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On extended interpolative Ćirić–Reich–Rus type F-contractions and an application

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Abstract

The goal of this work is to introduce an extended interpolative Ćirić–Reich–Rus type contraction by the approach of Wardowski. We establish some related fixed point results (for single and multivalued-mappings). Some examples are presented to illustrate the main result. Moreover, we give an application to integral equations.

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1 Introduction

A Banach couple is two Banach spaces $\mathcal{A}$ and $\mathcal{B}$ topologically and algebraically imbedded in a separated topological linear space, and denoted by $(\mathcal{A},\mathcal{B})$. The Banach space $\mathcal{E}$ is called intermediate for the spaces of the Banach couple $(\mathcal{A},\mathcal{B})$ if the imbedding $\mathcal{A}\cap\mathcal{B}\subset\mathcal{E}\subset\mathcal{A}+\mathcal{B}$ holds.

Let $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ be two Banach couples. A linear mapping T acting from the space $\mathcal{A}+\mathcal{B}$ into $\mathcal{C}+\mathcal{D}$ is said to be a bounded operator from $(\mathcal{A},\mathcal{B})$ into $(\mathcal{C},\mathcal{D})$ if the restrictions of T to $\mathcal{A}$ and $\mathcal{B}$ are bounded operators from $\mathcal{A}$ into $\mathcal{C}$ and $\mathcal{B}$ into $\mathcal{D}$, respectively.

Let $L(\mathcal{AB},\mathcal{CD})$ be the linear space of all bounded operators from $(\mathcal{A},\mathcal{B})$ into $(\mathcal{C},\mathcal{D})$. Consider,

$$\|T\|_{L(\mathcal{AB},\mathcal{CD})}=\max\left\{\|T\|_{\mathcal{A}\to\mathcal{B}},\|T\|_{\mathcal{C}\to\mathcal{D}}\right\}.$$

Note that $(L(\mathcal{AB},\mathcal{CD}),\|\|)$ is a Banach space.

Definition 1.1 ([1]) Let $(\mathcal{A},\mathcal{B})$ and $(\mathcal{C},\mathcal{D})$ be two Banach couples, and $\mathcal{E}$ (respectively $\mathcal{F}$) be intermediate for the spaces of the Banach couple $(\mathcal{A},\mathcal{B})$ (respectively $(\mathcal{C},\mathcal{D})$). The triple $(\mathcal{A},\mathcal{B},\mathcal{E})$ is called an interpolation triple, relative to $(\mathcal{C},\mathcal{D},\mathcal{F})$, if every bounded operator from $(\mathcal{A},\mathcal{B})$ to $(\mathcal{C},\mathcal{D})$ maps $\mathcal{E}$ to $\mathcal{F}$.

A triple $(\mathcal{A},\mathcal{B},\mathcal{E})$ is said to be an interpolation triple of type α $(0\le\alpha\le 1)$ relative to $(\mathcal{C},\mathcal{D},\mathcal{F})$ if it is an interpolation triple and the following

$$\|T\|_{\mathcal{E}\to\mathcal{F}}\le c\|T\|^{\alpha}_{\mathcal{A}\to\mathcal{B}}\cdot\|T\|^{1-\alpha}_{\mathcal{C}\to\mathcal{D}},$$

holds for some constant c.

Inspired by the definition above, the interpolative Kannan contraction has been described in [2] as follows: Given a metric space (X,d), the mapping $\Upsilon : X \to X$ is called an interpolative Kannan contraction if

$$d(\Upsilon\theta, \Upsilon\vartheta) \leq \lambda\big[d(\theta, \Upsilon\theta)\big]^{\alpha} \cdot \big[d(\vartheta, \Upsilon\vartheta)\big]^{1-\alpha}, \tag{1.1}$$

for all $\theta, \vartheta \in X$ with $\theta \neq \Upsilon\theta$, where $\lambda \in [0,1)$ and $\alpha \in (0,1)$. The main result in [2] is stated as follows.

Theorem 1.2 ([2]) *Let (X,d) be a complete metric space and Υ be an interpolative Kannan type contraction. Then Υ possesses a unique fixed point in X.*

Karapınar, Agarwal and Aydi [3] gave a counter-example to Theorem 1.2, showing that the fixed point may be not unique. The following result is the corrected version of Theorem 1.2.

Theorem 1.3 ([3]) *Let Υ be a self-mapping on the complete metric space (X,d). Suppose that*

$$d(\Upsilon\theta, \Upsilon\vartheta) \leq \lambda\big[d(\theta, \Upsilon\theta)\big]^{\alpha} \cdot \big[d(\vartheta, \Upsilon\vartheta)\big]^{1-\alpha},$$

for all $\theta, \vartheta \in X \setminus \mathrm{Fix}(\Upsilon)$, where $\mathrm{Fix}(\Upsilon) = \{\eta \in X, \Upsilon\eta = \eta\}$. Then there is a unique fixed point of Υ.

