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Existence of solutions for equations and inclusions of multiterm fractional q-integro-differential with nonseparated and initial boundary conditions

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Abstract

The goal of this paper is to investigate existence of solutions for the multiterm nonlinear fractional q-integro-differential ${}^cD_q^\alpha u(t)$ in two modes equations and inclusions of order $\alpha \in (n-1, n]$, with non-separated boundary and initial boundary conditions where the natural number n is more than or equal to five. We consider a Carathéodory multivalued map and use Leray–Schauder and Covitz–Nadler famous fixed point theorems for finding solutions of the inclusion problems. Besides, we present results whenever the multifunctions are convex and nonconvex. Lastly, we give some examples illustrating the primary effects.

MSC: Primary 34A08; 39A12; secondary 39A13

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1 Introduction

Fractional calculus and q-calculus are the significant branches in mathematical analysis. The field of fractional calculus has countless applications, and the subject of fractional differential equations ranges from the theoretical views of existence and uniqueness of solutions to the analytical and mathematical methods for finding solutions (for instance, see [1–4]). There has been an intensive development in fractional differential equations and inclusion (for example, see [5–11]). During the last two decades, the fractional differential equations and inclusions, both differential and q-differential, were developed intensively by many authors for a variety of subjects (for instance, consider [12–20]). In recent years, there are many published papers about differential and integro-differential equations and inclusions which are valuable tools in the modeling of many phenomena in various fields of science (for more details, see [21–32] and references therein).

In this article, motivated by [7, 21, 27], among these achievements and following results, we are working to stretch out the analytical and computational methods of checking of positive solutions for fractional q-integro-differential equation

$$\begin{aligned}{}^cD_q^{\alpha}u(t) = f\big(t, u(t), u'(t), u''(t), u'''(t), \varphi_1 u(t), \varphi_2 u(t),\\ {}^cD_q^{\beta_{11}}u(t), {}^cD_q^{\beta_{12}}u(t), \dots, {}^cD_q^{\beta_{1k_1}}u(t),\\ {}^cD_q^{\beta_{21}}u(t), {}^cD_q^{\beta_{22}}u(t), \dots, {}^cD_q^{\beta_{2k_2}}u(t),\\ {}^cD_q^{\beta_{31}}u(t), {}^cD_q^{\beta_{32}}u(t), \dots, {}^cD_q^{\beta_{3k_3}}u(t)\big),\end{aligned} \tag{1}$$

for almost all $t \in \overline{J} = [0,\delta]$, with $\delta > 0$, and the inclusion case

$$\begin{aligned}{}^cD_q^{\alpha}u(t) \in T\big(t, u(t), u'(t), u''(t), u'''(t), \varphi_1 u(t), \varphi_2 u(t),\\ {}^cD_q^{\beta_{11}}u(t), {}^cD_q^{\beta_{12}}u(t), \dots, {}^cD_q^{\beta_{1k_1}}u(t),\\ {}^cD_q^{\beta_{21}}u(t), {}^cD_q^{\beta_{22}}u(t), \dots, {}^cD_q^{\beta_{2k_2}}u(t),\\ {}^cD_q^{\beta_{31}}u(t), {}^cD_q^{\beta_{32}}u(t), \dots, {}^cD_q^{\beta_{3k_3}}u(t)\big),\end{aligned} \tag{2}$$

for each $t \in I = [0,1]$, with the initial and antiperiodic boundary conditions as follows:

$$u^{(4)}(0) = \cdots = u^{(n-1)}(0) = 0, \tag{3}$$

$$a_1 u(0) + a_2 u(\delta) = 0, \tag{4}$$

$${}^cD_q^{p_j}u(0) = -{}^cD_q^{p_j}u(\delta) \quad (j = 1,2,3), \tag{5}$$

where ${}^cD_q^{\alpha}$ denotes the Caputo fractional q-derivative, $\alpha \in (n-1, n]$, with the natural number n more than or equal to five, β_{1j_1}, β_{2j_2}, β_{3j_3}, in problems (1) and (2), belonging to $J_0 = (0,1)$, $J_1 = (1,2)$, $J_2 = (2,3)$, for $j_1 \in N_{k_1}$, $j_2 \in N_{k_2}$, $j_3 \in N_{k_3}$, respectively, with $N_k = \{1,2,\dots,k\}$, while the map φ_i, in problem (1) and (2), is defined by

$$\begin{aligned}\varphi_i u(t) = \int_0^t \mu_i(t,s)\theta_i\big(t, s, u(s), u'(s), u''(s), u'''(s),\\ {}^cD_q^{\gamma_{i1}}u(s), {}^cD_q^{\gamma_{i2}}u(s), {}^cD_q^{\gamma_{i3}}u(s)\big)\,ds,\end{aligned} \tag{6}$$

where the real-valued functions μ_i, θ_i defined on $\overline{J}^2, \overline{J}^2 \times \mathbb{R}^7$, respectively, are continuous, for $i = 1,2$, γ_{i1}, γ_{i2}, γ_{i3} belong to J_0, J_1, J_2, respectively, a continuous function f maps $\overline{J} \times \mathcal{R}^m$ to $\mathbb{R}$, a map $T : I \times \mathcal{R}^m \to P(\mathbb{R})$ is a multifunction, here $\mathcal{R}^m = \mathbb{R}^{6+k_1+k_2+k_3}$ and $P(\mathbb{R}) = \{A \subseteq \mathbb{R} \mid A \neq \emptyset\}$, in (4), $a_1 + a_2 \neq 0$, and the constants p_1, p_2, p_3 in (5) belong to J_0, J_1, J_2, respectively.

As before, we remind some of the previous works briefly. In 1910, the subject of q-difference equations was introduced by Jackson [33–35]. After that, at the beginning of the last century, studies on q-difference equations appeared in many works, especially in Carmichael [36], Mason [37], Adams [38], Trjitzinsky [39], Agarwal [40]. An excellent account in the study of fractional differential and q-differential equations can be found in [1, 3, 41–43]. In 2009, Su and Zhang investigated the problem

$$\begin{cases} {}^cD_{0^+}^{\eta}u(t) = f(t, u(t), {}^cD_{0^+}^{\nu}u(t)),\\ a_1 u(0) - a_2 u'(0) = C_1, \qquad b_1 u(1) + b_2, u'(1) = C_2,\end{cases}$$

for all $t \in (0,1)$, where $a_1b_1 + a_1b_2 + a_2b_1 > 0$, $\eta \in (1,2]$, $\nu \in (0,1]$, for $i = 1,2$, $a_i, b_i \geq 0$, a continuous function f maps $I = [0,1] \times \mathbb{R}^2$ to $\mathbb{R}$, and ${}^cD^{\eta}_{0^+}$ is the Caputo's fractional derivative [19]. In 2011, Agarwal, O'regan and Staněk investigated the problem ${}^cD^{\alpha}f(x) + T(x, f(x), f'(x), {}^cD^{\mu}f(x)) = 0$, $f(1) = f'(0) = 0$, for all $t \in [0,1]$, where $\mu \in (0,1)$, and as always ${}^cD^{\alpha}$ is the Caputo fractional derivative of order α with $\alpha \in (1,2)$, positive function T is a scalar L^{κ}-Carathéodory on $I \times E$ with $E = (0,\infty)^3$ and $\kappa(\alpha - 1) > 1$, such that $T(t, x_1, x_2, x_3)$ may be singular at 0 in one dimension of its space variables x_1, x_2, and x_3 [44]. In 2012, Ahmad et al. discussed the existence and uniqueness of solutions for the fractional q-difference equations ${}^cD^{\alpha}_q u(t) = T(t, u(t))$, $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$ and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, for $t \in I$, where $\alpha \in (1,2]$, $\alpha_i, \beta_i, \gamma_i, \eta_i \in \mathbb{R}$, for $i = 1,2$ and $T \in C([0,1] \times \mathbb{R}, \mathbb{R})$ [13].

In 2013, Baleanu, Rezapour and Mohammadi et al., by using fixed-point methods, studied the existence and uniqueness of a solution for the nonlinear fractional differential equation boundary-value problem $D^{\alpha}u(t) = f(t, u(t))$ with a Riemann–Liouville fractional derivative via the different boundary-value conditions: $u(0) = u(\delta)$, as well as the three-point boundary condition $u(0) = \beta_1 u(\eta)$ and $u(\delta) = \beta_2 u(\eta)$, where $\delta > 0$, $t \in I = [0,\delta]$, $\alpha \in (0,1)$ $\eta \in (0,\delta)$ and $0 < \beta_1 < \beta_2 < 1$ [12]. In 2016, Ahmad et al. investigated solutions of the problem

$$\begin{cases} {}^cD^{\eta}_q f(x) \in F(x, u(x), D_q u(x), D^{\nu}_q x(x)), \\ u(0) + u(1) = 0, \qquad D_q u(0) + D_q u(1) = 0, \qquad D^2_q u(0) + D^2_q u(1) = 0, \end{cases}$$

for each $x \in I$, where $\eta \in (2,3]$, $\nu \in [0,3]$, ${}^cD^{\eta}_q$ denotes Caputo fractional q-derivative, $q \in (0,1)$, and F mapping $I \times A$ to $\mathcal{P}(\mathbb{R})$ is a multivalued map, here $\mathcal{P}(\mathbb{R})$ is a power set of $\mathbb{R}$ and $A = \mathbb{R}^3$ [15]. In 2017, Baleanu, Mousalou and Rezapour presented a new method to investigate some fractional integro-differential equations involving the Caputo–Fabrizio derivative

$${}^{\mathrm{CF}}D^{\alpha}u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \exp\left(\frac{\alpha}{\alpha - 1}(t-s)\right) u'(s)\,ds,$$

where $t \in [0,1]$, $M(\alpha)$ is a normalization constant depending on α such that $M(0) = M(1) = 1$, and proved the existence of approximate solutions for these problems [10]. Also in the same year, they introduced a new operator called the infinite coefficient-symmetric Caputo–Fabrizio fractional derivative and applied it to investigate the approximate solutions for two infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential problems [11]. In addition to this, Akbari and Rezapour, by using the shifted Legendre and Chebyshev polynomials, discussed the existence of solutions for a sum-type fractional integro-differential problem under the Caputo differentiation [6]. Over the past three years, Baleanu, Rezapour and many others, by using the Caputo–Fabrizio derivative, achieved innovative and remarkable results for solutions of fractional differential equations [22, 23, 25, 28, 30, 32]. In the following year, Rezapour and Hedayati investigated the existence of solutions for the inclusion ${}^cD^{\alpha}x(t) \in F(x, f(x), {}^cD^{\beta}f(x), f'(x))$ for each $x \in I$ with the conditions ${}^cD^{\beta}f(0) - \int_0^{\eta_1} f(r)\,dr = f(0) + f'(0)$ and ${}^cD^{\beta}f(1) - \int_0^{\eta_2} f(r)\,dr = f(1) + f'(1)$, where multifunction F maps $[0,1] \times \mathbb{R}^3$ to $2^{\mathbb{R}}$ and is compact-valued, while ${}^cD^{\alpha}$ is the Caputo differential operator [16]. In 2019, Samei et al. discussed the fractional hybrid q-differential inclusions ${}^cD^{\alpha}_q(x/F(t, x, I^{\alpha_1}_q x, \ldots, I^{\alpha_n}_q x)) \in T(t, x, I^{\beta_1}_q x, \ldots, I^{\beta_k}_q x)$, with the boundary conditions $x(0) = x_0$ and $x(1) = x_1$, where $1 < \alpha \leq 2$, $q \in (0,1)$, $x_0, x_1 \in \mathbb{R}$, $\alpha_i > 0$, for

$i = 1, 2, \dots, n$, $\beta_j > 0$, for $j = 1, 2, \dots, k$, $n, k \in \mathbb{N}$, ${}^cD_q^\alpha$ denotes Caputo type q-derivative of order α, I_q^β denotes Riemann–Liouville type q-integral of order β, $F : J \times \mathbb{R}^n \to (0, \infty)$ is continuous, and T mapping $J \times \mathbb{R}^k$ to $P(\mathbb{R})$ is a multifunction [17].

2 Preliminaries

As before, we point out some of the fundamental facts on the fractional q-calculus which are needed in the next sections (for more information, consider [1–3, 33]). Then, some well-known fixed point theorems and definitions are presented.

Assume that $q \in (0, 1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ and consider the power function $(a-b)_q^{(n)} = \prod_{k=0}^{n-1}(a - bq^k)$ whenever $n \in \mathbb{N}$ and $(a-b)_q^{(n)} = 1$ where $n = 0$, and $a, b \in \mathbb{R}$ [1, 3, 33]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we define [33]

$$(a-b)_q^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \left(a - bq^k\right) / \left(a - bq^{\alpha+k}\right).$$

If $b = 0$, then it is clear that $a^{(\alpha)} = a^\alpha$ (Algorithm 1). The q-Gamma function is defined by $\Gamma_q(x) = ((1-q)^{(x-1)})/((1-q)^{x-1})$, where $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$ [3, 33, 45, 46]. The value of q-Gamma function, $\Gamma_q(x)$, for input values q and x will have a counting number of sentences n in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating q-Gamma function of order n which is shown in Algorithm 2. For any positive numbers α and β, the

Algorithm 1 The proposed method for calculating $(a-b)_q^{(\alpha)}$

```
Input: a, b, α, n, q
s ← 1
if n = 0 then
    p ← 1
else
    for k = 0 to n do
        s ← s * (a − b * a^k)/(a − b * q^(α+k))
    end for
    p ← a^α * s
end if
Output: (a − b)^(α)
```

Algorithm 2 The proposed method for calculating $\Gamma_q(x)$

```
Input: n, q ∈ (0,1), x ∈ ℝ \ {0, −1, −2, ...}
p ← 1
for k = 0 to n do
    p ← p(1 − q^(k+1))(1 − q^(x+k))
end for
Γ_q(x) ← p/(1 − q)^(x−1)
Output: Γ_q(x)
```

Algorithm 3 The proposed method for calculating $(D_qf)(x)$

Input: $q \in (0,1), f(x), x$

syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q * z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q * x))/((1-q)x)$
end if

Output: $(D_qf)(x)$

q-Beta function is defined by [41]

$$B_q(\alpha,\beta) = \int_0^1 (1-qs)_q^{(\alpha-1)} s^{\beta-1}\, d_qs. \tag{7}$$

The q-derivative of function f is defined by $(D_qf)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_qf)(0) = \lim_{x\to 0}(D_qf)(x)$, which is shown in Algorithm 3 [3, 38, 41]. Also, the higher order q-derivative of a function f is defined by $(D_q^nf)(x) = D_q(D_q^{n-1}f)(x)$ for all $n \geq 1$, where $(D_q^0f)(x) = f(x)$ [38, 41]. The q-integral of a function f defined in the interval $[0,b]$ is defined by $I_qf(x) = \int_0^x f(s)\, d_qs = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k)$, for $0 \leq x \leq b$, provided that the sum converges absolutely [38, 47]. If $a \in [0,b]$, then

$$\int_a^b f(u)\, d_qu = I_qf(b) - I_qf(a) = (1-q)\sum_{k=0}^{\infty} q^k\bigl[bf\bigl(bq^k\bigr) - af\bigl(aq^k\bigr)\bigr],$$

whenever the series exists. The operator I_q^n is given by $(I_q^0f)(x) = f(x)$ and $(I_q^nf)(x) = (I_q(I_q^{n-1}f))(x)$ for all $n \geq 1$ [3, 38, 47]. It has been proved that $(D_q(I_qf))(x) = f(x)$ and $(I_q(D_qf))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [38, 48]. The fractional Riemann–Liouville type q-integral of the function f on $[0,1]$, of $\alpha \geq 0$ is given by $(I_q^0f)(t) = f(t)$ and

$$\bigl(I_q^\alpha f\bigr)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)\, d_qs,$$

for $t \in [0,1]$ and $\alpha > 0$ [15, 18, 45]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned}\bigl({}^cD_q^\alpha f\bigr)(t) &= \bigl(I_q^{[\alpha]-\alpha}\bigl(D_q^{[\alpha]}f\bigr)\bigr)(t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)}\int_0^t (t-qs)^{([\alpha]-\alpha-1)}\bigl(D_q^{[\alpha]}f\bigr)(s)\, d_qs,\end{aligned} \tag{8}$$

for $t \in [0,1]$, $\alpha > 0$, and $[\alpha]$ denotes the smallest integer greater or equal to α [3, 15, 18, 45]. It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta}f)(x)$ and $(D_q^\alpha(I_q^\alpha f))(x) = f(x)$, where $\alpha, \beta \geq 0$ [18]. By employing Algorithm 2, we can calculate $(I_q^\alpha f)(x)$; this is shown in Algorithm 4.

Let us consider a normed space $(\mathcal{X}, \|\cdot\|)$. We denote the set of all nonempty subsets, all nonempty closed subsets, all nonempty bounded subsets, all nonempty compact subsets,

Algorithm 4 The proposed method for calculating $(I_q^\alpha f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x

```
s ← 0
for i = 0 to n do
    pf ← (1 − q^{i+1})^{α−1}
    s ← s + pf ∗ q^i ∗ f(x ∗ q^i)
end for
g ← (x^α ∗ (1 − q) ∗ s)/(Γ_q(x))
```

Output: $(I_q^\alpha f)(x)$

and all nonempty compact and convex subsets of $\mathcal{X}$, by $P(\mathcal{X})$, $P_{cl}(\mathcal{X})$, $P_b(\mathcal{X})$, $P_{cp}(\mathcal{X})$, and $P_{cp,c}(\mathcal{X})$, respectively. We say that a multivalued map $\Theta : \mathcal{X} \to P(\mathcal{X})$ is convex(closed)-valued whenever for any $x \in \mathcal{X}$, $\Theta(\mathcal{X})$ is convex (closed) [29]. If for all $\mathcal{A} \in P_b(\mathcal{X})$, we have $\Theta(\mathcal{A}) = \bigcup_{a\in\mathcal{A}} \Theta(a)$ is a bounded subset of $\mathcal{X}$, then multifunction Θ is called bounded on bounded sets, where $\sup_{a\in\mathcal{A}}\{\sup\{|b| : b \in \Theta(a)\}\}$ is finite [29]. We use the concepts of upper semicontinuous, compact, completely continuous for the multifunction $\Theta : \mathcal{X} \to P(\mathcal{X})$ as in [49, 50]. For investigating the nonlinear problem (1) and (2) under conditions (3), (4), and (5), we need the following lemma, which can be found in [51] and [52].

