4+\lambda)p_1^2p_2 - 192p_2^2 + 256p_1p_3\bigr]. \tag{20}$$

Using (3), (4) in (20) and, for some x and y such that $|x| \leq 1$, $|y| \leq 1$, by setting $|p_1| =: s \in [0,2]$ and $|x| =: t \in [0,1]$, we can write

$$\begin{aligned} &12{,}288\left|a_2a_4 - a_3^2\right| \\ &\quad \leq \lambda^2\left[\lambda^2 s^4 + 16\left(4-s^2\right)\left(s^2-8s+12\right)t^2 + 8\lambda\left(4-s^2\right)s^2 t + 128\left(4-s^2\right)s\right]. \end{aligned} \tag{21}$$

Let us consider the function

$$G_2(s,t) := \lambda^2 s^4 + 16\left(4-s^2\right)\left(s^2-8s+12\right)t^2 + 8\lambda\left(4-s^2\right)s^2 t + 128\left(4-s^2\right)s$$

defined on the domain $[0,2] \times [0,1]$. It can be verified that the function G_2 is an increasing function of t, it follows that $G_2(s,\cdot)$ has its maximum at $t = 1$, and

$$G_2(s,1) = \left(\lambda^2 - 8\lambda - 16\right)s^4 + 32(\lambda-4)s^2 + 768.$$

Furthermore, since $\lambda^2 - 8\lambda - 16 < 0$ and $\lambda - 4 < 0$, it follows that $|G_2(s,1)| \leq 768$ for $s \in [0,2]$. Using this conclusion in (21), we get the asserted bound on $|a_2a_4 - a_3^2|$. The equality holds in the case of the function $g_{3,\lambda}$ defined by (19). Hence the bound thus obtained is sharp.

We now find the bound on $|a_2a_3 - a_4|$. Using (16), we have

$$384(a_2a_3 - a_4) = \lambda\left[\left(\lambda^2 - 4\lambda - 8\right)p_1^3 + 8(\lambda+4)p_1p_2 - 32p_3\right]. \tag{22}$$

Using Lemma 1.1 in (22) and setting $|p_1| =: s \in [0,2]$ and $|x| =: t \in [0,1]$, we have

$$384|a_2a_3 - a_4| \leq G_3(s,t),$$

where the function G_3 is defined on $[0,2] \times [0,1]$ by

$$G_3(s,t) := \lambda\left[\lambda^2 s^3 + 8(s-2)\left(4-s^2\right)t^2 + 4\lambda\left(4-s^2\right)st + 16\left(4-s^2\right)\right].$$

It is easy to check that there is only one critical point of G_3 in $(0,2) \times (0,1)$, viz.

$$(s_3,t_3) := \left(\frac{4(\lambda^2-16)}{9\lambda^2}, \frac{\lambda(\lambda^2-16)}{64-22\lambda^2}\right).$$

Further computation shows that

$$G_3(s_3,t_3) = \frac{\lambda^6 + 924\lambda^4 + 768\lambda^2 - 4096}{5832\lambda^3} \leq \frac{\lambda}{6}.$$

On the boundary of rectangular domain $(0,2) \times (0,1)$, we have

(i) $G_3(0,t) = \lambda(1-t^2)/6 \leq \lambda/6$, $t \in [0,1]$;
(ii) $G_3(2,t) = \lambda^3/48 < \lambda/6$, $t \in [0,1]$;
(iii) $G_3(s,0) = \lambda(\lambda^2 s^3 - 16s^2 + 64)/384 =: H_2(s)$, $s \in [0,2]$;
(iv) $G_3(s,1) = \lambda s(16(s+2) + (s^2 - 4s - 8)s^2)/384 =: H_3(s)$, $s \in [0,2]$.

The function H_2 is decreasing on $(0,2)$, so $H_2(s) \le H_2(0) = \lambda/6$. Now a computation shows that the function H_3 is increasing in $(0, s_4)$ and decreasing in $(s_4, 1)$, and

$$H_3(s_4) = \frac{\lambda+2}{9}\sqrt{\frac{\lambda+2}{3(8+4\lambda-\lambda^2)}} < \frac{\lambda}{6},$$

where $s_4 := 4\sqrt{(\lambda+2)/(3(8+4\lambda-\lambda^2))}$. Thus, we have $|a_2a_3 - a_4| \le \lambda/6$. Sharpness of the result could be seen in the case of the function $g_{4,\lambda}$ defined by (19). This completes the proof. □

Conjecture 3.2 *Since the function $g_{n,\lambda}$ given by* (19) *is extremal for the first five coefficients for functions in the class $\mathcal{S}^*_{l_\lambda}$, one may expect naturally $|a_{n+1}| \le \lambda/2n$, for all $n \ge 6$.*

Theorem 3.3 *Let $f \in \mathcal{S}^*_{l_\lambda}$. Then*

$$\sum_{k=2}^{\infty}\left(k^2 - \lambda - 1\right)|a_k|^2 \le 1.$$

Proof Since $f \in \mathcal{S}^*_{l_\lambda}$, it follows from (15) that $\lambda f(z)^2 w(z) = (zf'(z))^2 - f(z)^2$. For $|z| = r \in [0,1)$ and $t \in [0, 2\pi]$, we have

$$\begin{aligned} 2\pi \sum_{k=1}^{\infty}|a_k|^2 r^{2k} &= \int_0^{2\pi}\left|f\left(re^{it}\right)\right|^2 dt \\ &\ge \frac{1}{\lambda}\int_0^{2\pi}\left|\left(re^{it}f'\left(re^{it}\right)\right)^2 - f\left(re^{it}\right)^2\right| dt \\ &= \frac{2\pi}{\lambda}\sum_{k=1}^{\infty}\left(k^2-1\right)|a_k|^2 r^{2k}. \end{aligned}$$

This on simplification after letting $r \to 1^-$ gives the required result. □

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