On the other hand, Ćirić–Reich–Rus [4–9] generalized the Banach contraction principle [10].

Theorem 1.4 *Let (X,d) be a complete metric space. Let $\Upsilon : X \to X$ so that the following:*

$$d(\Upsilon\theta, \Upsilon\vartheta) \leq \alpha d(\theta, \vartheta) + \beta d(\theta, \Upsilon\theta) + \gamma d(\vartheta, \Upsilon\vartheta)$$

holds, for all $\theta, \vartheta \in X$, where $\alpha, \beta, \gamma \geq 0$ such that $\alpha + \beta + \gamma < 1$. Then Υ admits a unique fixed point.

Recently, Karapinar et al. [3] initiated the notion of interpolative Ćirić–Reich–Rus type contractions.

Definition 1.5 ([11]) Let (X,d) be a metric space. We say that the self-mapping Υ on X is an *interpolative Ćirić–Reich–Rus type contraction* if there are $\lambda \in [0,1)$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$ so that

$$d(\Upsilon\theta, \Upsilon\vartheta) \leq \lambda\big[d(\theta, \vartheta)\big]^{\alpha} \cdot \big[d(\theta, \Upsilon\theta)\big]^{\beta} \cdot \big[d(\vartheta, \Upsilon\vartheta)\big]^{1-\alpha-\beta}, \tag{1.2}$$

for all $\theta, \vartheta \in X \setminus \mathrm{Fix}(\Upsilon)$.

Theorem 1.6 ([3]) *An interpolative Ćirić–Reich–Rus type contraction mapping on the complete metric space (X,d) possesses a fixed point in X.*

For other results dealing with interpolate approach, see [11–14]. On the other hand in 2012, Wardowski [15] gave a new generalization of the Banach contraction by introducing the notion of F-contractions. For related results, see [16–20]. Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of all natural numbers, real numbers and positive real numbers, respectively. $\mathcal{F}$ represents the collection of all functions $F:(0,\infty)\to\mathbb{R}$ so that:

(F1) F is strictly increasing.

(F2) For each sequence $\{\alpha_n\}$ in $(0,\infty)$, $\lim_{n\to\infty}\alpha_n=0$ iff $\lim_{n\to\infty}F(\alpha_n)=-\infty$.

(F3) There is $k\in(0,1)$ so that $\lim_{\alpha\to0^+}\alpha^kF(\alpha)=0$.

Definition 1.7 ([15]) Let (X,d) be a metric space. A mapping $\Upsilon:X\to X$ is said to be an F-contraction if there exist $\tau>0$ and $F\in\mathcal{F}$ such that for all $\Omega,\omega\in X$,

$$d(\Upsilon\Omega,\Upsilon\omega)>0 \quad\Longrightarrow\quad \tau+F\bigl(d(\Upsilon\Omega,\Upsilon\omega)\bigr)\le F\bigl(d(\Omega,\omega)\bigr). \tag{1.3}$$

Example 1.8 ([15]) The functions $F:(0,\infty)\to\mathbb{R}$ defined by

(1) $F(\alpha)=\ln\alpha$,

(2) $F(\alpha)=\ln\alpha+\alpha$,

(3) $F(\alpha)=\frac{-1}{\sqrt{\alpha}}$,

(4) $F(\alpha)=\ln(\alpha^2+\alpha)$,

belong to $\mathcal{F}$.

Wardowski [15] introduced a new proper generalization of Banach contraction as follows.

Theorem 1.9 ([15]) *Let (X,d) be a complete metric space and let $T:X\to X$ be an F-contraction. Then Υ has a unique fixed point, say z, in X and for any point $\sigma\in X$, the sequence $\{\Upsilon^j\sigma\}$ converges to z.*

By using the approach of Wardowski [15] (for single and multi-valued mappings), we initiate the concept of extended interpolative Ćirić–Reich–Rus type contractions. Some related fixed point results are also presented.

2 Main results

First, we introduce the notion of *extended interpolative Ćirić–Reich–Rus type F-contractions.*

Definition 2.1 Let (X,d) be a metric space. We say that the self-mapping Υ on X is an *extended interpolative Ćirić–Reich–Rus type F-contraction* if there exist $\alpha,\beta\in(0,1)$ with $\alpha+\beta<1$, $\tau>0$ and $F\in\mathcal{F}$ such that

$$\tau+F\bigl(d(\Upsilon\theta,\Upsilon\vartheta)\bigr)\le\alpha F\bigl(d(\theta,\vartheta)\bigr)+\beta F\bigl(d(\theta,\Upsilon\theta)\bigr)+(1-\alpha-\beta)F\bigl(d(\vartheta,\Upsilon\vartheta)\bigr), \tag{2.1}$$

for all $\theta,\vartheta\in X\setminus\operatorname{Fix}(\Upsilon)$ with $d(\Upsilon\theta,\Upsilon\vartheta)>0$.