Lemma 1 *The general solution of the fractional q-differential equation* ${}^cD_q^\alpha u(t) = 0$ *is given by* $u(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_{n-1}t^{n-1}$, *for* $\alpha > 0$, *where* $b_i \in \mathbb{R}$ *for* $i = N_{n-1}$ *and* $n = [\alpha] + 1$.

In fact, by using Lemma 1, for the solution of the fractional q-differential equation ${}^cD_q^\alpha u(t) = 0$ we have $I_q^\alpha {}^cD_q^\alpha u(t) = u(t) + b_0 + b_1 t + b_2 t^2 + \cdots + b_{n-1}t^{n-1}$. Now, we prove the next key result.

Lemma 2 *Consider the boundary value problem with the antiperiodic conditions*

$$\begin{cases} {}^cD_q^\alpha u(t) = v(t), \\ u^{(4)}(0) = \cdots = u^{(n-1)}(0) = 0, \\ a_1 u(0) + a_2 u(\delta) = 0 \quad (a_1 + a_2 \neq 0), \\ {}^cD_q^{p_i} u(0) = -{}^cD_q^{p_i} u(\delta), \quad i = 1, 2, 3, \end{cases} \tag{9}$$

for $n - 1 < \alpha \leq n$, *with the natural number* n *being more than or equal to five*, $q \in (0,1)$, $v \in L^1(\bar{J}, \mathbb{R})$ *and each* $t \in [0,\delta]$ *with* $\delta > 0$, *where* p_1, p_2, p_3 *belong to* $(0,1)$, $(1,2)$, *and* $(2,3)$, *respectively. Then, the problem has at least one solution, namely,* $u(t) = \int_0^\delta G(t,qs)v(s)\,d_qs$, *where*

$$\begin{aligned} G(t,qs) &= \frac{1}{\Gamma_q(\alpha)}\left[(t-qs)^{(\alpha-1)} - \frac{a_2}{a_1+a_2}(\delta-qs)^{(\alpha-1)}\right] \\ &\quad + B_1(t,s,\delta) - B_2(t,s,\delta) - B_3(t,s,\delta), \\ G(t,qs) &= -\frac{a_2(\delta-qs)^{(\alpha-1)}}{(a_1+a_2)\Gamma_q(\alpha)} + B_1(t,s,\delta) - B_2(t,s,\delta) - B_3(t,s,\delta), \end{aligned}$$

whenever $s \le t$ or $t \le s$, respectively, here

$$
\begin{aligned}
B_1(t,s,\delta) &= \frac{[a_2\delta - (a_1+a_2)t]\Gamma_q(2-p_1)(\delta - qs)^{(\alpha-p_1-1)}\delta^{p_1-1}}{(a_1+a_2)\Gamma_q(\alpha-p_1)}, \\
B_2(t,s,\delta) &= \frac{\Gamma_q(3-p_2)(\delta-qs)^{(\alpha-p_2-1)}\delta^{p_2-2}}{2(a_1+a_2)(2-p_1)} \\
&\quad \times \bigl[a_2p_1\delta^2 - (a_1+a_2)\bigl(2\delta t - (2-p_1)t^2\bigr)\bigr], \\
B_3(t,s,\delta) &= \frac{\Gamma_q(4-p_3)(\delta-qs)^{(\alpha-p_3-1)}}{\Gamma_q(\alpha-p_3)} \\
&\quad \times \biggl[\frac{a_2[-6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\delta^{p_3}}{6(a_2+a_2)(2-p_1)(3-p_1)(3-p_2)} \\
&\quad + \frac{[6(p_2-p_1)\delta^2 t + (2-p_1)(3-p_1)(-3\delta t^2 + (3-p_2)t^3)]\delta^{p_3-3}}{6(2-p_1)(3-p_1)(3-p_2)}\biggr].
\end{aligned}
$$

Proof We assume that u is one of the solutions of (9). By applying Lemma 1, there exist $c_i \in \mathbb{R}$ for $i \in N_{n-1}$ such that

$$
u(t) = I_q^{\alpha} v(t) - \left(\sum_{i=0}^{n-1} c_i t^i\right) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} v(s)\, d_q s - \left(\sum_{i=0}^{n-1} c_i t^i\right).
$$

By using conditions (3) for problem (1), we obtain $b_4 = \cdots = b_{n-1} = 0$. Since ${}^cD_q^{p_1} k = 0$ for all constant k, ${}^cD_q^{p_1} t$, ${}^cD_q^{p_1} t^2$, ${}^cD_q^{p_1} t^3$ are equal to $\frac{t^{1-p_1}}{\Gamma_q(2-p_1)}$, $\frac{2t^{2-p_1}}{\Gamma_q(3-p_1)}$, $\frac{6t^{3-p_1}}{\Gamma_q(4-p_1)}$, respectively, ${}^cD_q^{p_2} t$, ${}^cD_q^{p_2} t^2$, ${}^cD_q^{p_2} t^3$ are equal to 0, $\frac{2t^{2-p_1}}{\Gamma_q(3-p_2)}$, $\frac{6t^{3-p_2}}{\Gamma_q(4-p_2)}$, respectively, ${}^cD_q^{p_3} t = {}^cD_q^{p_3} t^2 = 0$, ${}^cD_q^{p_3} t^3 = \frac{6t^{3-p_3}}{\Gamma_q(4-p_3)}$ and ${}^cD_q^{p_i} I_q^{\alpha} v(t) = I_q^{\alpha-p_i} v(t)$, for $i = 1,2,3$, we get

$$
\begin{aligned}
{}^cD_q^{p_1} u(t) &= I_q^{\alpha-p_1} v(t) - c_1 \frac{t^{1-p_1}}{\Gamma_q(2-p_1)} - c_2 \frac{2t^{2-p_1}}{\Gamma_q(3-p_1)} - c_3 \frac{6t^{3-p_1}}{\Gamma_q(4-p_1)}, \\
{}^cD_q^{p_2} u(t) &= I_q^{\alpha-p_2} v(t) - c_2 \frac{2t^{2-p_2}}{\Gamma_q(3-p_2)} - c_3 \frac{6t^{3-p_2}}{\Gamma_q(4-p_2)}, \\
{}^cD_q^{p_3} u(t) &= I_q^{\alpha-p_3} v(t) - c_3 \frac{6t^{3-p_3}}{\Gamma_q(4-p_3)}.
\end{aligned}
$$

By applying conditions (4) and (5), we obtain

$$
\begin{aligned}
c_0 &= \frac{a_2}{(a_1+a_2)} \biggl[I_q^{\alpha} v(\delta) - \Gamma_q(2-p_1)\delta^{p_1} I_q^{\alpha-p_1} v(\delta) \\
&\quad + \frac{p_1\Gamma_q(3-p_2)\delta^{p_2}}{2(2-p_1)} I_q^{\alpha-p_2} v(\delta) \\
&\quad + \frac{[-6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)\delta^{p_3}}{6(2-p_1)(3-p_1)(3-p_2)} I_q^{\alpha-p_3} v(\delta) \biggr], \\
c_1 &= \Gamma_q(2-p_1)\delta^{p_1-1} I_q^{\alpha-p_1} v(\delta)\, d_q s - \frac{\Gamma_q(3-p_2)\delta^{p_2-1}}{(2-p_1)} I_q^{\alpha-p_2} v(\delta) \\
&\quad + \frac{(p_2-p_1)\Gamma_q(4-p_3)\delta^{p_3-1}}{(2-p_1)(3-p_1)(3-p_2)} I_q^{\alpha-p_3} v(\delta),
\end{aligned}
$$

$$c_2 = \frac{\Gamma_q(3-p_2)\delta^{p_2-2}}{2} I_q^{\alpha-p_2} v(\delta) - \frac{\Gamma_q(4-p_3)\delta^{p_3-2}}{2(3-p_2)} I_q^{\alpha-p_3} v(\delta),$$
$$c_3 = \frac{\Gamma_q(4-p_3)\delta^{p_3-3}}{6} I_q^{\alpha-p_3} v(\delta).$$

Thus, substituting the values of c_i, for $i \in N_{n-1}$ in condition (9), we get the unique solution of the problem. □

3 Main results

At present, we are ready, by using the above results and basic definitions, to investigate positive solutions of problems (1) and (2) with conditions (3), (4), and (5) in the subsequent two subsections. For brevity, we denote the space of all $x \in C^3(\bar{J})$ by $\mathcal{X}$. We consider the norm

$$\|x\| = \sup_{t\in\bar{J}} |x(t)| + \sup_{t\in\bar{J}} |x'(t)| + \sup_{t\in\bar{J}} |x''(t)| + \sup_{t\in\bar{J}} |x'''(t)|$$

on $\mathcal{X}$. As we know, $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

3.1 Positive solutions for problem (1)

We first give the following theorem which can be found in [26].

Theorem 3 *The completely continuous operator Θ defined on a Banach space A has a fixed point in A whenever the set of all $a \in A$ such that $a = \lambda\Theta(a)$ is bounded, for* $0 < \lambda < 1$.

Theorem 4 *The operator $\Theta : \mathcal{X} \to \mathcal{X}$ defined by*

$$\begin{aligned}(\Theta u)(t) &= I_q^{\alpha}\widetilde{f}(t,u(t)) - \frac{a_2}{(a_1+a_2)} I_q^{\alpha}\widetilde{f}(\delta,u(\delta)) + B_1(t,\delta) I_q^{\alpha-p_1}\widetilde{f}(\delta,u(\delta)) \\ &\quad - B_2(t,\delta) I_q^{\alpha-p_2}\widetilde{f}(\delta,u(\delta)) - B_3(t,\delta) I_q^{\alpha-p_3}\widetilde{f}(\delta,u(\delta)),\end{aligned}$$

is completely continuous, where

$$\begin{aligned}\widetilde{f}(s,u(s)) = f\big(&s,u(s),u'(s),u''(s),u'''(s),\varphi_1 u(s),\varphi_2 u(s), \\ &{}^cD_q^{\beta_{11}} u(s), {}^cD_q^{\beta_{12}} u(s), \dots, {}^cD_q^{\beta_{1k_1}} u(s), \\ &{}^cD_q^{\beta_{21}} u(s), {}^cD_q^{\beta_{22}} u(s), \dots, {}^cD_q^{\beta_{2k_2}} u(s), \\ &{}^cD_q^{\beta_{31}} u(s), {}^cD_q^{\beta_{32}} u(s), \dots, {}^cD_q^{\beta_{3k_3}} u(s)\big),\end{aligned}$$

and

$$B_1(t,\delta) = \frac{[a_2\delta - (a_1+a_2)t]\Gamma_q(2-p_1)\delta^{p_1-1}}{(a_1+a_2)},$$
$$B_2(t,\delta) = \frac{[a_2 p_1 \delta^2 - (a_1+a_2)(2\delta t - (2-p_1)t^2)]\Gamma_q(3-p_2)\delta^{p_2-2}}{2(a_1+a_2)(2-p_1)},$$

$$B_3(t,\delta) = \frac{a_2[-6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)\delta^{p_3}}{6(a_2+a_2)(2-p_1)(3-p_1)(3-p_2)} + \frac{\Gamma_q(4-p_3)\delta^{p_3-3}}{6(2-p_1)(3-p_1)(3-p_2)} \times \bigl[6(p_2-p_1)\delta^2 t + (2-p_1)(3-p_1)\bigl(-3\delta t^2 + (3-p_2)t^3\bigr)\bigr].$$

Proof To begin, consider a sequence $\{u_n\}$ in $\mathcal{X}$ such that u_n tends to u_0 and $\beta_{1j} \in (0,1)$ for $j \in N_{k_1}$. By using assumptions, we get

$$\begin{aligned}\sup_{t\in\overline{J}}\bigl|{}^cD_q^{\beta_{1j}}u_n(t) - {}^cD_q^{\beta_{1j}}u_0(t)\bigr| &= \sup_{t\in\overline{J}}\bigl|I_q^{1-\beta_{1j}}u_n'(t) - I_q^{1-\beta_{1j}}u_0'(t)\bigr| \\ &= \sup_{t\in\overline{J}}\bigl|I_q^{1-\beta_{1j}}\bigl[u_n'(t)-u_0'(t)\bigr]\bigr| \\ &\le \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\sup_{t\in\overline{J}}\bigl|u_n'(t)-u_0'(t)\bigr| \\ &\le \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\|u_n-u_0\|.\end{aligned}$$

Since $\|u_n - u\| \to 0$, $\lim_{n\to\infty}{}^cD_q^{\beta_{1j}}u_n(t) = {}^cD_q^{\beta_{1j}}u_0(t)$ uniformly on $\overline{J}$. Again with the same method, we have $\lim_{n\to\infty}{}^cD_q^{\beta_{2j}}u_n(t) = {}^cD_q^{\beta_{2j}}u_0(t)$ and

$$\lim_{n\to\infty}{}^cD_q^{\beta_{3j}}u_n(t) = {}^cD_q^{\beta_{3j}}u_0(t),$$

uniformly on $\overline{J}$ for $j \in N_{k_2}$ and $j \in N_{k_3}$, respectively. Also, we obtain $\lim_{n\to\infty}{}^cD_q^{\gamma_{i1}}u_n(t) = {}^cD_q^{\gamma_{i1}}u_0(t)$, $\lim_{n\to\infty}{}^cD_q^{\gamma_{2i}}u_n(t) = {}^cD_q^{\gamma_{i2}}u_0(t)$, and

$$\lim_{n\to\infty}{}^cD_q^{\gamma_{i3}}u_n(t) = {}^cD_q^{\gamma_{i3}}u_0(t),$$

uniformly on $\overline{J}$ for $i \in N_2$. On the other hand,

$$\begin{aligned}\|\Theta u_n - \Theta u_0\| &= \sup_{t\in\overline{J}}\bigl|\Theta u_n(t) - \Theta u_0(t)\bigr| + \sup_{t\in\overline{J}}\bigl|(\Theta u_n)'(t) - (\Theta u_0)'(t)\bigr| \\ &\quad + \sup_{t\in\overline{J}}\bigl|(\Theta u_n)''(t) - (\Theta u_0)''(t)\bigr| \\ &\quad + \sup_{t\in\overline{J}}\bigl|(\Theta u_n)'''(t) - (\Theta u_0)'''(t)\bigr|.\end{aligned}$$

Thus, by employing the continuity of f, θ_1, θ_2, we conclude that $\|\Theta u_n - \Theta u\| \to 0$. Therefore, Θ is continuous on $\mathcal{X}$. At present, suppose that $\mathcal{B} \subseteq \mathcal{X}$ is bounded. So there exists $L \in (0,\infty)$ such that $|\widetilde{f}(t,u(t))| \le L$ for each t and u belonging to $\overline{J}$ and $\mathcal{B}$, respectively. Due to the assumptions, we get

$$\begin{aligned}\bigl|(\Theta x)(t)\bigr| &\le I_q^{\alpha}\bigl|\widetilde{f}\bigl(t,u(t)\bigr)\bigr| + \frac{|a_2|}{|a_1+a_2|}I_q^{\alpha}\bigl|\widetilde{f}\bigl(\delta,u(\delta)\bigr)\bigr| \\ &\quad + \bigl|B_1(t,\delta)\bigr|I_q^{\alpha-p_1}\bigl|\widetilde{f}\bigl(\delta,u(\delta)\bigr)\bigr| + \bigl|B_2(t,\delta)\bigr|I_q^{\alpha-p_2}\bigl|\widetilde{f}\bigl(\delta,u(\delta)\bigr)\bigr| \\ &\quad + \bigl|B_3(t,\delta)\bigr|I_q^{\alpha-p_3}\bigl|\widetilde{f}\bigl(\delta,u(\delta)\bigr)\bigr|\end{aligned}$$