Theorem 2.2 *An extended interpolative Ćirić–Reich–Rus type F-contraction self-mapping on a complete metric space admits a fixed point in X.*

Proof Starting from $\theta_0 \in X$, consider $\{\theta_n\}$, given as $\theta_n = T^n(\theta_0)$ for each positive integer n. If there is n_0 so that $\theta_{n_0} = \theta_{n_0+1}$, then θ_{n_0} is a fixed point of T. Suppose that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. Taking $\theta = \theta_n$ and $\vartheta = \theta_{n-1}$ in (2.1), one writes

$$\begin{aligned}
&\tau + F\bigl(d(\theta_{n+1},\theta_n)\bigr) \\
&\quad = \tau + F\bigl(d(\Upsilon\theta_n,\Upsilon\theta_{n-1})\bigr) \\
&\quad \leq \alpha F(d(\theta_n,\theta_{n-1}) + \beta F\bigl(d(\theta_n,\Upsilon\theta_n)\bigr) + (1-\alpha-\beta)F\bigl(d(\theta_{n-1},\Upsilon\theta_{n-1})\bigr) \\
&\quad = \alpha F\bigl(d(\theta_n,\theta_{n-1})\bigr) + \beta F\bigl(d(\theta_n,\theta_{n+1})\bigr) + (1-\alpha-\beta)F\bigl(d(\theta_{n-1},\theta_n)\bigr).
\end{aligned} \tag{2.2}$$

Suppose that $d(\theta_{n-1},\theta_n) < d(\theta_n,\theta_{n+1})$ for some $n \geq 1$. The inequality (2.2) yields

$$\tau + F\bigl(d(\theta_n,\theta_{n+1})\bigr) \leq F\bigl(d(\theta_n,\theta_{n+1})\bigr), \tag{2.3}$$

which is a contradiction. Therefore, $d(\theta_n,\theta_{n+1}) \leq d(\theta_{n-1},\theta_n)$ for all $n \geq 1$. Again from (2.2), we get

$$\tau + F\bigl(d(\theta_n,\theta_{n+1})\bigr) \leq F\bigl(d(\theta_{n-1},\theta_n)\bigr). \tag{2.4}$$

Consequently

$$F\bigl(d(\theta_n,\theta_{n+1})\bigr) \leq F\bigl(d(\theta_{n-1},\theta_n)\bigr) - \tau \leq \cdots \leq F\bigl(d(\theta_0,\theta_1)\bigr) - n\tau, \tag{2.5}$$

for all $n \geq 1$. Therefore $d(\theta_n,\theta_{n+1}) < d(\theta_{n-1},\theta_n)$ for all $n \geq 1$. Taking $n \to \infty$ in (2.5) yields $\lim_{n\to\infty} F(d(\theta_n,\theta_{n+1})) = -\infty$. From (F2), we get $\lim_{n\to\infty} d(\theta_n,\theta_{n+1}) = 0$. Put $\gamma_n = d(\theta_n,\theta_{n+1})$. Thus, $\lim_{n\to\infty} \gamma_n = 0$. Then for any $n \in \mathbb{N}$, we have ${\gamma_n}^k(F(\gamma_n) - F(\gamma_0)) \leq -{\gamma_n}^k n\tau < 0$. Thus, $\lim_{n\to\infty} {\gamma_n}^k n = 0$. So, there is $N \in \mathbb{N}$ so that $\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq N$. Now, for any $m,n \in \mathbb{N}$ with $m > n$, we get

$$d(\theta_n,\theta_m) \leq \sum_{i=n}^{m-1} d(\theta_i,\theta_{i+1}) = \sum_{i=n}^{m-1} \gamma_i \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}}.$$

Since the last term of the above inequality tends to zero as $m,n \to \infty$, we have $d(\theta_n,\theta_m) \to 0$ as $m,n \to \infty$, that is, $\{\theta_n\}$ is a Cauchy sequence, and so $\theta_n \to \theta$ as $n \to \infty$. Suppose to the contrary $\theta \neq \Upsilon\theta$.

We consider two cases.

Case 1: There is a subsequence $\{\theta_{n_k}\}$ such that $\Upsilon\theta_{n_k} = \Upsilon\theta$ for all $k \in \mathbb{N}$. In this case,

$$d(\theta,\Upsilon\theta) = \lim_{k\to\infty} d(\theta_{n_k+1},\Upsilon\theta) = \lim_{k\to\infty} d(\Upsilon\theta_{n_k},\Upsilon\theta) = 0.$$

Case 2: There is a natural number N such that $\Upsilon\theta_n \neq \Upsilon\theta$ for all $n \geq N$. In this case, applying (2.1), for $\theta = \theta_n$ and $\vartheta = \theta$, we have

$$\begin{aligned}
\tau + F\bigl(d(\theta_{n+1},\Upsilon\theta)\bigr) &= \tau + F\bigl(d(\Upsilon\theta_n,\Upsilon\theta)\bigr) \\
&\leq \alpha F\bigl(d(\theta_n,\theta)\bigr) + \beta F\bigl(d(\theta_n,\theta_{n+1})\bigr) + (1-\alpha-\beta)F\bigl(d(\theta,\Upsilon\theta)\bigr).
\end{aligned} \tag{2.6}$$