$$\begin{aligned}
&\leq \frac{(|a_1| + 2|a_2|)L\delta^\alpha}{|a_1 + a_2|\Gamma_q(\alpha + 1)} + \frac{(|a_1| + 2|a_2|)\Gamma_q(2 - p_1)L\delta^\alpha}{|a_1 + a_2|\Gamma_q(\alpha - p_1 + 1)} \\
&\quad + \frac{(|a_2|p_1 + |a_1 + a_2|(4 - p_1))\Gamma_q(3 - p_2)L\delta^\alpha}{2|a_1 + a_2|(2 - p_1)\Gamma_q(\alpha - p_2 + 1)} \\
&\quad + \frac{|a_2|[6(p_2 - p_1) + (2 - p_1)(3 - p_1)p_2]\Gamma_q(4 - p_3)L\delta^\alpha}{6|a_1 + a_2|(2 - p_1)(3 - p_1)(3 - p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad + \frac{[6(p_2 - p_1) + (2 - p_1)(3 - p_1)(6 - p_2)]\Gamma_q(4 - p_3)L\delta^\alpha}{6(2 - p_1)(3 - p_1)(3 - p_2)\Gamma_q(\alpha - p_3 + 1)}, \\
\bigl|(\Theta u)'(t)\bigr| &\leq I_q^{\alpha-1}\bigl|\widetilde{f}\bigl(t, u(t)\bigr)\bigr| + \Gamma(2 - p_1)\delta^{p_1-1}I_q^{\alpha-p_1}\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr)\bigr| \\
&\quad + \frac{|\delta - (2 - p_1)t|\Gamma_q(3 - p_2)\delta^{p_2-2}}{(2 - p_1)} I_q^{\alpha-p_2}\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr)\bigr| \\
&\quad + \frac{\Gamma_q(4 - p_3)\delta^{p_3-3}}{2(2 - p_1)(3 - p_1)(3 - p_2)} \\
&\quad \times \bigl|2(p_2 - p_1)\delta^2 + (2 - p_1)(3 - p_1)\bigl(-2\delta t + (3 - p_2)t^2\bigr)\bigr| I_q^{\alpha-p_3}\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr)\bigr| \\
&\leq \frac{\delta^{\alpha-1}L}{\Gamma_q(\alpha)} + \frac{\Gamma_q(2 - p_1)L\delta^{\alpha-1}}{\Gamma_q(\alpha - p_q + 1)} + \frac{(3 - p_1)\Gamma_q(3 - p_2)L\delta^{\alpha-1}}{(2 - p_1)\Gamma_q(\alpha - p_2 + 1)} \\
&\quad + \frac{[2(p_2 - p_1) + (2 - p_1)(3 - p_1)(5 - p_2)]\Gamma_q(4 - p_3)L\delta^{\alpha-1}}{2(2 - p_1)(3 - p_1)(3 - p_2)\Gamma_q(\alpha - p_3 + 1)}, \\
\bigl|(\Theta u)''(t)\bigr| &\leq I_q^{\alpha-2}\bigl|\widetilde{f}\bigl(t, u(t)\bigr)\bigr| + \Gamma(3 - q)\delta^{2-p_2}I_q^{\alpha-p_2}\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr)\bigr| \\
&\quad + \frac{|-\delta + (3 - p_2)t|\Gamma_q(4 - p_3)\delta^{p_3-3}}{(3 - p_2)} I_q^{\alpha-p_3}\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr)\bigr| \\
&\leq \frac{L\delta^{\alpha-2}}{\Gamma_q(\alpha - 1)} + \frac{\Gamma_q(3 - p_2)L\delta^{\alpha-2}}{\Gamma_q(\alpha - p_2 + 1)} + \frac{(4 - p_2)\Gamma_q(4 - p_3)L\delta^{\alpha-2}}{(3 - p_2)\Gamma_q(\alpha - p_3 + 1)}, \\
\bigl|(\Theta u)'''(t)\bigr| &\leq I_q^{\alpha-3}\bigl|\widetilde{f}\bigl(t, u(t)\bigr)\bigr| + \Gamma_q(4 - p_3)\delta^{p_3-3}I_q^{\alpha-p_3}\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr)\bigr| \\
&\leq \frac{L\delta^{\alpha-3}}{\Gamma_q(\alpha - 2)} + \frac{\Gamma_q(4 - p_3)L\delta^{\alpha-3}}{\Gamma_q(\alpha - p_3 + 1)},
\end{aligned}$$

for almost all $u \in \mathcal{B}$. Hence, we have

$$\begin{aligned}
\|\Theta u\| &= \sup_{t\in\overline{J}}\bigl|(\Theta u)(t)\bigr| + \sup_{t\in\overline{J}}\bigl|(\Theta u)'(t)\bigr| + \sup_{t\in\overline{J}}\bigl|(\Theta u)''(t)\bigr| + \sup_{t\in\overline{J}}\bigl|(\Theta u)'''(t)\bigr| \\
&\leq \biggl[\frac{|a_1| + 2|a_2|}{|a_1 + a_2|\Gamma_q(\alpha + 1)} + \frac{(|a_1| + 2|a_2|)\Gamma_q(2 - p_1)}{|a_1 + a_2|\Gamma_q(\alpha - p_1 + 1)} \\
&\quad + \frac{(|a_2|p_1 + |a_1 + a_2|(4 - p_1))\Gamma_q(3 - p_2)}{2|a_1 + a_2|(2 - p_1)\Gamma_q(\alpha - p_2 + 1)} \\
&\quad + \frac{|a_2|[6(p_2 - p_1) + (2 - p_1)(3 - p_1)p_2]\Gamma_q(4 - p_3)}{6|a_1 + a_2|(2 - p_1)(3 - p_1)(3 - p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad + \frac{[6(p_2 - p_1) + (2 - p_1)(3 - p_1)(6 - p_2)]\Gamma_q(4 - p_3)}{6(2 - p_1)(3 - p_1)(3 - p_2)\Gamma_q(\alpha - p_3 + 1)}\biggr]L\delta^\alpha \\
&\quad + \biggl[\frac{1}{\Gamma_q(\alpha)} + \frac{\Gamma_q(2 - p_1)}{\Gamma_q(\alpha - p_1 + 1)} + \frac{(3 - p_1)\Gamma_q(3 - p_2)}{(2 - p_1)\Gamma_q(\alpha - p_2 + 1)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{[2(p_2-p_1)+(2-p_1)(3-p_1)(5-p_2)]\Gamma_q(4-p_3)}{2(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg]L\delta^{\alpha-1} \\
&+\left[\frac{1}{\Gamma_q(\alpha-1)}+\frac{\Gamma_q(3-p_2)}{\Gamma_q(\alpha-p_2+1)}+\frac{(4-p_2)\Gamma_q(4-p_3)}{(3-p_2)\Gamma_q(\alpha-p_3+1)}\right]L\delta^{\alpha-2} \\
&+\left[\frac{1}{\Gamma_q(\alpha-2)}+\frac{\Gamma_q(4-p_3)}{\Gamma_q(\alpha-p_3+1)}\right]L\delta^{\alpha-3} \\
&=\Lambda_1 L,
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
\Lambda_1 = {}&\Bigg[\frac{|a_1|+2|a_2|}{|a_1+a_2|\Gamma_q(\alpha+1)}+\frac{(|a_1|+2|a_2|)\Gamma_q(2-p_1)}{|a_1+a_2|\Gamma_q(\alpha-p_1+1)} \\
&+\frac{(|a_2|p_1+|a_1+a_2|(4-p_1))\Gamma_q(3-p_2)}{2|a_1+a_2|(2-p_1)\Gamma_q(\alpha-p_2+1)} \\
&+\frac{|a_2|[6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)}{6|a_1+a_2|(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)} \\
&+\frac{[6(p_2-p_1)+(2-p_1)(3-p_1)(6-p_2)]\Gamma_q(4-p_3)}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg]\delta^{\alpha} \\
&+\Bigg[\frac{1}{\Gamma_q(\alpha)}+\frac{\Gamma_q(2-p_1)}{\Gamma_q(\alpha-p_1+1)}+\frac{(3-p_1)\Gamma_q(3-p_2)}{(2-p_1)\Gamma_q(\alpha-p_2+1)} \\
&+\frac{[2(p_2-p_1)+(2-p_1)(3-p_1)(5-p_2)]\Gamma_q(4-p_3)}{2(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg]\delta^{\alpha-1} \\
&+\left[\frac{1}{\Gamma_q(\alpha-1)}+\frac{\Gamma_q(3-p_2)}{\Gamma_q(\alpha-p_2+1)}+\frac{(4-p_2)\Gamma_q(4-p_3)}{(3-p_2)\Gamma_q(\alpha-p_3+1)}\right]\delta^{\alpha-2} \\
&+\left[\frac{1}{\Gamma_q(\alpha-2)}+\frac{\Gamma_q(4-p_3)}{\Gamma_q(\alpha-p_3+1)}\right]\delta^{\alpha-3}.
\end{aligned}
\tag{11}
$$

Equation (10) implies that $\Theta(\mathcal{B})$ is a bounded set. Now, we demonstration that the sets of all Θu, $(\Theta u)'$, $(\Theta u)''$, and $(\Theta u)'''$ are equicontinuous on $\overline{J}$ for all $u \in \mathcal{B}$. Let t_1 and t_2 in $\overline{J}$. If $t_1 \leq t_2$, then we get

$$
\begin{aligned}
\big|(\Theta u)(t_2)-(\Theta u)(t_1)\big| = {}&\Bigg|I_q^{\alpha}\widetilde{f}\big(t_2,u(t_2)\big)-I_q^{\alpha}\widetilde{f}\big(t_1,u(t_1)\big) \\
&-(t_2-t_1)\Gamma_q(2-p_1)\delta^{p_1-1}I_q^{\alpha-p_1}\widetilde{f}\big(\delta,u(\delta)\big) \\
&+\frac{[2\delta(t_2-t_1)-(2-p_1)(t_2^2-t_1^2)]\Gamma_q(3-p_2)\delta^{p_2-2}}{2(2-p_1)} \\
&\times I_q^{\alpha-p_2}\widetilde{f}\big(\delta,u(\delta)\big) \\
&-\frac{6(p_2-p_1)\delta^2(t_2-t_1)\Gamma_q(4-p_3)\delta^{p_3-3}}{6(2-p_1)(3-p_1)(3-p_2)} \\
&-\frac{(2-p_1)(3-p_1)\Gamma_q(4-p_3)\delta^{p_3-3}}{6(2-p_1)(3-p_1)(3-p_2)} \\
&\times\big[-3\delta\big(t_2^2-t_1^2\big)+(3-p_2)\big(t_2^3-t_1^3\big)\big]I_q^{\alpha-p_3}\widetilde{f}\big(\delta,u(\delta)\big)\Bigg|
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{L}{\Gamma_q(\alpha)} \int_0^{t_1} \bigl[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}\bigr]\, d_q s \\
&\quad + \frac{L}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)}\, d_q s \\
&\quad + \frac{(t_2 - t_1)\Gamma_q(2-p_1)L\delta^{\alpha-1}}{\Gamma_q(\alpha - p_1 + 1)} \\
&\quad + \frac{[2\delta(t_2 - t_1) + (2-p_1)(t_2^2 - t_1^2)]\Gamma_q(3-p_2)L\delta^{\alpha-2}}{2(2-p_1)\Gamma_q(\alpha - p_2 + 1)} \\
&\quad + \frac{(p_2 - p_1)\delta^2(t_2 - t_1)\Gamma_q(4-p_3)L\delta^{\alpha-3}}{(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad + \frac{(2-p_1)(3-p_1)\Gamma_q(4-p_3)L\delta^{\alpha-3}}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad \times \bigl[3\delta\bigl(t_2^2 - t_1^2\bigr) + (3-p_2)\bigl(t_2^3 - t_1^3\bigr)\bigr] \\
&\le \frac{L}{\Gamma_q(\alpha+1)}\bigl(t_2^\alpha - t_1^\alpha\bigr) + \frac{(t_2 - t_1)\Gamma_q(2-p_1)L\delta^{\alpha-1}}{\Gamma_q(\alpha - p_1 + 1)} \\
&\quad + \frac{[2\delta(t_2 - t_1) + (2-p_1)(t_2^2 - t_1^2)]\Gamma_q(3-p_2)L\delta^{\alpha-2}}{2(2-p_1)\Gamma_q(\alpha - p_2 + 1)} \\
&\quad + \frac{(p_2 - p_1)\delta^2(t_2 - t_1)\Gamma_q(4-p_3)L\delta^{\alpha-3}}{(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad + \frac{(2-p_1)(3-p_1)\Gamma_q(4-p_3)L\delta^{\alpha-3}}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad \times \bigl[3\delta\bigl(t_2^2 - t_1^2\bigr) + (3-p_2)\bigl(t_2^3 - t_1^3\bigr)\bigr]. \qquad (12)
\end{aligned}$$

Again, by using a similar technique, we have

$$\begin{aligned}
\bigl|(\Theta u)'(t_2) - (\Theta u)'(t_1)\bigr| &\le \frac{L}{\Gamma_q(\alpha)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr) \\
&\quad + \frac{(t_2 - t_1)\Gamma_q(3-p_2)L\delta^{\alpha-2}}{\Gamma_q(\alpha - p_2 + 1)} \\
&\quad + \frac{\Gamma_q(4-p_3)L\delta^{\alpha-3}}{2(3-p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&\quad \times \bigl[2\delta(t_2 - t_1) + (3-p_2)\bigl(t_2^2 - t_1^2\bigr)\bigr], \qquad (13) \\
\bigl|(\Theta u)''(t_2) - (\Theta u)''(t_1)\bigr| &\le \frac{L}{\Gamma_q(\alpha-1)}\bigl(t_2^{\alpha-2} - t_1^{\alpha-2}\bigr) \\
&\quad + \frac{(t_2 - t_1)\Gamma_q(4-p_3)L\delta^{\alpha-3}}{\Gamma_q(\alpha - p_3 + 1)}, \qquad (14) \\
\bigl|(\Theta u)'''(t_2) - (\Theta u)'''(t_1)\bigr| &\le \frac{L}{\Gamma_q(\alpha-2)}\bigl(t_2^{\alpha-3} - t_1^{\alpha-3}\bigr). \qquad (15)
\end{aligned}$$

If $t_2 \to t_1$, then right-hand sides of all inequalities (12)–(15) tend to zero, and so Θ is completely continuous. This completes the proof. □

Theorem 5 *Problem* (1) *under conditions* (3), (4), *and* (5), *has at least one solution whenever function* f *mapping* $\overline{J} \times \mathcal{R}^m$ *into* $\mathbb{R}$ *is continuous and the following assumptions hold for each* $t, s \in \overline{J}$, ${}_ix_j \in \mathbb{R}$:

(1) *There exists positive constants* $d_0 > 0$ *and* ${}_0d_{j_0}, {}_1d_{j_1}, {}_2d_{j_2}, {}_3d_{j_3} \in [0, \infty)$ *such that*

$$\begin{aligned} &\bigl|f(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_0x_6, {}_1x_1, {}_1x_2, \dots, \\ &\quad {}_1x_{k_1}, {}_2x_1, {}_2x_2, \dots, {}_2x_{k_2}, {}_3x_1, {}_3x_2, \dots, {}_3x_{k_3})\bigr| \\ &\quad \le d_0 + \sum_{j=1}^{6} {}_0d_j|{}_0x_j| + \sum_{j=1}^{k_1} {}_1d_j|{}_1x_j| + \sum_{j=1}^{k_2} {}_2d_j|{}_2x_j| + \sum_{j=1}^{k_3} {}_3d_j|{}_3x_j|, \end{aligned}$$

for j_0, j_1, j_2, j_3 *belonging to* N_6, N_{k_1}, N_{k_2}, *and* N_{k_3}, *respectively.*

(2) *There exist constants* ${}_0c_1, {}_0c_2$ *in* $(0, \infty)$ *and* ${}_i\eta_j \in [0, \infty)$ *such that*

$$\bigl|\theta_i(t, s, x_1, x_2, x_3, x_4, x_5, x_6, x_7)\bigr| \le {}_0c_i + \sum_{j=1}^{7} {}_i\eta_j|x_j|,$$

for each $x_i \in \mathbb{R}$, *where* $i = 1, 2$ *and* $j \in N_7$. *In addition,*

$$\begin{aligned} \Lambda_1' = {}& \Lambda_1\Biggl[\Biggl(\sum_{j=1}^{4} {}_0d_j\Biggr) \\ &+ {}_0d_5\gamma_1^0\Biggl(\Biggl(\sum_{j=1}^{4} {}_1c_j\Biggr) + {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Biggr) \\ &+ {}_0d_6\gamma_2^0\Biggl(\Biggl(\sum_{j=1}^{4} {}_2c_j\Biggr) + {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Biggr) \\ &+ \sum_{j=1}^{k_1} {}_1d_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2d_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} + \sum_{j=1}^{k_3} {}_3d_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Biggr] \\ < {}& 1, \end{aligned}$$

where $\gamma_i^0 = \sup_{t\in\overline{J}} \int_0^t |\mu_i(t,s)|\,ds$ *and* Λ_1 *is defined in Eq.* (11) *for* $i = 1, 2$.