Letting $n \to \infty$ in the inequality (2.6), we find that $\lim_{n\to\infty} F(d(\theta_{n+1}, \Upsilon\theta)) = -\infty$ and so $\lim_{n\to\infty} d(\theta_{n+1}, \Upsilon\theta) = 0$. Therefore,

$$d(\theta, \Upsilon\theta) = \lim_{n\to\infty} d(\theta_{n+1}, \Upsilon\theta) = \lim_{n\to\infty} d(\Upsilon\theta_n, \Upsilon\theta) = 0.$$

Thus, $d(\theta, \Upsilon\theta) = 0$ and so $\theta = \Upsilon\theta$. Hence, $\Upsilon\theta = \theta$. □

We illustrate Theorem 2.2 by the following examples.

Example 2.3 let $X = \{-1, 0, 1\}$ be endowed with the metric

$$d(\theta, \vartheta) = \begin{cases} 0 & \text{if } \theta = \vartheta, \\ \frac{3}{2} & \text{if } (\theta, \vartheta) \in \{(1, -1), (-1, 1)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, (X, d) is complete. Take $\Upsilon 0 = \Upsilon(-1) = 0$ and $\Upsilon 1 = -1$.

First, letting $\theta = 0$ and $\vartheta = 1$, we have

$$F\bigl(d(\Upsilon\theta, \Upsilon\vartheta)\bigr) = F\bigl(d(0, -1)\bigr) = F(1) \quad \text{and} \quad F\bigl(d(\theta, \vartheta)\bigr) = F\bigl(d(0, 1)\bigr) = F(1).$$

Thus, we cannot find $\tau > 0$ such that $\tau + F(d(\Upsilon\theta, \Upsilon\vartheta)) \le F(d(\theta, \vartheta))$, that is, Theorem 1.9 is not applicable.

On the other hand, let $\theta, \vartheta \in X \setminus \mathrm{Fix}(\Upsilon)$ with $d(\Upsilon\theta, \Upsilon\vartheta) > 0$. Hence $(\theta, \vartheta) \in \{(1, -1), (-1, 1)\}$. Without loss of generality, take $(\theta, \vartheta) = (1, -1)$. Choose $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, $\tau = \frac{1}{2}\ln(\frac{3}{2})$ and $F(t) = \ln(t)$ for $t > 0$. We have

$$\begin{aligned} \tau + F\bigl(d(\Upsilon\theta, \Upsilon\vartheta)\bigr) &= \frac{1}{3}\ln\left(\frac{3}{2}\right) + \ln(1) \\ &= \frac{1}{3}\ln\left(\frac{3}{2}\right) \\ &\le \frac{1}{2}\ln\left(\frac{3}{2}\right) \\ &= \alpha F\bigl(d(\theta, \vartheta)\bigr) + \beta F\bigl(d(\theta, \Upsilon\theta)\bigr) + (1 - \alpha - \beta) F\bigl(d(\vartheta, \Upsilon\vartheta)\bigr), \end{aligned}$$

that is, (2.1) holds for all $\theta, \vartheta \in X \setminus \mathrm{Fix}(\Upsilon)$ with $d(\Upsilon\theta, \Upsilon\vartheta) > 0$. Here, Υ admits a fixed point ($u = 0$).

Example 2.4 Let $X = [0, 1]$. We endow X with the metric d defined by

$$d(\theta, \vartheta) = \begin{cases} \max\{\theta, \vartheta\} & \text{if } \theta \neq \vartheta, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the mapping $\Upsilon : X \to X$ given as

$$\Upsilon\theta = \begin{cases} 0 & \text{if } \theta \in [0, \frac{1}{4}), \\ \frac{1}{8} & \text{if } \theta \in [\frac{1}{4}, \frac{1}{2}], \\ \frac{1}{4} & \text{if } \theta \in (\frac{1}{2}, 1]. \end{cases}$$

Take $F(t) = \ln(t)$ and $\alpha = \beta = \frac{1}{4}$. Choose $\tau \in (0, \ln(2))$. Let $\theta, \vartheta \in X \setminus \mathrm{Fix}(\Upsilon)$ such that $d(\Upsilon\theta, \Upsilon\vartheta) > 0$. Without loss of generality, we have the following cases: $(\theta, \vartheta) \in \{((0, \frac{1}{4}) \times [\frac{1}{4}, \frac{1}{2}]), ((0, \frac{1}{4}) \times (\frac{1}{2}, 1]), ([\frac{1}{4}, \frac{1}{2}] \times (\frac{1}{2}, 1])\}$. *Case 1*: $(\theta, \vartheta) \in ((0, \frac{1}{4}) \times [\frac{1}{4}, \frac{1}{2}])$. Here, we have

$$\begin{aligned}
\tau + F\bigl(d(\Upsilon\theta, \Upsilon\vartheta)\bigr) &= \tau + \ln\left(\frac{1}{8}\right) \\
&\leq \ln(2) + \ln\left(\frac{1}{8}\right) = \ln\left(\frac{1}{4}\right) \\
&\leq \frac{1}{2}\ln\left(\frac{1}{4}\right) \\
&\leq \frac{1}{2}\ln(\vartheta) \\
&\leq \alpha F\bigl(d(\theta, \vartheta)\bigr) + \beta F\bigl(d(\theta, \Upsilon\theta)\bigr) + (1 - \alpha - \beta) F\bigl(d(\vartheta, \Upsilon\vartheta)\bigr).
\end{aligned}$$

Case 2: $(\theta, \vartheta) \in ((0, \frac{1}{4}) \times (\frac{1}{2}, 1])$. Here, we have

$$\begin{aligned}
\tau + F\bigl(d(\Upsilon\theta, \Upsilon\vartheta)\bigr) &= \tau + F\left(\frac{1}{4}\right) \\
&\leq \ln(2) + \ln\left(\frac{1}{4}\right) = \ln\left(\frac{1}{2}\right) \\
&\leq \frac{1}{2}\ln\left(\frac{1}{2}\right) \\
&\leq \frac{1}{2}\ln(\vartheta) \\
&\leq \alpha F\bigl(d(\theta, \vartheta)\bigr) + \beta F\bigl(d(\theta, \Upsilon\theta)\bigr) + (1 - \alpha - \beta) F\bigl(d(\vartheta, \Upsilon\vartheta)\bigr).
\end{aligned}$$

Case 3: $(\theta, \vartheta) \in ([\frac{1}{4}, \frac{1}{2}] \times (\frac{1}{2}, 1])$. Again, we have

$$\begin{aligned}
\tau + F\bigl(d(\Upsilon\theta, \Upsilon\vartheta)\bigr) &= \tau + F\left(\frac{1}{4}\right) \\
&\leq \alpha F\bigl(d(\theta, \vartheta)\bigr) + \beta F\bigl(d(\theta, \Upsilon\theta)\bigr) + (1 - \alpha - \beta) F\bigl(d(\vartheta, \Upsilon\vartheta)\bigr).
\end{aligned}$$

All assumptions of Theorem 2.2 hold. Here, T has a fixed point, which is, $u = 0$.

On the other, the Wardowski contraction is not satisfied. Indeed, for $\theta = \frac{1}{5}$ and $\vartheta = \frac{1}{4}$, we have, for the standard metric $d(\theta, \vartheta) = |\theta - \vartheta|$, the following inequality:

$$d(\Upsilon\theta, \Upsilon\vartheta) = \frac{1}{8} > \frac{5}{100} = d(\theta, \vartheta),$$

so one writes

$$\tau + F\bigl(d(\Upsilon\theta, \Upsilon\vartheta)\bigr) > F\bigl(d(\theta, \vartheta)\bigr),$$

for all $\tau > 0$ and $F \in \mathcal{F}$.

Remark 2.5 If we consider $F(t) = \ln(t)$ (for $t > 0$) in Theorem 1.6, the contraction (2.1) becomes

$$d(\Upsilon\theta,\Upsilon\vartheta) \leq e^{-\tau}\big[d(\theta,\vartheta)\big]^{\alpha}\cdot\big[d(\theta,\Upsilon\theta)\big]^{\beta}\cdot\big[d(\vartheta,\Upsilon\vartheta)\big]^{1-\alpha-\beta}, \tag{2.7}$$

for all $\theta,\vartheta \in X\setminus \mathrm{Fix}(\Upsilon)$. That is, (2.7) corresponds to the main contraction (1.2). Hence, Υ possesses a fixed point, i.e., Theorem 1.6 is a particular case of Theorem 2.2.

In what follows, we consider the multi-valued version of Theorem 2.2. Denote by $CB(X)$ the set of all nonempty closed bounded subsets of X. Define the Pompeiu–Hausdorff metric H induced by d on $CB(X)$ as follows:

$$H(\mathcal{A},\mathcal{B}) = \max\Big\{\sup_{\theta\in\mathcal{A}} d(\theta,\mathcal{B}),\sup_{\vartheta\in\mathcal{B}} d(\vartheta,\mathcal{A})\Big\},$$

for all $\mathcal{A},\mathcal{B}\in CB(X)$ where $d(\theta,\mathcal{B}) = \inf_{\vartheta\in\mathcal{B}} d(\theta,\vartheta)$. An element $\varsigma\in X$ is called a fixed point of the multi-valued mapping $\Upsilon : X \to CB(X)$ whenever $\varsigma\in\Upsilon\varsigma$.