Proof Thus, Theorem 4 implies that operator Θ of $\mathcal{X}$ to itself is completely continuous. Now, we prove that $\mathcal{B} \subset \mathcal{X}$ which contains all $u \in \mathcal{X}$ such that $u = \lambda\Theta(u)$ where $\lambda \in (0, 1)$ is bounded. Let $u \in \mathcal{B}$ and $t \in \overline{J}$. Then, we obtain

$$\begin{aligned} u(t) = {}& \lambda I_q^{\alpha}\widetilde{f}\bigl(t, u(t)\bigr) - \frac{a_2}{(a_1 + a_2)}\lambda I_q^{\alpha}\widetilde{f}\bigl(\delta, u(\delta)\bigr) + B_1(t,\delta)\lambda I_q^{\alpha-p_1}\widetilde{f}\bigl(\delta, u(\delta)\bigr) \\ &- B_2(t,\delta)\lambda I_q^{\alpha-p_2}\widetilde{f}\bigl(\delta, u(\delta)\bigr) - B_3(t,\delta)\lambda I_q^{\alpha-p_3}\widetilde{f}\bigl(\delta, u(\delta)\bigr), \\ u'(t) = {}& \lambda I_q^{\alpha-1}\widetilde{f}\bigl(t, u(t)\bigr) - \Gamma_q(2-p_1)\delta^{1-p_1}\lambda I_q^{\alpha-p_1}\widetilde{f}\bigl(\delta, u(\delta)\bigr) \\ &+ \frac{[\delta - (2-p_1)t]\Gamma_q(3-p_2)\delta^{p_2-2}}{(2-p_1)}\lambda I_q^{\alpha-p_2}\widetilde{f}\bigl(\delta, u(\delta)\bigr) \\ &- \frac{\Gamma_q(4-p_3)\delta^{p_3-3}}{2(2-p_1)(3-p_1)(3-p_2)} \end{aligned}$$

$$\begin{aligned}
&\times \big[2(p_2-p_1)\delta^2+(2-p_1)(3-p_1)\big(-2\delta t+(3-p_2)t^2\big)\big]\\
&\times \lambda I_q^{\alpha-p_3}\widetilde{f}\big(\delta,u(\delta)\big),\\
u''(t)={}&\lambda I_q^{\alpha-2}\widetilde{f}\big(t,u(t)\big)-\Gamma_q(3-p_2)\delta^{2-p_2}\lambda I_q^{\alpha-p_2}\widetilde{f}\big(\delta,u(\delta)\big)\\
&-\frac{[-\delta+(3-p_2)t]\Gamma_q(4-p_3)\delta^{p_3-3}}{(3-p_2)}\lambda I_q^{\alpha-p_3}\widetilde{f}\big(\delta,u(\delta)\big),\\
u'''(t)={}&\lambda I_q^{\alpha-3}\widetilde{f}\big(t,u(t)\big)-\Gamma_q(4-p_3)\delta^{p_3-3}\lambda I_q^{\alpha-p_3}\widetilde{f}\big(\delta,u(\delta)\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\big|u(t)\big|={}&\lambda\big|\Theta u(t)\big|\\
\le{}&\Bigg[d_0+\Bigg(\sum_{j=1}^{4}{}_0d_j\|u\|\Bigg)+{}_0d_5\gamma_1^0\Bigg({}_0c_1+\Bigg(\sum_{j=1}^{4}{}_1c_j\|u\|\Bigg)\\
&+{}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}\|u\|+{}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}\|u\|\\
&+{}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\|u\|\Bigg)+{}_0d_6\gamma_2^0\Bigg({}_0c_2+\Bigg(\sum_{j=1}^{4}{}_2c_j\|x\|\Bigg)\\
&+{}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\|u\|+{}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}\|u\|\\
&+{}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\|u\|\Bigg)+\sum_{j=1}^{k_1}{}_1d_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\|u\|\\
&+\sum_{j=1}^{k_2}{}_2d_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}\|u\|+\sum_{j=1}^{k_3}{}_3d_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\|u\|\Bigg]\\
&\times\Bigg[\frac{(|a_1|+2|a_2|)\delta^\alpha}{|a_1+a_2|\Gamma_q(\alpha+1)}+\frac{(|a_1|+2|a_2|)\Gamma_q(2-p_1)\delta^\alpha}{|a_1+a_2|\Gamma_q(\alpha-p_1+1)}\\
&+\frac{(|a_2|p_1+|a_1+a_2|(4-p_1))\Gamma_q(3-p_2)\delta^\alpha}{2|a_1+a_2|(2-p_1)\Gamma_q(\alpha-p_2+1)}\\
&+\frac{|a_2|[6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)\delta^\alpha}{6|a_1+a_2|(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\\
&+\frac{[6(p_2-p_1)+(2-p_1)(3-p_1)(6-p_2)]\Gamma_q(4-p_3)\delta^\alpha}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg],\\
\big|u'(t)\big|={}&\lambda\big|(\Theta u)'(t)\big|\\
\le{}&\Bigg[d_0+\Bigg(\sum_{j=1}^{4}{}_0d_j\|u\|\Bigg)+{}_0d_5\gamma_1^0\Bigg({}_0c_1+\Bigg(\sum_{j=1}^{4}{}_1c_j\|u\|\Bigg)\\
&+{}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}\|u\|+{}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}\|u\|\\
&+{}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\|u\|\Bigg)+{}_0d_6\gamma_2^0\Bigg({}_0c2+\Bigg(\sum_{j=1}^{4}{}_2c_j\|u\|\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\|u\| + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}\|u\| \\
&+ {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\|u\|\Bigg) + \sum_{j=1}^{k_1} {}_1d_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\|u\| \\
&+ \sum_{j=1}^{k_2} {}_2d_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}\|u\| + \sum_{j=1}^{k_3} \beta_{2j} \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\|u\|\Bigg] \\
&\times \Bigg[\frac{\delta^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{\Gamma_q(2-p_1)\delta^{\alpha-1}}{\Gamma_q(\alpha-p_1+1)} + \frac{(3-p_1)\Gamma_q(3-p_2)\delta^{\alpha-1}}{(2-p_1)\Gamma_q(\alpha-p_2+1)} \\
&+ \frac{[2(p_2-p_1)+(2-p_1)(3-p_1)(5-p_2)]\Gamma_q(4-p_3)\delta^{\alpha-1}}{2(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\big|u''(t)\big| &= \lambda\big|(\Theta u)''(t)\big| \\
&\leq \Bigg[d_0 + \Bigg(\sum_{j=1}^{4} {}_0d_j\|u\|\Bigg) + {}_0d_5\gamma_1^0\Bigg({}_0c_1 + \Bigg(\sum_{j=1}^{4} {}_1c_j\|u\|\Bigg) \\
&+ {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}\|u\| + {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}\|u\| \\
&+ {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\|u\|\Bigg) + {}_0d_6\gamma_2^0\Bigg({}_0c_2 + \Bigg(\sum_{j=1}^{4} {}_2c_j\|u\|\Bigg) \\
&+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\|u\| + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}\|u\| \\
&+ {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\|u\|\Bigg) + \sum_{j=1}^{k_1} {}_1d_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\|u\| \\
&+ \sum_{j=1}^{k_2} {}_2d_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}\|u\| + \sum_{k=1}^{k_3} {}_3d_j \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\|u\|\Bigg] \\
&\times \Bigg[\frac{\delta^{\alpha-2}}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-p_2)\delta^{\alpha-2}}{\Gamma_q(\alpha-p_2+1)} + \frac{(4-p_2)\Gamma_q(4-p_3)\delta^{\alpha-2}}{(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\big|u'''(t)\big| &= \lambda\big|(\Theta u)'''(t)\big| \\
&\leq \Bigg[d_0 + \Bigg(\sum_{j+1}^{4} {}_0d_j\|u\|\Bigg) + {}_0d_5\gamma_1^0\Bigg({}_0c_1 + \Bigg(\sum_{j=1}^{4} {}_1c_j\|x\|\Bigg) \\
&+ {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_1 1)}\|u\| + {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}\|u\| \\
&+ {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\|u\|\Bigg) + {}_0d_6\gamma_2^0\Bigg({}_0c_2 + \Bigg(\sum_{j=1}^{4} {}_2c_j\|u\|\Bigg) \\
&+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\|u\| + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}\|u\| \\
&+ {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\|u\|\Bigg) + \sum_{j=1}^{k_1} {}_1d_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\|u\|
\end{aligned}
$$

$$+\sum_{j=1}^{k_2} {}_2d_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}\|u\| \sum_{j=1}^{k_3} {}_3d_j \frac{\delta^{3-\beta_{2j}}}{\Gamma(4-\beta_{3j})}\|u\|\Bigg]$$
$$\times\left[\frac{\delta^{\alpha-3}}{\Gamma_q(\alpha-2)}+\frac{\Gamma_q(4-p_3)\delta^{\alpha-3}}{\Gamma_q(\alpha-p_3+1)}\right].$$

Thus, we conclude that

$$\begin{aligned}\|u\| \le{}& \Lambda_1\Bigg[\left(\sum_{j=1}^{4} {}_0d_j\right) + {}_0d_5\gamma_1^0\left(\left(\sum_{j=1}^{4} {}_1c_j\right)\right.\\
&+ {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg)\\
&+ {}_0d_6\gamma_2^0\left(\left(\sum_{j=1}^{4} {}_2c_j\right) + {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}\right.\\
&+ {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) + \sum_{j=1}^{k_1} {}_1d_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\\
&+ \sum_{j=1}^{k_2} {}_2d_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} + \sum_{j=1}^{k_3} {}_3d_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg]\|u\|\\
&+ \Lambda_1\big(d_0 + {}_0d_5\gamma_1^0{}_0c_1 + {}_0d_6\gamma_1^0{}_0c_2\big).\end{aligned}$$

Hence, $(1-\Lambda_1')\|u\| \le \Lambda_1(d_0 + {}_0d_5\gamma_1^0{}_0c_1 + {}_0d_6\gamma_2^0{}_0c_2)$. Therefore, the set $\mathcal{B}$ is bounded. At present, by employing Theorem 3, the operator Θ has at least one fixed point. By a simple review, we conclude that each fixed point of the operator Θ is a solution for problem (1). □

Theorem 6 *Assume that the real-valued functions f and θ_i, defined on $\overline{J}\times\mathcal{R}^m$ and $\overline{J}^2\times\mathbb{R}^7$, respectively, are continuous. Then problem (1) under conditions (3), (4), and (5) has a unique solution whenever the following assumptions hold for each $t,s\in\overline{J}$, ${}_ix_j\in\mathbb{R}$:*

(1) *There exists constants ${}_0\eta_j > 0$ and ${}_1\eta_{j_0}, {}_2\eta_{j_0}, {}_3\eta_{j_0} \ge 0$ such that*

$$\begin{aligned}&\big|f(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_0x_6,\\
&\quad {}_1x_1, {}_1x_2, \ldots, {}_1x_{k_1}, {}_2x_1, {}_2x_2, \ldots, {}_2x_{k_2}, {}_3x_1, {}_3x_2, \ldots, {}_3x_{k_3})\\
&\quad -f\big(t, {}_0x_1', {}_0x_2', {}_0x_3', {}_0x_4', {}_0x_5', {}_0x_6',\\
&\quad {}_1x_1', {}_1x_2', \ldots, {}_1x_{k_1}', {}_2x_1', {}_2x_2', \ldots, {}_2x_{k_2}', {}_3x_1', {}_3x_2', \ldots, {}_3x_{k_3}'\big)\big|\\
&\le \sum_{j=1}^{6} {}_0\eta_j\big|{}_0x_j - {}_0x_j'\big| + \sum_{j=1}^{k_1} {}_1\eta_j\big|{}_1x_j - {}_1x_j'\big|\\
&\quad + \sum_{j=1}^{k_2} {}_2\eta_j\big|{}_2x_j - {}_2x_j'\big| + \sum_{j=1}^{k_3} {}_3\eta_j\big|{}_3x_j - {}_3x_j'\big|,\end{aligned}$$

for j_0, j_1, j_2, j_3 belonging to N_6, N_{k_1}, N_{k_2}, and N_{k_3}, respectively.

(2) *There exist constants* ${}_0c_i$ *in* $(0,\infty)$ *and* ${}_i\eta_j \in [0,\infty)$ *such that*

$$\begin{aligned}&\left|\theta_i(t,s,x_1,x_2,x_3,x_4,x_5,x_6,x_7)-\theta_i\left(t,s,x'_1,x'_2,x'_3,x'_4,x'_5,x'_6,x'_7\right)\right|\\&\quad\le\sum_{j=1}^{7}{}_ic_j\left|x_j-x'_j\right|,\end{aligned}$$

for each $i=1,2$, $x_j, x'j \in \mathbb{R}$, *where* $j \in N_7$ *and*

$$\begin{aligned}\Lambda'_1={}&\Lambda_1\Bigg[\left(\sum_{j=1}^{4}{}_0\eta_j\right)\\&+{}_0\eta_5\gamma_1^0\left(\left(\sum_{j=1}^{4}{}_1c_j\right)+{}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}+{}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}+{}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\right)\\&+{}_0\eta_6\gamma_2^0\left(\left(\sum_{j=1}^{4}{}_2c_j\right)+{}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}+{}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}+{}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\right)\\&+\sum_{j=1}^{k_1}{}_1\eta_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}+\sum_{j=1}^{k_2}{}_2\eta_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}+\sum_{j=1}^{k_3}{}_3\eta_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg]\\&<1.\end{aligned}$$

Proof We choose a positive constant r such that

$$r\left(1-\Lambda'_1\right)\ge\left(\eta_0+{}_0\eta_5\gamma_1^0\vartheta_1+{}_0\eta_6\gamma_2^0\vartheta_2\right)\Lambda_1,$$

where $\eta_0=\sup_{t\in\bar{J}}|f(t,0,0,\dots,0)|$, $\vartheta_j=\sup_{t,s\in\bar{J}}|\theta_j(t,s,0,0,\dots,0)|$ are finite for $j=1,2$. We claim that $\Theta(\mathcal{B}_r)\subseteq\mathcal{B}_r$, where $\mathcal{B}_r$ is the set of all $u\in X$ such that $\|u\|\le r$. In this case, considering $u\in\mathcal{B}_r$, we get

$$\begin{aligned}\left|(\Theta u)(t)\right|\le{}&I_q^\alpha\left|\widetilde{f}\left(t,u(t)\right)\right|+\frac{|a_2|}{|a_1+a_2|}I_q^\alpha\left|\widetilde{f}\left(\delta,u(\delta)\right)\right|\\&+\left|B_1(t,\delta)\right|I_q^{\alpha-p_1}\left|\widetilde{F}\left(\delta,u(\delta)\right)\right|+\left|B_2(t,\delta)\right|I_q^{\alpha-p_2}\left|\widetilde{f}\left(\delta,u(\delta)\right)\right|\\&+\left|B_3(t,\delta)\right|I_q^{\alpha-p_3}\left|\widetilde{f}\left(\delta,u(\delta)\right)\right|\\\le{}&I_q^\alpha\left[\left|\widetilde{f}\left(t,u(t)\right)-f(t,0,0,\dots,0)\right|+\left|f(t,0,0,\dots,0)\right|\right]\\&+\frac{|a_2|}{|a_1+a_2|}I_q^\alpha\left[\left|\widetilde{f}\left(\delta,u(\delta)\right)-f(\delta,0,0,\dots,0)\right|+\left|f(\delta,0,0,\dots,0)\right|\right]\\\le{}&I_q^\alpha\left[\left|\widetilde{f}\left(t,u(t)\right)-f(t,0,0,\dots,0)\right|+\left|f(t,0,0,\dots,0)\right|\right]\\&+\frac{|a_2|}{|a_1+a_2|}I_q^\alpha\left[\left|\widetilde{f}\left(\delta,u(\delta)\right)-f(\delta,0,0,\dots,0)\right|+\left|f(\delta,0,0,\dots,0)\right|\right]\\&+\left|B_2(t,\delta)\right|I_q^{\alpha-p_1}\\&\times\left[\left|\widetilde{f}\left(\delta,u(\delta)\right)-f(\delta,0,0,\dots,0)\right|+\left|f(\delta,0,0,\dots,0)\right|\right]\\&+\left|B_3(t,\delta)\right|I_q^{\alpha-p_2}\end{aligned}$$

$$
\begin{aligned}
&\times \bigl[\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr) - f(\delta, 0, 0, \ldots, 0)\bigr| + \bigl|f(\delta, 0, 0, \ldots, 0)\bigr|\bigr] \\
&+ \bigl|B_3(t, \delta)\bigr| I_q^{\alpha - p_3} \\
&\times \bigl[\bigl|\widetilde{f}\bigl(\delta, u(\delta)\bigr) - f(\delta, 0, 0, \ldots, 0)\bigr| + \bigl|f(\delta, 0, 0, \ldots, 0)\bigr|\bigr] \\
\le{}& \Biggl[\Biggl(\Biggl(\sum_{j=1}^{4} {}_0\eta_j\Biggr) + {}_0\eta_5 \gamma_1^0 \Biggl(\Biggl(\sum_{j=1}^{4} {}_1c_j\Biggr) + {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\
&+ {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Biggr) + {}_0\eta_6 \gamma_2^0 \Biggl(\Biggl(\sum_{j=1}^{4} {}_2c_j\Biggr) \\
&+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Biggr) \\
&+ \sum_{j=1}^{k_1} {}_1\eta_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\
&+ \sum_{j=1}^{k_3} {}_3\eta_j \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Biggr) r + \eta_0 + {}_0\eta_5 \gamma_1^0 \vartheta_1 + {}_0\eta_6 \gamma_2^0 \vartheta_2\Biggr] \\
&\times \Biggl[\frac{(|a_1| + 2|a_2|)\delta^\alpha}{|a_1 + a_2| \Gamma_q(\alpha+1)} + \frac{(|a_1| + 2|a_2|)\Gamma_q(2-p_1)\delta^\alpha}{|a_1 + a_2| \Gamma_q(\alpha - p_1 + 1)} \\
&+ \frac{(|a_2| p_1 + |a_1 + a_2|(4-p_1))\Gamma_q(3-p_2)\delta^\alpha}{2|a_1 + a_2|(2-p_1)\Gamma_q(\alpha - p_2 + 1)} \\
&+ \frac{|a_2|[6(p_2 - p_1) + (2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)\delta^\alpha}{6|a_1 + a_2|(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha - p_3 + 1)} \\
&+ \frac{[6(p_2 - p_1) + (2-p_1)(3-p_1)(6-p_2)]\Gamma_q(4-p_3)\delta^\alpha}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha - p_3 + 1)}\Biggr].
\end{aligned}
$$