Definition 2.6 Let (X,d) be a metric space. We say that the multi-valued mapping $\Upsilon : X\to CB(X)$ is an *extended interpolative multi-valued Ćirić–Reich–Rus type F-contraction* if there are $\alpha,\beta > 0$ with $\alpha+\beta < 1$, $\tau > 0$ and $F\in\mathcal{F}$ so that

$$\tau + F\big(H(\Upsilon\theta,\Upsilon\vartheta)\big) \leq \alpha F\big(d(\theta,\vartheta)\big) + \beta F\big(d(\theta,\Upsilon\theta)\big) + (1-\alpha-\beta)F\big(d(\vartheta,\Upsilon\vartheta)\big), \tag{2.8}$$

for all $\theta,\vartheta\in X\setminus\mathrm{Fix}(\Upsilon)$ with $H(\Upsilon\theta,\Upsilon\vartheta) > 0$.

Theorem 2.7 *Let (X,d) be a complete metric space and Υ be an extended interpolative multi-valued Ćirić–Reich–Rus type F-contraction. Assume in addition that*

$$(H):\quad F(\inf\mathcal{A}) = \inf\big(F(\mathcal{A})\big).$$

Then Υ possesses a fixed point.

Proof Choose two arbitrary points $\theta_0\in X$ and $\theta_1\in\Upsilon\theta_0$. If $\theta_0\in\Upsilon\theta_0$ or $\theta_1\in\Upsilon\theta_1$, we have nothing to prove. Let $\theta_0\notin\Upsilon\theta_0$ and $\theta_1\notin\Upsilon\theta_1$. Then $\Upsilon\theta_0\neq\Upsilon\theta_1$. Now,

$$\begin{aligned}\frac{\tau}{2} + F\big(d(\theta_1,\Upsilon\theta_1)\big) &< \tau + F\big(H(\Upsilon\theta_0,\Upsilon\theta_1)\big)\\ &\leq \alpha F\big(d(\theta_0,\theta_1)\big) + \beta F\big(d(\theta_0,\Upsilon\theta_0)\big) + (1-\alpha-\beta)F\big(d(\theta_1,\Upsilon\theta_1)\big)\\ &\leq \alpha F\big(d(\theta_0,\theta_1)\big) + \beta F\big(d(\theta_0,\theta_1)\big) + (1-\alpha-\beta)F\big(d(\theta_1,\Upsilon\theta_1)\big).\end{aligned} \tag{2.9}$$

In the case where $d(\theta_0,\theta_1) < d(\theta_1,\Upsilon\theta_1)$, we obtain from (2.9), $\frac{\tau}{2} + F(d(\theta_1,\Upsilon\theta_1)) < F(d(\theta_1,\Upsilon\theta_1))$, which is a contradiction. Now, let $d(\theta_1,\Upsilon\theta_1)\leq d(\theta_0,\theta_1)$. Substituting in (2.9), we have

$$\frac{\tau}{2} + F\big(d(\theta_1,\Upsilon\theta_1)\big) < F\big(d(\theta_0,\theta_1)\big).$$

From this inequality and using (H), we can conclude that there is $\theta_2 \in \Upsilon\theta_1$ so that

$$\frac{\tau}{2} + F\big(d(\theta_1, \theta_2)\big) < F\big(d(\theta_0, \theta_1)\big).$$

Continuing this process, we obtain a sequence $\{\theta_n\}$ in X such that $\theta_{n+1} \in \Upsilon\theta_n$, $\theta_n \notin \Upsilon\theta_n$ and

$$\frac{\tau}{2} + F\big(d(\theta_n, \theta_{n+1})\big) < F\big(d(\theta_{n-1}, \theta_n)\big), \tag{2.10}$$

for all $n \geq 1$.

If there is n_0 so that $\theta_{n_0} = \theta_{n_0+1}$, then θ_{n_0} is a fixed point of T. So, assume that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. Consequently

$$F\big(d(\theta_n, \theta_{n+1})\big) \leq F\big(d(\theta_{n-1}, \theta_n)\big) - \frac{\tau}{2} \leq \cdots \leq F\big(d(\theta_0, \theta_1)\big) - n\left(\frac{\tau}{2}\right), \tag{2.11}$$

for all $n \geq 1$. Similar to Theorem 1.6, we find that $\{\theta_n\}$ is a cauchy sequence. Suppose $\theta_n \to \theta$. suppose to the contrary $\theta \notin \Upsilon\theta$.

We consider two cases.