In a similar manner, we conclude that

$$
\begin{aligned}
\bigl|(\Theta u)'(t)\bigr| \le{}& \Biggl[\Biggl(\Biggl(\sum_{j=1}^{4} {}_0\eta_j\Biggr) + {}_0\eta_5 \gamma_1^0 \Biggl(\Biggl(\sum_{j=1}^{4} {}_1c_j\Biggr) + {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\
&+ {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Biggr) + {}_0\eta_6 \gamma_2^0 \Biggl(\Biggl(\sum_{j=1}^{4} {}_2c_j\Biggr) \\
&+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Biggr) \\
&+ \sum_{j=1}^{k_1} {}_1\eta_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\
&+ \sum_{j=1}^{k_3} {}_3\eta_j \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Biggr) r + \eta_0 + {}_0\eta_5 \gamma_1^0 \vartheta_1 + {}_0\eta_6 \gamma_2^0 \vartheta_2\Biggr] \\
&\times \Biggl[\frac{\delta^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{\Gamma_q(2-p_1)\delta^{\alpha-1}}{\Gamma_q(\alpha - p_1 + 1)} + \frac{(3-p_1)\Gamma_q(3-p_2)\delta^{\alpha-1}}{(2-p_1)\Gamma_q(\alpha - p_2 + 1)}
\end{aligned}
$$

$$+ \frac{[2(p_2 - p_1) + (2 - p_1)(3 - p_1)(5 - p_2)]\Gamma_q(4 - p_3)\delta^{\alpha-1}}{2(2 - p_1)(3 - p_1)(3 - p_2)\Gamma_q(\alpha - p_3 + 1)}\Bigg],$$

$$\begin{aligned}\big|(\Theta u)''(t)\big| \leq {}& \Bigg[\Bigg(\Bigg(\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) + {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\ & + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg) \\ & + {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) \\ & + \sum_{j=1}^{k_1} {}_1\eta_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\ & + \sum_{j=1}^{k_3} {}_3\eta_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg)r + \eta_0 + {}_0\eta_5\gamma_1^0\vartheta_1 + {}_0\eta_6\gamma_2^0\vartheta_2\Bigg] \\ & \times\Bigg[\frac{\delta^{\alpha-2}}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-p_2)\delta^{\alpha-2}}{\Gamma_q(\alpha-p_2+1)} + \frac{(4-p_2)\Gamma_q(4-p_3)\delta^{\alpha-2}}{(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg],\end{aligned}$$

$$\begin{aligned}\big|(\Theta u)'''(t)\big| \leq {}& \Bigg[\Bigg(\Bigg(\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) + {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\ & + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg) \\ & + {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) \\ & + \sum_{j=1}^{k_1} {}_1\eta_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\ & + \sum_{j=1}^{k_3} {}_3\eta_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg)r + \eta_0 + {}_0\eta_5\gamma_1^0\vartheta_1 + {}_0\eta_6\gamma_2^0\vartheta_2\Bigg] \\ & \times\Bigg[\frac{\delta^{\alpha-2}}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-p_2)\delta^{\alpha-2}}{\Gamma_q(\alpha-p_2+1)} + \frac{(4-p_2)\Gamma_q(4-p_3)\delta^{\alpha-2}}{(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg].\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Theta u\| \leq {}& \Bigg[\Bigg(\Bigg(\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) + {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\ & + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg) \\ & + {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k_1} {}_1\eta_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\
&+\sum_{j=1}^{k_3} {}_3\eta_j \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg) r + \eta_0 + {}_0\eta_5\gamma_1^0\vartheta_1 + {}_0\eta_6\gamma_2^0\vartheta_2\Bigg]\Lambda_1 \\
&\le r.
\end{aligned}
$$

Indeed, $\|\Theta u\| \le r$. On the other hand, we obtain

$$
\begin{aligned}
\big|(\Theta u)(t)-(\Theta v)(t)\big| &\le I_q^\alpha\big[\big|\widetilde{f}\big(t,u(t)\big)-\widetilde{f}\big(t,v(t)\big)\big|\big] \\
&\quad+\frac{|a_2|}{|a_1+a_2|} I_q^\alpha\big[\big|\widetilde{f}\big(\delta,u(\delta)\big)-\widetilde{f}\big(\delta,v(\delta)\big)\big|\big] \\
&\quad+\big|B_1(t,\delta)\big| I_q^{\alpha-p_1}\big[\big|\widetilde{f}\big(\delta,u(\delta)\big)-\widetilde{f}\big(\delta,v(\delta)\big)\big|\big] \\
&\quad+\big|B_2(t,\delta)\big| I_q^{\alpha-p_2}\big[\big|\widetilde{f}\big(\delta,u(\delta)\big)-\widetilde{f}\big(\delta,v(\delta)\big)\big|\big] \\
&\quad+\big|B_3(t,\delta)\big| I_q^{\alpha-p_3}\big[\big|\widetilde{f}\big(\delta,u(\delta)\big)-\widetilde{f}\big(\delta,v(\delta)\big)\big|\big] \\
&\le\Bigg[\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) + {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\
&\quad+{}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg) \\
&\quad+{}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) \\
&\quad+\sum_{j=1}^{k_1} {}_1\eta_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} + \sum_{j=1}^{k_3} {}_3\eta_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg] \\
&\quad\times\Bigg[\frac{(|a_1|+2|a_2|)\delta^\alpha}{|a_1+a_2|\Gamma_q(\alpha+1)} + \frac{(|a_1|+2|a_2|)\Gamma_q(2-p_1)\delta^\alpha}{|a_1+a_2|\Gamma_q(\alpha-p_1+1)} \\
&\quad+\frac{(|a_2|p_1+|a_1+a_2|(4-p_1))\Gamma_q(3-p_2)\delta^\alpha}{2|a_1+a_2|(2-p_1)\Gamma_q(\alpha-p_2+1)} \\
&\quad+\frac{|a_2|[6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)\delta^\alpha}{6|a_1+a_2|(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)} \\
&\quad+\frac{[6(p_2-p_1)+(2-p_1)(3-p_1)(6-p_2)]\Gamma_q(4-p_3)\delta^\alpha}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg] \\
&\quad\times\|u-v\|,
\end{aligned}
$$

for each $t \in \overline{J}$ and each $u, v \in \mathcal{X}$. By considering similar arguments, we obtain

$$
\begin{aligned}
\big|(\Theta u)'(t)-(\Theta v)'(t)\big| &\le\Bigg[\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) + {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} \\
&\quad+{}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg) + {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+ {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) \\
&+ \sum_{j=1}^{k_1} {}_1\eta_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\
&+ \sum_{j=1}^{k_3} {}_3\eta_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg]\Bigg[\frac{\delta^{\alpha-1}}{\Gamma_q(\alpha)} \\
&+ \frac{\Gamma_q(2-p_1)\delta^{\alpha-1}}{\Gamma_q(\alpha-p_1+1)} + \frac{(3-p_1)\Gamma_q(3-p_2)\delta^{\alpha-1}}{(2-p_1)\Gamma_q(\alpha-p_2+1)} \\
&+ \frac{\Gamma_q(4-p_3)\delta^{\alpha-1}}{2(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3+1)} \\
&\times \big[2(p_2-p_1)+(2-p_1)(3-p_1)(5-p_2)\big]\Bigg]\|u-v\|,
\end{aligned}
$$

$$
\begin{aligned}
\big|(\Theta u)''(t)-(\Theta v)''(t)\big| \leq{}& \Bigg[\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) \\
&+ {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} \\
&+ {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg) \\
&+ {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} \\
&+ {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) + \sum_{j=1}^{k_1} {}_1\eta_j\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} \\
&+ \sum_{j=1}^{k_2} {}_2\eta_j\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} + \sum_{j=1}^{k_3} {}_3\eta_j\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg] \\
&\times \Bigg[\frac{\delta^{\alpha-2}}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(2-p_2)\delta^{\alpha-2}}{\Gamma_q(\alpha-p_2+1)} \\
&+ \frac{(4-p_2)\Gamma_q(4-p_3)\delta^{\alpha-2}}{(3-p_2)\Gamma_q(\alpha-p_3+1)}\Bigg]\|u-v\|,
\end{aligned}
$$

$$
\begin{aligned}
\big|(\Theta u)'''(t)-(\Theta v)'''(t)\big| \leq{}& \Bigg[\Bigg(\sum_{j=1}^{4} {}_0\eta_j\Bigg) + {}_0\eta_5\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg) \\
&+ {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} \\
&+ {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg)
\end{aligned}
$$

$$+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}$$

$$+ {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) + \sum_{j=1}^{k_1} {}_1\eta_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}$$

$$+ \sum_{j=1}^{k_2} {}_2\eta_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} + \sum_{j=1}^{k_3} {}_3\eta_j \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg]$$

$$\times \left[\frac{\delta^{\alpha-3}}{\Gamma_q(\alpha-2)} + \frac{\Gamma_q(4-p_3)\delta^{\alpha-3}}{\Gamma_q(\alpha-p_3+1)}\right]\|u-v\|,$$

for each $t \in \overline{J}$ and each $u, v \in \mathcal{X}$. Hence, we conclude that

$$\begin{aligned}
\|\Theta u - \Theta v\| \le \Lambda_1 \Bigg[&\left(\sum_{j=1}^{4} {}_0\eta_j\right) + {}_0\eta_5 \gamma_1^0 \left(\left(\sum_{j=1}^{4} {}_1c_j\right) + {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}\right. \\
&+ {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + {}_0\eta_6 \gamma_2^0 \left(\left(\sum_{j=1}^{4} {}_2c_j\right)\right. \\
&+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) \\
&+ \sum_{j=1}^{k_1} {}_1\eta_j \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} {}_2\eta_j \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\
&+ \sum_{j=1}^{k_3} {}_3\eta_j \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg] \|u-v\| \\
= \Lambda_1' \|u-v\|.&
\end{aligned}$$

Therefore, Θ is a contraction, because $\Lambda_1' < 1$, and so, by employing the Banach contraction principle, Θ has a unique fixed point, which is a solution of problem (1). □

3.2 Positive solutions for inclusion problem (2)

In the second section of main results, we look into the positive solutions for the inclusion problem (2) with the antiperiodic boundary conditions (3), (4), and (5). Now, we recall some definitions and concepts which are needed in the sequel, and also we use the same definitions of the previous section. As one knows, a multivalued map $T : \overline{J} \times \mathcal{R}^m \to P(\mathbb{R})$ is said to be Carathéodory whenever the map $t \mapsto T(t, r_1, r_2, \ldots, r_m)$ is measurable and the map $(r_1, r_2, \ldots, r_m) \mapsto T(t, r_1, r_2, \ldots, r_m)$ is upper semicontinuous, and we say that a Carathéodory function T is L^1-Carathéodory whenever for each $l > 0$ there exists $\varphi_l \in L^1(\overline{J}, \mathbb{R}^+)$ such that

$$\left\|T(t, r_1, r_2, \ldots, r_m)\right\|_p = \sup\left\{|s| : s \in T(t, r_1, r_2, \ldots, r_m)\right\} \le \psi_l(t),$$

where $|r_i| \le l$, for each $r_i \in \mathbb{R}$ with $i \in N_m$, each $t \in \overline{J}$, respectively [29, 50]. One can find the following lemma in [31].

Lemma 7 *The composite operator* $N \circ S_G : C(\bar{J}, \mathcal{A}) \to P_{cp,c}(C(\bar{J}, \mathcal{A}))$ *defined by* $N \circ S_G(r) = N(S_{G,r})$ *is a closed-graph operator, whenever* $G : \bar{J} \times \mathcal{A} \to P_{cp,c}(\mathcal{A})$ *is an* L^1*-Carathéodory multifunction and* N *is a linear continuous mapping from* $L^1(\bar{J}, \mathcal{A})$ *to* $C(\bar{J}, \mathcal{A})$*, where* $\mathcal{A}$ *is a Banach space and* $S_{G,r}$ *is the set of all* $w \in L^1(\bar{J}, \mathcal{A})$ *such that* $w(t) \in G(t, x(t))$ *for each* $t \in \bar{J}$*.*

The multivalued map $G : \bar{J} \times \mathcal{A} \to P_{cp}(\mathcal{A})$ is said to be of lower semicontinuous type whenever $S_G : C(\bar{J}, \mathcal{A}) \to P(L^1(\bar{J}, \mathcal{A}))$ is lower semicontinuous and has nonempty closed and decomposable values [51]. Also, one can see the following lemma in [51].

Lemma 8 *The lower semicontinuous multivalued map* $N : \mathcal{A} \to P(L^1(\bar{J}, \mathbb{R}))$ *has a continuous selection, i.e., there exists a continuous mapping* $H : \mathcal{A} \to L^1(\bar{J}, \mathbb{R})$ *such that* $H(a) \in N(a)$ *for each* $a \in \mathcal{A}$*, whenever* N *has closed decomposable values, where* $\mathcal{A}$ *be a separable metric space.*

Theorem 9 ([53]) *Suppose that* $(\mathcal{A}, \rho)$ *be a complete metric space. Then each contraction multivalued map* $T : \mathcal{A} \to P_{cl}(\mathcal{A})$ *has a fixed point.*

Theorem 10 ([54]) *Let* $\mathcal{C}$ *be a closed and convex subset of a Banach space* $\mathcal{A}$ *and* $\mathcal{O}$ *be an open subset of* $\mathcal{C}$ *such that* $0 \in \mathcal{O}$*. Then either* T *has a fixed point in* $\overline{\mathcal{O}}$ *or there are* $a \in \partial\mathcal{O}$ *and* $\kappa \in (0, 1)$ *such that* $a \in \kappa T(a)$*, whenever* $T : \overline{\mathcal{O}} \to P_{cp,c}(\mathcal{C})$ *is an upper semicontinuous compact map.*

Theorem 11 *If a multivalued map* T *mapping* $\bar{J} \times \mathcal{R}^m$ *into* $P_{cp,c}(\mathbb{R})$ *is Carathéodory, then problem* (2) *has at least one positive solution whenever the following assumptions are hold for each* $t, s \in \bar{J}$, ${}_ix_j \in \mathbb{R}$:

(1) *There exist positive real-valued and continuous nondecreasing functions* ϕ_{j_0} *and* $\psi_{1j_1}, \psi_{2j_2}, \psi_{3j_3}$ *defined on* $[0, \infty)$ *and nonnegative functions* $g_{0j_0}, g_{1j_1}, g_{2j_2}, g_{3j_3}$ *in* $L^1(\bar{J})$ *such that*

$$\begin{aligned}
&\big\| T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_0x_6, \\
&\quad {}_1x_1, {}_1x_2, \dots, {}_1x_{k_1}, {}_2x_1, {}_2x_2, \dots, {}_2x_{k_2}, {}_3x_1, {}_3x_2, \dots, {}_3x_{k_3}) \big\|_p \\
&\quad = \sup\big\{ |x| : x \in T(t, {}_0w_1, {}_0w_2, {}_0w_3, {}_0w_4, {}_0w_5, {}_0w_6, {}_1w_1, {}_1w_2, \dots, \\
&\quad {}_1w_{k_1}, {}_2w_1, {}_2w_2, \dots, {}_2w_{k_2}, {}_3w_1, {}_3w_2, \dots, {}_3w_{k_3}) \big\} \\
&\quad \le \sum_{j=1}^{6} g_{0j}(t)\phi_j\big(|{}_0x_j|\big) + \sum_{j=1}^{k_1} g_{1j}(t)\psi_{1j}\big(|{}_1x_j|\big) \\
&\qquad + \sum_{j=1}^{k_2} g_{2j}(t)\psi_{2j}\big(|{}_2x_j|\big) + \sum_{j=1}^{k_3} g_{3j}(t)\psi_{3j}\big(|{}_3x_j|\big),
\end{aligned}$$

for j_0, j_1, j_2 *and* j_3 *in* $N_6, N_{k_1}, N_{k_2}, N_{k_3}$*, respectively.*

(2) *There exist constants* ${}_0c_1, {}_0c_2$ *in* $(0, \infty)$ *and* ${}_i\eta_j \in [0, \infty)$ *such that*

$$\big|\theta_i(t, s, x_1, x_2, x_3, x_4, x_5, x_6, x_7)\big| \le {}_0c_i + \sum_{j=1}^{7} {}_i\eta_j |x_j|,$$

for each $x_i \in \mathbb{R}$*, where* $i = 1, 2$ *and* $j \in N_7$*.*

(3) *There exists a constant* $\Delta > 0$ *such that* $\Lambda_2 A(\Delta) < \Delta$, *where*

$$\begin{aligned} A(\Delta) = {}& \left(\sum_{j=1}^{4} \|g_{0j}\|_1 \phi_j(\Delta)\right) \\ & + \|g_{05}\|_1 \phi_5 \left({}_0c_1 \gamma_1^0 + \Delta \gamma_1^0 \left[\left(\sum_{j=1}^{4} {}_1\eta_j \right) \right.\right. \\ & \left.\left. + {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})} \right] \right) \\ & + \|g_{06}\|_1 \phi_6 \left({}_0c_2 \gamma_2^0 + \Delta \gamma_2^0 \left[\left(\sum_{j=1}^{4} {}_2\eta_j \right) \right.\right. \\ & \left.\left. + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})} \right] \right) \\ & + \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j} \left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} \Delta \right) \\ & + \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j} \left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \Delta \right) \\ & + \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j} \left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} \Delta \right), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \Lambda_2 = {}& \left[\frac{|a_1| + 2|a_2|}{|a_1 + a_2| \Gamma_q(\alpha)} + \frac{(|a_1| + 2|a_2|) \Gamma_q(2-p_1)}{|a_1 + a_2| \Gamma_q(\alpha - p_1)} \right. \\ & + \frac{(|a_2| p_1 + |a_1 + a_2|(4 - p_1)) \Gamma_q(3-p_2)}{2|a_1 + a_2|(2-p_1) \Gamma_q(\alpha - p_2)} \\ & + \frac{|a_2| [6(p_2 - p_1) + (2-p_1)(3-p_1) p_2] \Gamma_q(4-p_3)}{6|a_1 + a_2|(2-p_1)(3-p_1)(3-p_2) \Gamma_q(\alpha - p_3)} \\ & \left. + \frac{[6(p_2 - p_1) + (2-p_1)(3-p_1)(6-p_2)] \Gamma_q(4-p_3)}{6(2-p_1)(3-p_1)(3-p_2) \Gamma_q(\alpha - p_3)} \right] \delta^{\alpha-1} \\ & + \left[\frac{1}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(2-p_1)}{\Gamma_q(\alpha-p_1)} + \frac{(3-p_1)\Gamma_q(3-p_2)}{(2-p_1)\Gamma_q(\alpha-p_2)} \right. \\ & \left. + \frac{[2(p_2 - p_1) + (2-p_1)(3-p_1)(5-p_2)] \Gamma_q(4-p_3)}{2(2-p_1)(3-p_1)(3-p_2) \Gamma_q(\alpha - p_3)} \right] \delta^{\alpha-2} \\ & + \left[\frac{1}{\Gamma_q(\alpha-2)} + \frac{\Gamma_q(3-p_2)}{\Gamma_q(\alpha-p_2)} + \frac{(4-p_2)\Gamma_q(4-p_3)}{(3-p_2)\Gamma_q(\alpha-p_3)} \right] \delta^{\alpha-3} \\ & + \left[\frac{1}{\Gamma_q(\alpha-3)} + \frac{\Gamma_q(4-p_3)}{\Gamma_q(\alpha-p_3)} \right] \delta^{\alpha-4}. \end{aligned} \tag{17}$$