Case 1: There is a subsequence $\{\theta_{n_k}\}$ such that $\Upsilon\theta_{n_k} = \Upsilon\theta$ for all $k \in \mathbb{N}$. In this case,

$$d(\theta, \Upsilon\theta) = \lim d(\theta_{n_k+1}, \Upsilon\theta) = \lim H(\Upsilon\theta_{n_k}, \Upsilon\theta) = 0.$$

Case 2: There is a natural number N such that $\Upsilon\theta_n \neq \Upsilon\theta$ for all $n \geq N$. In this case, applying (2.8), for $\theta = \theta_n$ and $\vartheta = \theta$, we have

$$\begin{aligned} \tau + F\big(d(\theta_{n+1}, \Upsilon\theta)\big) &= \tau + F\big(H(\Upsilon\theta_n, \Upsilon\theta)\big) \\ &\leq \alpha F\big(d(\theta_n, \theta)\big) + \beta F\big(d(\theta_n, \theta_{n+1})\big) + (1 - \alpha - \beta) F\big(d(\theta, \Upsilon\theta)\big). \end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in the inequality (2.12), we find that $\lim_{n\to\infty} F(d(\theta_{n+1}, \Upsilon\theta)) = -\infty$ and so $\lim_{n\to\infty} d(\theta_{n+1}, \Upsilon\theta) = 0$. Therefore,

$$d(\theta, \Upsilon\theta) = \lim_{n\to\infty} d(\theta_{n+1}, \Upsilon\theta) \leq \lim_{n\to\infty} H(\Upsilon\theta_n, \Upsilon\theta) = 0.$$

Thus, $d(\theta, \Upsilon\theta) = 0$ and so $\theta \in \Upsilon\theta$. Thus, $\theta \in \Upsilon\theta$. □

Remark 2.8 Some corollaries could be derived for particular choices of F in Theorem 2.7.

3 An application to integral equations

Take $I = [0, T]$. Let $X = C(I, \mathbb{R})$ be the set of all real valued continuous functions with domain I. Consider

$$d(\theta, \vartheta) = \sup_{t\in I}\big(\big|\theta(t) - \vartheta(t)\big|\big) = \|\theta - \vartheta\|.$$

Consider the integral equation:

$$\theta(t) = q(t) + \int_0^T G(t, \omega) f\big(\omega, \theta(\omega)\big)\, d\omega, \quad t \in [0, T], \tag{3.1}$$

where

(C1) $q: I \to \mathbb{R}$ and $f: I \times \mathbb{R} \to \mathbb{R}$ are continuous;
(C2) $G: I \times I \to \mathbb{R}$ is continuous and measurable at $\omega \in I$ for all $t \in I$;
(C3) $G(t,\omega) \geq 0$ for all $t, \omega \in I$ and $\int_0^T G(t,\omega)\,d\omega \leq 1$ for all $t \in I$.

Theorem 3.1 *Assume that the conditions (C1)–(C3) hold. Suppose that there are $\tau > 0$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$ so that*

$$\begin{aligned} &\bigl|f\bigl(t,\theta(t)\bigr) - f\bigl(t,\vartheta(t)\bigr)\bigr| \\ &\quad \leq \frac{|\theta(t) - \vartheta(t)|}{[\tau\sqrt{\|\theta - \vartheta\|} + \alpha + \beta\sqrt{\frac{\|\theta-\vartheta\|}{\|\theta - \int_0^T G(t,\omega) f(\omega,\theta(\omega))\,d\omega\|}} + (1-\alpha-\beta)\sqrt{\frac{\|\theta-\vartheta\|}{\|\vartheta - \int_0^T G(t,\omega) f(\omega,\vartheta(\omega))\,d\omega\|}}]^2}, \end{aligned} \tag{3.2}$$

for each $t \in I$ and for all $\theta, \vartheta \in C(I,\mathbb{R})$ such that

$$\theta(t) \neq \int_0^T G(t,\omega)\bigl(f\bigl(\omega,\theta(\omega)\bigr)\bigr)\,d\omega,$$
$$\vartheta(t) \neq \int_0^T G(t,\omega) f\bigl(\omega,\vartheta(\omega)\bigr)\,d\omega,$$

and

$$\int_0^T G(t,\omega) f\bigl(\omega,\theta(\omega)\bigr)\,d\omega \neq \int_0^T G(t,\omega) f\bigl(\omega,\vartheta(\omega)\bigr)\,d\omega.$$

Then the integral equation (3.1) has a solution in $C(I,\mathbb{R})$.

Proof Define $\Upsilon: C(I,\mathbb{R}) \to C(I,\mathbb{R})$ as

$$\Upsilon\theta(t) = q(t) + \int_0^T G(t,\omega) f\bigl(\omega,\theta(\omega)\bigr)\,d\omega, \quad t \in [0,T].$$