Proof To begin, we define the set of selections of T for an arbitrary element $u \in \mathcal{X}$ which contains all $v \in L^1(\bar{J}, \mathbb{R})$ such that $v(t)$ belongs to the multifunction $\widetilde{T}(t, u(t))$ for each $t \in \bar{J}$

and is denoted by $S_{T,u}$, where

$$\begin{aligned}\widetilde{T}\big(t,u(t)\big) = T\big(&t,u(t),u'(t),u''(t),u'''(t),\varphi_1 u(t),\varphi_2 u(t),\\ &{}^cD_q^{\beta_{11}}u(t),{}^cD_q^{\beta_{12}}u(t),\dots,{}^cD_q^{\beta_{1k_1}}u(t),\\ &{}^cD_q^{\beta_{21}}u(t),{}^cD_q^{\beta_{22}}u(t),\dots,{}^cD_q^{\beta_{2k_2}}u(t),\\ &{}^cD_q^{\beta_{31}}u(t),{}^cD_q^{\beta_{32}}u(t),\dots,{}^cD_q^{\beta_{3k_3}}u(t)\big).\end{aligned}$$

By considering the first property of the multifunction T and using Theorem 1.3.5 in [8], we know that $S_{T,u}$ is nonempty. Defining an operator $H: X \to P(X)$ on the set of all $h \in X$ for which there exists $\nu \in S_{T,u}$ such that $h(t) = T_\nu(t)$ for $t \in \overline{J}$ and denoting by $H(x)$ where

$$\begin{aligned}T_\nu(t) &= I_q^\alpha \nu(t) - \frac{a_2}{a_1+a_2} I_q^\alpha \nu(\delta) + B_1(t,\delta)_q^{\alpha-p_1}\nu(\delta)\\ &\quad - B_2(t,\delta)I_q^{\alpha-p_2}\nu(\delta) - B_3(t,\delta)I_q^{\alpha-p_3}\nu(\delta),\end{aligned}$$

we claim that $H(x)$ is convex for all $u \in \mathcal{X}$. Assume that $h_1, h_2 \in H(x)$ and $\tau \in [0,1]$. Choose $\nu_1, \nu_2 \in S_{T,u}$ such that

$$\begin{aligned}h_i(t) &= I_q^\alpha \nu_i(t) - \frac{a_2}{a_1+a_2} I_q^\alpha \nu_i(\delta) + B_1(t,\delta)_q^{\alpha-p_1}\nu_i(\delta)\\ &\quad - B_2(t,\delta)I_q^{\alpha-p_2}\nu_i(\delta) - B_3(t,\delta)I_q^{\alpha-p_3}\nu_i(\delta),\end{aligned}$$

for each t in $\overline{J}$. Then, we obtain

$$\begin{aligned}\big[\tau h_1 + (1-\tau)h_2\big](t) &= I_q^\alpha\big[\tau\nu_1(t) + (1-\tau)\nu_2(t)\big]\\ &\quad - \frac{a_2}{a_1+a_2} I_q^\alpha\big[\tau\nu_1(\delta) + (1-\tau)\nu_2(\delta)\big]\\ &\quad + B_1(t,\delta)I_q^{\alpha-p_1}\big[\tau\nu_1(\delta) + (1-\tau)\nu_2(\delta)\big]\\ &\quad - B_2(t,\delta)I_q^{\alpha-p_2}\big[\tau\nu_1(\delta) + (1-\tau)\nu_2(\delta)\big]\\ &\quad - B_3(t,\delta)I_q^{\alpha-p_3}\big[\tau\nu_1(\delta) + (1-\tau)\nu_2(\delta)\big].\end{aligned}$$

Since T has convex values, by simple calculation, we can see that $S_{T,u}$ is convex and so $\tau h_1 + (1-\tau)h_2 \in H(x)$. At present, we prove that H maps bounded sets into bounded sets in $\mathcal{X}$. Suppose that B_r is the set of all $u \in \mathcal{X}$ such that $\|u\|$ is less than or equal to r, $u \in B_r$ and $h \in H(x)$. We select $\nu \in S_{T,u}$ such that

$$\begin{aligned}\big|h(t)\big| &\le I_q^\alpha \nu(t) - \frac{a_2}{a_1+a_2} I_q^\alpha \nu(\delta) + B_1(t,\delta)I_q^{\alpha-p_1}\nu(\delta) - B_2(t,\delta)I_q^{\alpha-p_2}\nu(\delta)\\ &\quad - B_3(t,\delta)I_q^{\alpha-p_3}\nu(\delta)\\ &\le I_q^\alpha\Bigg[g_{01}(t)\phi_1\big(\big|u(t)\big|\big) + g_{02}(t)\phi_2\big(\big|u'(t)\big|\big) + g_{03}(t)\phi_3\big(\big|u''(t)\big|\big)\\ &\quad + g_{04}(t)\phi_4\big(\big|u'''(t)\big|\big) + g_{05}(t)\phi_5\big(\big|\varphi_1 u(t)\big|\big) + g_{06}(t)\phi_6\big(\big|\varphi_2 u(t)\big|\big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k_1} g_{1j}(t)\psi_{1j}\left(\left|{}^cD_q^{\beta_{1j}}u(t)\right|\right)+\sum_{j=1}^{k_2} g_{2j}(t)\psi_{2j}\left(\left|{}^cD_q^{\beta_{2j}}u(t)\right|\right)\\
&+\sum_{j=1}^{k_3} g_{3j}(t)\psi_{3j}\left(\left|{}^cD_q^{\beta_{3j}}u(t)\right|\right)\Bigg]\\
&+\frac{|a_2|}{|a_1+a_2|}I_q^{\alpha}\Bigg[g_{01}(\delta)\phi_1\left(\left|u(\delta)\right|\right)+g_{02}(\delta)\phi_2\left(\left|u'(\delta)\right|\right)\\
&+g_{03}(\delta)\phi_3\left(\left|u''(\delta)\right|\right)+g_{04}(\delta)\phi_4\left(\left|u'''(\delta)\right|\right)\\
&+g_{05}(\delta)\phi_5\left(\left|\varphi_1 u(\delta)\right|\right)+g_{06}(\delta)\phi_6\left(\left|\varphi_2 u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_1} g_{1j}(\delta)\psi_{1j}\left(\left|{}^cD_q^{\beta_{1j}}u(\delta)\right|\right)+\sum_{j=1}^{k_2} g_{2j}(\delta)\psi_{2j}\left(\left|{}^cD_q^{\beta_{2j}}u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_3} g_{3j}(t)\psi_{3j}\left(\left|{}^cD_q^{\beta_{3j}}u(t)\right|\right)\Bigg]\\
&+\left|B_1(t,\delta)\right|I_q^{\alpha-p_1}\Bigg[g_{01}(\delta)\phi_1\left(\left|u(\delta)\right|\right)+g_{02}(\delta)\phi_2\left(\left|u'(\delta)\right|\right)\\
&+g_{03}(\delta)\phi_3\left(\left|u''(\delta)\right|\right)+g_{04}(\delta)\phi_4\left(\left|u'''(\delta)\right|\right)\\
&+g_{05}(\delta)\phi_5\left(\left|\varphi_1 u(\delta)\right|\right)+g_{06}(\delta)\phi_6\left(\left|\varphi_2 u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_1} g_{1j}(\delta)\psi_{1j}\left(\left|{}^cD_q^{\beta_{1j}}u(\delta)\right|\right)+\sum_{j=1}^{k_2} g_{2j}(\delta)\psi_{2j}\left(\left|{}^cD_q^{\beta_{2j}}u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_3} g_{3j}(t)\psi_{3j}\left(\left|{}^cD_q^{\beta_{3j}}u(t)\right|\right)\Bigg]\\
&+\left|B_2(t,\delta)\right|I_q^{\alpha-p_2}\Bigg[g_{01}(\delta)\phi_1\left(\left|u(\delta)\right|\right)+g_{02}(\delta)\phi_2\left(\left|u'(\delta)\right|\right)\\
&+g_{03}(\delta)\phi_3\left(\left|u''(\delta)\right|\right)+g_{04}(\delta)\phi_4\left(\left|u'''(\delta)\right|\right)\\
&+g_{05}(\delta)\phi_5\left(\left|\varphi_1 u(\delta)\right|\right)+g_{06}(\delta)\phi_6\left(\left|\varphi_2 u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_1} g_{1j}(\delta)\psi_{1j}\left(\left|{}^cD_q^{\beta_{1j}}u(\delta)\right|\right)+\sum_{j=1}^{k_2} g_{2j}(\delta)\psi_{2j}\left(\left|{}^cD_q^{\beta_{2j}}u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_3} g_{3j}(t)\psi_{3j}\left(\left|{}^cD_q^{\beta_{3j}}u(t)\right|\right)\Bigg]\\
&+\left|B_3(t,\delta)\right|\Bigg[g_{01}(\delta)\phi_1\left(\left|u(\delta)\right|\right)+g_{02}(\delta)\phi_2\left(\left|u'(\delta)\right|\right)\\
&+g_{03}(\delta)\phi_3\left(\left|u''(\delta)\right|\right)+g_{04}(\delta)\phi_4\left(\left|u'''(\delta)\right|\right)\\
&+g_{05}(\delta)\phi_5\left(\left|\varphi_1 u(\delta)\right|\right)+g_{06}(\delta)\phi_6\left(\left|\varphi_2 u(\delta)\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k_1} g_{1j}(\delta)\psi_{1j}\left(\left|{}^cD_q^{\beta_{1j}}u(\delta)\right|\right)+\sum_{j=1}^{k_2} g_{2j}(\delta)\psi_{2j}\left(\left|{}^cD_q^{\beta_{2j}}u(\delta)\right|\right)\\
&+\sum_{j=1}^{k_3} g_{3j}(t)\psi_{3j}\left(\left|{}^cD_q^{\beta_{3j}}u(t)\right|\right)\Bigg]\\
\le{}&\Bigg[\sum_{j=1}^{4}\|g_{0j}\|_1\phi_j(r)+\|g_{05}\|_1\phi_5\Bigg({}_0c_1\gamma_1^0+r\gamma_1^0\Bigg[\Bigg(\sum_{j=1}^{4}{}_1\eta_j\Bigg)\\
&+{}_1\eta_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}+{}_1\eta_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}+{}_1\eta_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg]\Bigg)\\
&+\|g_{06}\|_1\phi_6\Bigg({}_0c_2\gamma_2^0+r\gamma_2^0\Bigg[\sum_{j=1}^{4}{}_2\eta_j+{}_2\eta_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\\
&+{}_2\eta_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}+{}_2\eta_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg)\\
&+\sum_{j=1}^{k_1}\|g_{1j}\|_1\psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}r\right)\\
&+\sum_{j=1}^{k_2}\|g_{2j}\|_1\psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}r\right)\\
&+\sum_{j=1}^{k_3}\|g_{3j}\|_1\psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}r\right)\Bigg]\\
&\times\Bigg[\frac{(|a_1|+2|a_2|)\delta^{\alpha-1}}{|a_1+a_2|\Gamma_q(\alpha)}+\frac{(|a_1|+2|a_2|)\Gamma_q(2-p_1)\delta^{\alpha-1}}{|a_1+a_2|\Gamma_q(\alpha-p_1)}\\
&+\frac{(|a_2|p_1+|a_1+a_2|(4-p_1))\Gamma_q(3-p_2)\delta^{\alpha-1}}{2|a_1+a_2|(2-p_1)\Gamma_q(\alpha-p_2)}\\
&+\frac{|a_2|[6(p_2-p_1)+(2-p_1)(3-p_1)p_2]\Gamma_q(4-p_3)\delta^{\alpha-1}}{6|a_1+a_2|(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3)}\\
&+\frac{[6(p_2-p_1)+(2-p_1)(3-p_1)(6-p_2)]\Gamma_q(4-p_3)\delta^{\alpha-1}}{6(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3)}\Bigg],
\end{aligned}
\tag{18}
$$

for any $t\in\overline{J}$. Thus, similarly as for inequality (18), we get

$$
\begin{aligned}
\left|h'(t)\right|\le{}&\Bigg[\sum_{j=1}^{4}\|g_{0j}\|_1\phi_j(r)+\|g_{05}\|_1\phi_5\Bigg({}_0c_1\gamma_1^0+r\gamma_1^0\Bigg[\Bigg(\sum_{j=1}^{4}{}_1\eta_j\Bigg)\\
&+{}_1\eta_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}+{}_1\eta_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}+{}_1\eta_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg]\Bigg)\\
&+\|g_{06}\|_1\phi_6\Bigg({}_0c_2\gamma_2^0+r\gamma_2^0\Bigg[\sum_{j=1}^{4}{}_2\eta_j+{}_2\eta_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\\
&+{}_2\eta_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}+{}_2\eta_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} r\right) \\
&+ \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} r\right) \\
&+ \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} r\right)\Bigg] \\
&\times \Bigg[\frac{\delta^{\alpha-2}}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(2-p_1)\delta^{\alpha-2}}{\Gamma_q(\alpha-p_1)} + \frac{(3-p_1)\Gamma_q(3-p_2)\delta^{\alpha-2}}{(2-p_1)\Gamma_q(\alpha-p_2)} \\
&+ \frac{[2(p_2-p_1)+(2-p_1)(3-p_1)(5-p_2)]\Gamma_q(4-p_3)\delta^{\alpha-2}}{2(2-p_1)(3-p_1)(3-p_2)\Gamma_q(\alpha-p_3)}\Bigg],
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\left|h''(t)\right| \le{}& \Bigg[\sum_{j=1}^{4} \|g_{0j}\|_1 \phi_j(r) + \|g_{05}\|_1 \phi_5\Bigg({}_0c_1\gamma_1^0 + r\gamma_1^0\Bigg[\Bigg(\sum_{j=1}^{4} {}_1\eta_j\Bigg) \\
&+ {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg]\Bigg) \\
&+ \|g_{06}\|_1 \phi_6\Bigg({}_0c_2\gamma_2^0 + r\gamma_2^0\Bigg[\sum_{j=1}^{4} {}_2\eta_j + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+ {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg) \\
&+ \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} r\right) \\
&+ \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} r\right) \\
&+ \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} r\right)\Bigg] \\
&\times \Bigg[\frac{\delta^{\alpha-3}}{\Gamma_q(\alpha-2)} + \frac{\Gamma_q(3-p_2)\delta^{\alpha-3}}{\Gamma_q(\alpha-p_2)} + \frac{(4-p_2)\Gamma_q(4-p_3)\delta^{\alpha-3}}{(3-p_2)\Gamma_q(\alpha-p_3)}\Bigg],
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\left|h'''(t)\right| \le{}& \Bigg[\sum_{j=1}^{4} \|g_{0j}\|_1 \phi_j(r) + \|g_{05}\|_1 \phi_5\Bigg({}_0c_1\gamma_1^0 + r\gamma_1^0\Bigg[\Bigg(\sum_{j=1}^{4} {}_1\eta_j\Bigg) \\
&+ {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg]\Bigg) \\
&+ \|g_{06}\|_1 \phi_6\Bigg({}_0c_2\gamma_2^0 + r\gamma_2^0\Bigg[\sum_{j=1}^{4} {}_2\eta_j + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+ {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} r\right) && (21)\\
&+ \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} r\right)\\
&+ \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} r\right)\Bigg]\\
&\times \left[\frac{\delta^{\alpha-4}}{\Gamma_q(\alpha-3)} + \frac{\Gamma_q(4-p_3)\delta^{\alpha-4}}{\Gamma_q(\alpha-p_3)}\right]. && (22)
\end{aligned}
$$

Thus, from inequalities (18), (19), (20), and (22), we obtain

$$
\begin{aligned}
\|h\| \le \Lambda_2 \Bigg[& \sum_{j=1}^{4} \|g_{0j}\|_1 \phi_j(r) + \|g_{05}\|_1 \phi_5 \Bigg({}_0c_1 \gamma_1^0 + r\gamma_1^0 \Bigg[\Bigg(\sum_{j=1}^{4} {}_1\eta_j \Bigg)\\
& + {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})} \Bigg] \Bigg)\\
& + \|g_{06}\|_1 \phi_6 \Bigg({}_0c_2 \gamma_2^0 + r\gamma_2^0 \Bigg[\sum_{j=1}^{4} {}_2\eta_j + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})}\\
& + {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})} \Bigg] \Bigg)\\
& + \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} r\right)\\
& + \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} r\right)\\
& + \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} r\right)\Bigg].
\end{aligned}
$$