We have, for every $t \in [0,T]$,

$$\begin{aligned} &\bigl|\Upsilon\theta(t) - \Upsilon\vartheta(t)\bigr| \\ &\quad = \biggl|\int_0^T G(t,\omega)\bigl(f\bigl(\omega,\theta(\omega)\bigr) - f\bigl(\omega,\vartheta(\omega)\bigr)\bigr)\,d\omega\biggr| \\ &\quad \leq \int_0^T G(t,\omega)\bigl|f\bigl(\omega,\theta(\omega)\bigr) - f\bigl(\omega,\vartheta(\omega)\bigr)\bigr|\,d\omega \\ &\quad \leq \int_0^T \frac{G(t,\omega)|\theta(t) - \vartheta(t)|}{[\tau\sqrt{\|\theta-\vartheta\|} + \alpha + \beta\sqrt{\frac{\|\theta-\vartheta\|}{\|\theta - T\theta\|}} + (1-\alpha-\beta)\sqrt{\frac{\|\theta-\vartheta\|}{\|\vartheta - T\vartheta\|}}]^2}\,d\omega \\ &\quad \leq \frac{\|\theta-\vartheta\|}{[\tau\sqrt{\|\theta-\vartheta\|} + \alpha + \beta\sqrt{\frac{\|\theta-\vartheta\|}{\|\theta - T\theta\|}} + (1-\alpha-\beta)\sqrt{\frac{\|\theta-\vartheta\|}{\|\vartheta - T\vartheta\|}}]^2}\int_0^T G(t,\omega)\,d\omega \\ &\quad \leq \frac{\|\theta-\vartheta\|}{[\tau\sqrt{\|\theta-\vartheta\|} + \alpha + \beta\sqrt{\frac{\|\theta-\vartheta\|}{\|\theta - \Upsilon\theta\|}} + (1-\alpha-\beta)\sqrt{\frac{\|\theta-\vartheta\|}{\|\vartheta - \Upsilon\vartheta\|}}]^2}. \end{aligned}$$

Take the supremum to find that

$$
\begin{aligned}
d(\Upsilon\theta,\Upsilon\vartheta) &= \|\Upsilon\theta-\Upsilon\vartheta\| \\
&\leq \frac{\|\theta-\vartheta\|}{[\tau\sqrt{\|\theta-\vartheta\|}+\alpha+\beta\sqrt{\frac{\|\theta-\vartheta\|}{\|\theta-\Upsilon\theta\|}}+(1-\alpha-\beta)\sqrt{\frac{\|\theta-\vartheta\|}{\|\vartheta-\Upsilon\vartheta\|}}]^2} \\
&= \frac{d(\theta,\vartheta)}{[\tau\sqrt{d(\theta,\vartheta)}+\alpha+\beta\sqrt{\frac{d(\theta,\vartheta)}{d(\theta,\Upsilon\theta)}}+(1-\alpha-\beta)\sqrt{\frac{d(\theta,\vartheta)}{d(\vartheta,\Upsilon\vartheta)}}]^2}.
\end{aligned}
$$

From the above inequality, we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{d(\Upsilon\theta,\Upsilon\vartheta)}} \geq{}& \tau+\alpha\left(\frac{1}{\sqrt{d(\theta,\vartheta)}}\right)+\beta\left(\frac{1}{\sqrt{d(\theta,\Upsilon\theta)}}\right) \\
&+(1-\alpha-\beta)\left(\frac{1}{\sqrt{d(\vartheta,\Upsilon\vartheta)}}\right).
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
\tau+\left(\frac{-1}{\sqrt{d(\Upsilon\theta,\Upsilon\vartheta)}}\right) \leq{}& \alpha\left(\frac{-1}{\sqrt{d(\theta,\vartheta)}}\right)+\beta\left(\frac{-1}{\sqrt{d(\theta,\Upsilon\theta)}}\right) \\
&+(1-\alpha-\beta)\left(\frac{-1}{\sqrt{d(\vartheta,\Upsilon\vartheta)}}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tau+\left(\frac{-1}{\sqrt{d(\Upsilon\theta,\Upsilon\vartheta)}}+1\right) \leq{}& \alpha\left(\frac{-1}{\sqrt{d(\theta,\vartheta)}}+1\right)+\beta\left(\frac{-1}{\sqrt{d(\theta,\Upsilon\theta)}}+1\right) \\
&+(1-\alpha-\beta)\left(\frac{-1}{\sqrt{d(\vartheta,\Upsilon\vartheta)}}+1\right).
\end{aligned}
$$

Taking $F(t)=-\frac{1}{\sqrt{t}}+1$, we get

$$
\tau+F\bigl(d(\Upsilon\theta,\Upsilon\vartheta)\bigr) \leq \alpha F\bigl(d(\theta,\vartheta)\bigr)+\beta F\bigl(d(\theta,\Upsilon\theta)\bigr)+(1-\alpha-\beta)F\bigl(d(\vartheta,\Upsilon\vartheta)\bigr),
$$

for all $\theta,\vartheta\in X\setminus \mathrm{Fix}(\Upsilon)$ with $d(\Upsilon\theta,\Upsilon\vartheta)>0$, which is (2.8). Therefore, by Theorem 2.2, Υ has a fixed point. Hence there is a solution for (3.1). □

4 Conclusion

We aimed to enrich the fixed point theory by addressing interpolative approaches.

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