Thus, we conclude that H maps bounded sets into bounded sets in $\mathcal{X}$. Let $\tau_1, \tau_2 \in \overline{J}$ with $\tau_1 < \tau_2$, $u \in B_r$ and $h \in H(x)$. Then, we have

$$
\begin{aligned}
\left|h(\tau_2) - h(\tau_1)\right| \le{} & \frac{1}{\Gamma_q(\alpha)} \int_0^{\tau_1} \left[(\tau_2 - qs)^{(\alpha-1)} - (\tau_1 - qs)^{(\alpha-1)}\right] \left|\nu(s)\right| d_q s\\
& + \frac{1}{\Gamma_q(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - qs)^{(\alpha-1)} \left|\nu(s)\right| d_q s\\
& + (\tau_2 - \tau_1)\delta^{p_1-1} \Gamma_q(2-p_1) I_q^{\alpha-p_1} \left|\nu(\delta)\right|\\
& + \frac{[2\delta(\tau_2-\tau_1) + (2-p_1)(\tau_2^2 - \tau_1^2)]\Gamma_q(3-p_2)\delta^{p_2-2}}{2(2-p_1)} I_q^{\alpha-p_2} \left|\nu(\delta)\right|\\
& + \Bigg(\frac{(p_2-p_1)\delta^2(\tau_2-\tau_1)\Gamma_q(4-p_3)\delta^{p_3-3}}{(2-p_1)(3-p_1)(3-p_2)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(2-p_1)(3-p_1)\Gamma_q(4-p_3)\delta^{p_3-3}}{6(2-p_1)(3-p_1)(3-p_2)} \\
&\times \bigl[3\delta\bigl(\tau_2^2-\tau_1^2\bigr)+(3-p_2)\bigl(\tau_2^3-\tau_1^3\bigr)\bigr]\biggr) I_q^{\alpha-p_3}\bigl|\nu(\delta)\bigr| \\
\le{}& \Biggl[\sum_{j=1}^{4}\phi_j(r)+\phi_5\Biggl({}_0c_1\gamma_1^0+r\gamma_1^0\Biggl[\Biggl(\sum_{j=1}^{4}{}_1\eta_j\Biggr) \\
&+ {}_1\eta_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}+{}_1\eta_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})}+{}_1\eta_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Biggr]\Biggr) \\
&+ \phi_6\Biggl({}_0c_2\gamma_2^0+r\gamma_2^0\Biggl[\sum_{j=1}^{4}{}_2\eta_j+{}_2\eta_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+ {}_2\eta_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}+{}_2\eta_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Biggr]\Biggr) \\
&+ \sum_{j=1}^{k_1}\psi_{1j}\biggl(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}r\biggr)+\sum_{j=1}^{k_2}\psi_{2j}\biggl(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}r\biggr) \\
&+ \sum_{j=1}^{k_3}\psi_{3j}\biggl(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}r\biggr)\Biggr] \\
&\times \Biggl[\frac{1}{\Gamma_q(\alpha)}\int_0^{\tau_1}\bigl[(\tau_2-qs)^{(\alpha-1)}-(\tau_1-qs)^{(\alpha-1)}\bigr]\mathcal{G}(s)\,d_qs \\
&+ \frac{1}{\Gamma_q(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-qs)^{(\alpha-1)}\mathcal{G}(s)\,d_qs \\
&+ (\tau_2-\tau_1)\Gamma(2-p_1)\delta^{p_1-1}I_q^{\alpha-p_1}\mathcal{G}(\delta) \\
&+ \frac{[2\delta(\tau_2-\tau_1)+(2-p_1)(\tau_2^2-\tau_1^2)]\Gamma_q(3-p_2)\delta^{p_2-2}}{2(2-p_1)}I_q^{\alpha-p_2}\mathcal{G}(\delta) \\
&+ \biggl(\frac{(p_2-p_1)\delta^2(\tau_2-\tau_1)\Gamma_q(4-p_3)\delta^{p_3-3}}{(2-p_1)(3-p_1)(3-p_2)} \\
&+ \frac{(2-p_1)(3-p_1)\Gamma_q(4-p_3)\delta^{p_3-3}}{6(2-p_1)(3-p_1)(3-p_2)} \\
&\times \bigl[3\delta\bigl(\tau_2^2-\tau_1^2\bigr)+(3-p_2)\bigl(\tau_2^3-\tau_1^3\bigr)\bigr]\biggr)I_q^{\alpha-p_3}\mathcal{G}(\delta)\Biggr],
\end{aligned}
\tag{23}
$$

where

$$
\mathcal{G}(z)=\sum_{j=1}^{6}g_{0j}(z)+\sum_{j=1}^{k_1}g_{1j}(z)+\sum_{j=1}^{k_2}g_{2j}(z)+\sum_{j=1}^{k_3}g_{3j}(z).
$$

Similarly, from inequality (23), we have

$$
\bigl|h'(\tau_2)-h'(\tau_1)\bigr| \le \Biggl[\sum_{j=1}^{4}\phi_j(r)+\phi_5\Biggl({}_0c_1\gamma_1^0+r\gamma_1^0\Biggl[\Biggl(\sum_{j=1}^{4}{}_1\eta_j\Biggr)
$$

$$
\begin{aligned}
&+ {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} \\
&+ {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})} \Bigg]\Bigg) \\
&+ \phi_6 \Bigg({}_0c_2\gamma_2^0 + r\gamma_2^0 \Bigg[\sum_{j=1}^{4} {}_2\eta_j + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+ {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})} \Bigg]\Bigg) \\
&+ \sum_{j=1}^{k_1} \psi_{1j} \bigg(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} r \bigg) + \sum_{j=1}^{k_2} \psi_{2j} \bigg(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} r \bigg) \\
&+ \sum_{j=1}^{k_3} \psi_{3j} \bigg(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} r \bigg) \Bigg] \bigg[\frac{1}{\Gamma_q(\alpha-1)} \\
&\times \int_0^{\tau_1} \big[(\tau_2 - qs)^{(\alpha-2)} - (\tau_1 - qs)^{(\alpha-2)} \big] \mathcal{G}(s)\, d_qs \\
&+ \frac{1}{\Gamma_q(\alpha-1)} \int_{\tau_1}^{\tau_2} (\tau_2 - qs)^{(\alpha-2)} \mathcal{G}(s)\, d_qs \\
&+ (\tau_2 - \tau_1)\Gamma_q(3-p_2)\delta^{p_2-2} I_q^{\alpha-p_2} \mathcal{G}(\delta) \\
&+ \frac{\Gamma_q(4-p_3)\delta^{p_3-3}}{2(3-p_2)} \\
&\times \big[2\delta(\tau_2-\tau_1) + (3-p_2)\big(t_2^2 - \tau_1^2\big) \big] I_q^{\alpha-p_3} \mathcal{G}(\delta) \bigg],
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\big| h''(\tau_2) - h''(\tau_1) \big| \leq{}& \Bigg[\sum_{j=1}^{4} \phi_j(r) + \phi_5 \Bigg({}_0c_1\gamma_1^0 + r\gamma_1^0 \Bigg[\Bigg(\sum_{j=1}^{4} {}_1\eta_j \Bigg) \\
&+ {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} \\
&+ {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})} \Bigg]\Bigg) \\
&+ \phi_6 \Bigg({}_0c_2\gamma_2^0 + r\gamma_2^0 \Bigg[\sum_{j=1}^{4} {}_2\eta_j + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+ {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})} \Bigg]\Bigg) \\
&+ \sum_{j=1}^{k_1} \psi_{1j} \bigg(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} r \bigg) + \sum_{j=1}^{k_2} \psi_{2j} \bigg(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} r \bigg) \\
&+ \sum_{j=1}^{k_3} \psi_{3j} \bigg(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} r \bigg) \Bigg] \bigg[\frac{1}{\Gamma_q(\alpha-1)} \\
&\times \int_0^{\tau_1} \big[(\tau_2 - qs)^{(\alpha-2)} - (\tau_1 - qs)^{(\alpha-2)} \big] \mathcal{G}(s)\, d_qs
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma_q(\alpha-2)}\int_{\tau_1}^{\tau_2}(\tau_2-qs)^{(\alpha-3)}\mathcal{G}(s)\,d_qs \\
&+(\tau_2-\tau_1)\Gamma_q(4-p_3)\delta^{p_3-3}I_q^{\alpha-p_3}\mathcal{G}(\delta)\Bigg], \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\big|h'''(\tau_2)-h'''(\tau_1)\big| \le{}& \Bigg[\sum_{j=1}^{4}\phi_j(r)+\phi_5\Bigg({}_0c_1\gamma_1^0+r\gamma_1^0\Bigg[\Bigg(\sum_{j=1}^{4}{}_1\eta_j\Bigg) \\
&+{}_1\eta_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})}+{}_1\eta_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} \\
&+{}_1\eta_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg]\Bigg) \\
&+\phi_6\Bigg({}_0c_2\gamma_2^0+r\gamma_2^0\Bigg[\sum_{j=1}^{4}{}_2\eta_j+{}_2\eta_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\
&+{}_2\eta_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})}+{}_2\eta_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg) \\
&+\sum_{j=1}^{k_1}\psi_{1j}\bigg(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}r\bigg)+\sum_{j=1}^{k_2}\psi_{2j}\bigg(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}r\bigg) \\
&+\sum_{j=1}^{k_3}\psi_{3j}\bigg(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}r\bigg)\Bigg]\Bigg[\frac{1}{\Gamma_q(\alpha-1)} \\
&\times\int_0^{\tau_1}\big[(\tau_2-qs)^{(\alpha-2)}-(\tau_1-qs)^{(\alpha-2)}\big]\mathcal{G}(s)\,d_qs \\
&+\frac{1}{\Gamma_q(\alpha-3)}\int_{\tau_1}^{\tau_2}(\tau_2-qs)^{(\alpha-4)}\mathcal{G}(\delta)\,d_qs\Bigg]. \qquad (26)
\end{aligned}
$$

Therefore, since $u\in B_r$, when $t_2-t_1\to 0$, the above inequalities (23)–(26) tend to zero. Therefore, by employing Arzelà–Ascoli theorem, we get that $H:\mathcal{X}\to P(\mathcal{X})$ is a compact multivalued map. Let $u_n\to u^*$, $h_n\in H(u_n)$ for all n and $h_n\to h^*$. We show that $h^*\in H(u^*)$. Since $h_n\in H(u_n)$ for all n, there exists $v_n\in S_{T,u_n}$ such that

$$
\begin{aligned}
h_n(t)={}&I_q^\alpha v_n(t)-\frac{a_2}{a_1+a_2}I_q^\alpha v_n(\delta)+B_1(t,\delta)I_q^{\alpha-p_1}v_n(\delta) \\
&-B_2(t,\delta)I_q^{\alpha-p_2}v_n(\delta)-B_3(t,\delta)I_q^{\alpha-p_3}v_n(\delta),
\end{aligned}
$$

for all $t\in\overline{J}$. We claim that there exists v^* belonging to S_{T,u^*} such that

$$
\begin{aligned}
h^*(t)={}&I_q^\alpha v^*(t)-\frac{a_2}{a_1+a_2}I_q^\alpha v^*(\delta)+B_1(t,\delta)I_q^{\alpha-p_1}v^*(\delta) \\
&-B_2(t,\delta)I_q^{\alpha-p_2}v^*(\delta)-B_3(t,\delta)I_q^{\alpha-p_3}v^*(\delta),
\end{aligned}
$$

for each t belonging to $\overline{J}$. In this case, we consider the linear operator $\Omega:L^1(\overline{J},\mathbb{R})\to\mathcal{X}$ defined by $v\mapsto\Omega(v)(t)$, where Ω is continuous and

$$
\begin{aligned}
\Omega(v)(t)={}&I_q^\alpha v(t)-\frac{a_2}{a_1+a_2}I_q^\alpha v(\delta)+B_1(t,\delta)I_q^{\alpha-p_1}v(\delta) \\
&-B_2(t,\delta)I_q^{\alpha-p_2}v(\delta)-B_3(t,\delta)I_q^{\alpha-p_3}v(\delta),
\end{aligned}
$$

for all t in $\bar{J}$. On the other hand, Ω is a linear continuous map and by applying Lemma 7, we obtain $\Omega \circ S_{T,u}$ is a closed-graph operator. Note that $h_n \in \Omega \circ S_{T,u_n}$ for all n. Since $u_n \to u^*$ and $h_n \to h^*$, there exists $v^* \in S_{T,u^*}$ such that

$$\begin{aligned} h^*(t) &= I_q^\alpha v^*(t) - \frac{a_2}{a_1+a_2} I_q^\alpha v^*(\delta) + B_1(t,\delta) I_q^{\alpha-p_1} v^*(\delta) \\ &\quad - B_2(t,\delta) I_q^{\alpha-p_2} v^*(\delta) - B_3(t,\delta) I_q^{\alpha-p_3} v^*(\delta). \end{aligned}$$

If $0<\kappa<1$ and $u \in \kappa H(x)$, then there exists $v \in S_{T,u}$ such that

$$\begin{aligned} u(t) &= \kappa I_q^\alpha v(t) - \frac{a_2}{a_1+a_2} \kappa I_q^\alpha v(\delta) + B_1(t,\delta) I_q^{\alpha-p_1} v(\delta) \\ &\quad - B_2(t,\delta) \kappa I_q^{\alpha-p_2} v(\delta) - B_3(t,\delta) \kappa I_q^{\alpha-p_3} v(\delta), \end{aligned}$$

for any $t \in \bar{J}$. Hence,

$$\begin{aligned} \|u\| &= \sup_{t\in\bar{J}} \bigl|u(t)\bigr| + \sup_{t\in\bar{J}} \bigl|u'(t)\bigr| + \sup_{t\in\bar{J}} \bigl|u''(t)\bigr| + \sup_{t\in\bar{J}} \bigl|u'''(t)\bigr| \\ &\le \Lambda_2 \Biggl[\sum_{j=1}^{4} \|g_{0j}\|_1 \phi_j\bigl(\|u\|\bigr) + \|g_{05}\|_1 \phi_5 \Biggl({}_0c_1 \gamma_1^0 + \|u\| \gamma_1^0 \Biggl[\Biggl(\sum_{j=1}^{4} {}_1\eta_j \Biggr) \\ &\quad + {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})} \Biggr] \Biggr) \\ &\quad + \|g_{06}\|_1 \phi_6 \Biggl({}_0c_2 \gamma_2^0 + \|u\| \gamma_2^0 \Biggl[\sum_{j=1}^{4} {}_2\eta_j + {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} \\ &\quad + {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})} \Biggr] \Biggr) \\ &\quad + \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j} \Biggl(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} \|u\| \Biggr) + \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j} \Biggl(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \|u\| \Biggr) \\ &\quad + \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j} \Biggl(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} \|u\| \Biggr) \Biggr] \\ &= \Lambda_2 A\bigl(\|u\|\bigr). \end{aligned}$$

Indeed, $\|u\| \le \Lambda_2 A(\|u\|)$. On the other hand, the operator $\Phi : \overline{D} \to P_{cp,c}(\mathcal{X})$ is upper semi-continuous and compact, where $D = \{u \in \mathcal{X} : \|u\| < \Delta\}$. By considering the choice of D, there is no $u \in \partial D$ such that $u \in \kappa H(u)$ for some $\kappa \in (0,1)$ and so H has a fixed point $u \in \overline{D}$ due to Theorem 10. Therefore H satisfies the assumptions of the nonlinear alternative of the Leray–Schauder-type result. It is easy to check that each fixed point of H is a solution of problem (2). This completes the proof. □

In the next case we will show that convex-valued condition of T is not necessary.

Theorem 12 *If T defined on $\bar{J} \times \mathcal{R}^m$ to $P_{cp,c}(\mathbb{R})$ is a multifunction such that the map $(t,x_1,x_2,\dots,x_m) \mapsto T(t,x_1,x_2,\dots,x_m)$ is both $L(\bar{J}) \otimes \mathcal{B}(R)$ measurable and lower semicon-*

tinuous for each $t \in \overline{J}$ *where* $m = 6 + k_1 + k_2 + k_3$ *and* $\mathcal{B}(R) = \bigotimes_{j=1}^{m} B(\mathbb{R})$, *then problem* (2) *has at least one positive solution whenever the assumptions* (1), (2), *and* (3) *in Theorem* 11 *hold*.

Proof By using the assumptions and Lemma 4.1 in [55], we conclude that T is lower semicontinuous. Also, Lemma 8 implies that there exists a continuous function $N : \mathcal{X} \to L^1(\overline{J}, \mathbb{R})$ such that $N(u) \in S_{T,u}$ for all $u \in \mathcal{X}$. Now consider the problem

$$ {}^cD_q^{\alpha} u(t) = N(u)(t) \tag{27}$$

with the boundary conditions (23)–(26). Obviously, each solution of problem (27) is a solution of problem (2). Define the operator $\overline{H} : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned} \overline{H}u(t) &= I_q^{\alpha} N(u)(t) - \frac{a_2}{a_1 + a_2} I_q^{\alpha} N(u)(\delta) + B_1(t,\delta)_q^{\alpha-p_1} N(u)(\delta) \\ &\quad - B_2(t,\delta) I_q^{\alpha-p_2} N(u)(\delta) - B_3(t,\delta) I_q^{\alpha-p_3} N(u)(\delta), \end{aligned}$$

for each $t \in \overline{J}$. Similar to proof of the last result, it can be shown that $\overline{H}$ is continuous, completely continuous, and satisfies all conditions of the nonlinear alternative of Leray–Schauder type for single-valued maps. Again by using a similar argument as for the last result, one can find a solution for problem (27). This completes the proof. □

Theorem 13 *If a multivalued* T *mapping* $\overline{J} \times \mathcal{R}^m$ *into* $P_{cp}(\mathbb{R})$ *is measurable and bounded for each* $t \in \overline{J}$, *then problem* (2) *has at least one solution whenever the following assumptions hold for each* $t, s \in \overline{J}$, ${}_ix_j, {}_ix'_j \in \mathbb{R}$:

(1) *There exist nonnegative functions* $m_{ij} \in L^1(\overline{J})$ *such that*

$$\begin{aligned} &d_H\bigl(T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_0x_6, \\ &\qquad {}_1x_1, {}_1x_2, \ldots, {}_1x_{k_1}, {}_2x_1, {}_2x_2, \ldots, {}_2x_{k_2}, {}_3x_1, {}_3x_2, \ldots, {}_3x_{k_3}), \\ &\qquad T\bigl(t, {}_0x'_1, {}_0x'_2, {}_0x'_3, {}_0x'_4, {}_0x'_5, {}_0x'_6, \\ &\qquad {}_1x'_1, {}_1x'_2, \ldots, {}_1x'_{k_1}, {}_2x'_1, {}_2x'_2, \ldots, {}_2x'_{k_2}, {}_3x'_1, {}_3x'_2, \ldots, {}_3x'_{k_3}\bigr)\bigr) \\ &\quad \le \sum_{j=1}^{6} m_{0j}(t)\bigl|{}_0x_j - {}_0x'_j\bigr| + \sum_{j=1}^{k_1} m_{1j}(t)\bigl|{}_1x_j - {}_1x'_j\bigr| \\ &\qquad + \sum_{j=1}^{k_2} m_{2j}(t)\bigl|{}_2x_j - {}_2x'_j\bigr| + \sum_{j=1}^{k_3} m_{3j}(t)\bigl|{}_3x_j - {}_3x'_j\bigr|. \end{aligned}$$

(2) *There exist* ${}_ic_j \ge 0$ *such that*

$$\begin{aligned} &\bigl|\theta_i(t, s, x_1, x_2, x_3, x_4, x_5, x_6, x_7) - \theta_i\bigl(t, s, x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, x'_7\bigr)\bigr| \\ &\quad \le \sum_{j=1}^{7} {}_ic_j\bigl|x_j - x'_j\bigr|, \end{aligned}$$

for $i = 1, 2$, $x_j, x'j \in \mathbb{R}$ *where* $j \in N_7$ *and*

$$\begin{aligned} \Lambda_2' = \Lambda_2 \Bigg[& \left(\sum_{j=1}^{4} \| m_{0j}(t) \|_1 \right) + \| m_{05}(t) \|_1 \gamma_1^0 \left(\left(\sum_{j=1}^{4} {}_1c_j \right) \right. \\ & \left. + {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})} \right) \\ & + \| m_{06}(t) \|_1 \gamma_2^0 \left(\left(\sum_{j=1}^{4} {}_2c_j \right) \right. \\ & \left. + {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})} \right) \\ & + \sum_{j=1}^{k_1} \| m_{1j}(t) \|_1 \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} \| m_{2j}(t) \|_1 \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\ & + \sum_{j=1}^{k_3} \| m_{3j}(t) \|_1 \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})} \Bigg] \\ < 1. & \end{aligned}$$

Proof By applying the hypothesis and Theorem III-6 (the measurable selection theorem in [52]), T admits a measurable selection $v : \bar{J} \to \mathbb{R}$. Since T is integrable and bounded, $v \in L^1(\bar{J}, \mathbb{R})$ and so $S_{T,u} \neq \emptyset$ for each $u \in \mathcal{X}$. We claim that the operator H satisfies the assumptions of Theorem 9. In this case, we prove that $H(u) \in P_{cl}(\mathcal{X})$ for any $u \in \mathcal{X}$. In this case, consider the sequence $\{u_n\} \subset H(u)$ such that $u_n \to u^*$ for some $u^* \in \mathcal{X}$. For each n, choose $w_n \in S_{T,u}$ such that

$$\begin{aligned} u_n(t) = {} & I_q^\alpha w_n(t) - \frac{a_2}{a_1 + a_2} I_q^\alpha w_n(\delta) + B_1(t,\delta) I_q^{\alpha - p_1} w_n(\delta) \\ & - B_2(t,\delta) I_q^{\alpha - p_2} w_n(\delta) - B_3(t,\delta) I_q^{\alpha - p_3} w_n(\delta), \end{aligned}$$

for all $t \in \bar{I}$. Hence, there exists a subsequence of $\{w_n\}$ that converges to w in $L^1(\bar{I}, \mathbb{R})$, because T has compact values. We denote this subsequence again by $\{w_n\}$. Thus, $w \in S_{T,u}$ and $u_n(t)$ tends to $u^*(t)$, where

$$\begin{aligned} u^*(t) = {} & I_q^\alpha w(t) - \frac{a_2}{a_1 + a_2} I_q^\alpha w(\delta) + B_1(t,\delta) I_q^{\alpha - p_1} w(\delta) \\ & - B_2(t,\delta) I_q^{\alpha - p_2} w(\delta) - B_3(t,\delta) I_q^{\alpha - p_3} w(\delta), \end{aligned}$$

for any $t \in \bar{I}$. Indeed, $u^* \in H(u)$. Now, we show that there exists $\Lambda_2' < 1$ such that $d_H(H(v), H(\tilde{v})) \leq \Lambda_2' \|v - \tilde{v}\|$, for all $v, \tilde{v} \in \mathcal{X}$. Let $v, \tilde{v} \in \mathcal{X}$ and $h_1 \in H(v)$. Choose v_1 belonging to $S_{T,v}$ such that

$$\begin{aligned} h_1(t) = {} & I_q^\alpha v_1(t) - \frac{a_2}{a_1 + a_2} I_q^\alpha v_1(\delta) + B_1(t,\delta) I_q^{\alpha - p_1} v_1(\delta) \\ & - B_2(t,\delta) I_q^{\alpha - p_2} v_1(\delta) - B_3(t,\delta) I_q^{\alpha - p_3} v_1(\delta), \end{aligned}$$

for almost all $t \in \overline{I}$. On the other hand, we get

$$\begin{aligned} d_H\big(\widetilde{T}\big(t,v(t)\big),\widetilde{T}\big(t,\tilde{v}(t)\big)\big) \le \Bigg[&\Bigg(\sum_{j=1}^{4}\big\|m_{0j}(t)\big\|_1\Bigg) + \big\|m_{05}(t)\big\|_1\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4}{}_1c_j\Bigg) \\ &+ {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} \\ &+ {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) + \big\|m_{06}(t)\big\|_1\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4}{}_2c_j\Bigg) \\ &+ {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} \\ &+ {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) + \sum_{j=1}^{k_1}\big\|m_{1j}(t)\big\|_1\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} \\ &+ \sum_{j=1}^{k_2}\big\|m_{2j}(t)\big\|_1\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\ &+ \sum_{j=1}^{k_3}\big\|m_{3j}(t)\big\|_1\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg]\|v-\tilde{v}\|, \end{aligned}$$

for each $t \in \overline{J}$. Hence, there exists $f_t \in \widetilde{T}(t,\tilde{v}(t))$ such that $|v_1(t) - f_t| < \Lambda'_t$, where

$$\begin{aligned} \Lambda'_t = \Bigg[&\Bigg(\sum_{j=1}^{4}\big\|m_{0j}(t)\big\|_1\Bigg) + \big\|m_{05}(t)\big\|_1\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4}{}_1c_j\Bigg) \\ &+ {}_1c_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg) \\ &+ \big\|m_{06}(t)\big\|_1\gamma_2^0\Bigg(\Bigg(\sum_{j=1}^{4}{}_2c_j\Bigg) \\ &+ {}_2c_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg) \\ &+ \sum_{j=1}^{k_1}\big\|m_{1j}(t)\big\|_1\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2}\big\|m_{2j}(t)\big\|_1\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})} \\ &+ \sum_{j=1}^{k_3}\big\|m_{3j}(t)\big\|_1\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg]\|v-\tilde{v}\|, \end{aligned}$$

for almost all $t \in \overline{J}$. Define $N : \overline{J} \to P(\mathbb{R})$ by $N(t) = \{x \in \mathbb{R} : |v_1(t) - x| \le \Lambda'_t\}$ for all $t \in \overline{J}$. By employing Theorem III-41 in [52], we get that N is measurable. Since the multivalued operator $t \mapsto N(t) \cap \widetilde{T}(t,\tilde{v}(t))$ is measurable (Proposition III-4 in [52]), there exists a function $v_2 \in S_{F,\tilde{z}}$ such that

$$\big|v_1(t) - v_2(t)\big| \le \Bigg[\Bigg(\sum_{j=1}^{4}\big\|m_{0j}(t)\big\|_1\Bigg) + \big\|m_{05}(t)\big\|_1\gamma_1^0\Bigg(\Bigg(\sum_{j=1}^{4}{}_1c_j\Bigg)$$

$$+ {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg)$$
$$+ \|m_{06}(t)\|_1 \gamma_2^0 \Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg)$$
$$+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg)$$
$$+ \sum_{j=1}^{k_1} \|m_{1j}(t)\|_1 \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} \|m_{2j}(t)\|_1 \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}$$
$$+ \sum_{j=1}^{k_3} \|m_{3j}(t)\|_1 \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg] \|\nu - \tilde{\nu}\|,$$

for almost all $t \in \overline{J}$. Define

$$h_2(t) = I_q^{\alpha} \nu_2(t) - \frac{a_2}{a_1 + a_2} I_q^{\alpha} \nu_2(\delta) + B_1(t,\delta) I_q^{\alpha - p_1} \nu_2(\delta)$$
$$- B_2(t,\delta) I_q^{\alpha - p_2} \nu_2(\delta) - B_3(t,\delta) I_q^{\alpha - p_3} \nu_2(\delta),$$

for all $t \in \overline{J}$. Then, we have

$$\big|h_1(t) - h_2(t)\big| = \sup_{t\in\overline{J}} \big|h_1(t) - h_2(t)\big| + \sup_{t\in\overline{J}} \big|h_1'(t) - h_2'(t)\big|$$
$$+ \sup_{t\in\overline{J}} \big|h_1''(t) - h_2''(t)\big| + \sup_{t\in\overline{J}} \big|h_1'''(t) - h_2'''(t)\big|$$
$$\leq \Lambda_2 \Bigg[\Bigg(\sum_{j=1}^{4} \|m_{0j}(t)\|_1\Bigg) + \|m_{05}(t)\|_1 \gamma_1^0 \Bigg(\Bigg(\sum_{j=1}^{4} {}_1c_j\Bigg)$$
$$+ {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg)$$
$$+ \|m_{06}(t)\|_1 \gamma_2^0 \Bigg(\Bigg(\sum_{j=1}^{4} {}_2c_j\Bigg)$$
$$+ {}_2c_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2c_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2c_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg)$$
$$+ \sum_{j=1}^{k_1} \|m_{1j}(t)\|_1 \frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})} + \sum_{j=1}^{k_2} \|m_{2j}(t)\|_1 \frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}$$
$$+ \sum_{j=1}^{k_3} \|m_{3j}(t)\|_1 \frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Bigg] \|\nu - \tilde{\nu}\|$$
$$= \Lambda_2' \|\nu - \tilde{\nu}\|.$$

By interchanging the roles of v and $\tilde{v}$, we get $d_H(H(v),H(\tilde{v})) \le \Lambda_2' \|v - \tilde{v}\|$. Since $\Lambda_2' < 1$, H is a contraction and so by using Theorem 9, H has a fixed point. It is easy to check that each fixed point of H is a solution of problem (2). □

4 Examples and numerical check technique for the problems

In this part, we give complete computational techniques for checking of the existence of solutions for the inclusion problem (1) in Theorems 11, 13, which cover all similar problems and present numerical examples for solving perfectly. Foremost, we show that a simplified analysis can be executed to calculate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in the summation. To this aim, we consider a pseudo-code description of the method for calculating q-Gamma function of order n in Algorithm 2 (for more details, see https://en.wikipedia.org/wiki/Q-gamma_function).

Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with smaller values of n. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values of q near 1 is obtained with more values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column, and line 29 of the third column of Table 2. Also, Table 3 is the same as Table 2, but x values increase in Table 3.

Note that all routines are written in MATLAB software with the variable Digits set to 16 (This environment variable controls the number of digits in MATLAB) and work on a PC with 2.90 GHz of Core 2 CPU and 4 GB of RAM. Furthermore, we provided Algorithm 3 which calculates $(D_q^{\alpha} f)(x)$.

Here, we give two examples to illustrate the inclusion problems (2) in Theorems 11 and 13.

Example 1 Consider the fractional q-differential inclusion

$$\begin{aligned} {}^cD_q^{\frac{16}{3}} u(t) \in T\big(&t, u(t), u'(t), u''(t), u'''(t), \varphi_1 u(t), \\ &{}^cD_q^{\frac{3}{4}} u(t), {}^cD_q^{\frac{7}{5}} u(t), {}^cD_q^{\frac{5}{2}} u(t), {}^cD_q^{\frac{8}{3}} u(t)\big), \end{aligned} \tag{28}$$

for $t \in \overline{J} = [0,1]$ ($\delta = 1$), with the conditions $u^{(4)}(0) = u^{(5)}(0) = 0$, $\frac{1}{4}u(0) + \frac{2}{3}u(1) = 0$ and

$$ {}^cD_q^{\frac{1}{5}} u(0) = -{}^cD_q^{\frac{1}{5}} u(1), \qquad {}^cD_q^{\frac{5}{3}} u(0) = -{}^cD_q^{\frac{5}{3}} u(1), \qquad {}^cD_q^{\frac{15}{7}} u(0) = -{}^cD_q^{\frac{15}{7}} u(1), $$

Table 1 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \dots, 15$ of Algorithm 2

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	<u>2.340263</u>	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	<u>11.257095</u>	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	<u>64.350881</u>
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2 Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

Table 3 Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	11.257095	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	259.967394
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

where

$$\begin{aligned}
\varphi_1 u(t) = \int_0^t \frac{e^{-(s-t)/2}}{50} \Bigg[& \frac{e^{-2\pi t}}{8(1+t^2)} + \frac{3u(s)}{514(1+t)(2+\sin(u(s)))} \\
&+ \frac{7e^{-st} u'(s)}{257(s^2+4)} + \frac{\sqrt[3]{\pi}\, u'(s) u''(s)}{771(1+|u'(s)|)} + \frac{4e^{-\cos^2(u(s))} u'''(s)}{1285(s^2+1)} \\
&+ \frac{\sin(u(s)) {}^c D_q^{\frac{1}{7}} u(s)}{1799\sqrt{1 + |{}^c D_q^{\frac{1}{7}} u(s)| + |{}^c D_q^{\frac{11}{6}} u(s)|}} + \frac{e^{-2\pi} \cos^2(u(s)) {}^c D_q^{\frac{11}{6}} u(s)}{2827(s^2+2s+1)} \\
&+ \frac{{}^c D_q^{\frac{16}{7}} u(s)}{4626(1+|u'(s)|)} \Bigg] \, ds.
\end{aligned}$$

Put $\alpha = \frac{16}{3} \in (5,6]$, when $n = 6$,

$$\beta_{ij} = \begin{bmatrix} \frac{3}{4} & 0 \\ \frac{7}{5} & 0 \\ \frac{5}{2} & \frac{8}{3} \end{bmatrix}, \qquad p_i = \begin{bmatrix} \frac{1}{5} \\ \frac{5}{3} \\ \frac{15}{7} \end{bmatrix}, \qquad \gamma_{1j} = \begin{bmatrix} \frac{1}{7} \\ \frac{11}{6} \\ \frac{16}{7} \end{bmatrix},$$

${}_0c_1 = \frac{10}{17} \in (0,\infty)$,

$${}_1\eta_j = \begin{bmatrix} \frac{3}{514} & \frac{7}{1028} & \frac{\sqrt[3]{\pi}}{771} & \frac{4}{1285} & \frac{1}{1799} & \frac{e^{-2\pi}}{2827} & \frac{2}{4626} \end{bmatrix},$$

where each ${}_1\eta_i$ in $[0,\infty)$ and $\gamma_1^0 = \frac{\sqrt{e}-1}{25}$. Define the multifunction $T : \bar{J} \times \mathbb{R}^9 \to P(\mathbb{R})$ by

$$T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_2x_1, {}_3x_1, {}_3x_2) = \{y \in \mathbb{R} : B_1 \le y \le B_2\},$$

where

$$\begin{aligned} B_1 &= \frac{|{}_0x_1|^3}{4(3 + |{}_0x_1|^3)} - \frac{e^{-\pi t}}{5}\sin^2({}_0x_2) - \frac{|{}_0x_5|}{13(4 + \sin^2({}_0x_1))^2} \\ &\quad - \frac{9|{}_1x_1|}{10(4 + |{}_1x_1|)} - \frac{t^3}{9}\cos^2({}_2x_1) - \frac{e^{-\pi t^2}}{81(t^4+3)}|{}_3x_1| - \frac{t^{\frac{1}{3}}}{20(1 + |{}_3x_2|)}, \\ B_2 &= \frac{1}{3}e^{-|{}_0x_1|} + \frac{7e^{-\pi t^2}}{18(1 + ({}_0x_2)^2 t)} + \frac{|{}_0x_4|}{10(1 + |{}_0x_4|)} + \frac{t^{\frac{3}{2}}|{}_0x_5 + \sin({}_0x_5)|}{119} \\ &\quad + \frac{e^{-t}}{t^4+2}\sin^4({}_1x_1) + \frac{e^{-\frac{3}{2}t}}{5\sqrt{1 + |{}_2x_1|^{\frac{5}{2}}}} + \frac{t}{135(t^2+1)}\left|{}_3x_1 + \frac{{}_0x_3}{1 + |{}_0x_3|}\right| \\ &\quad + \frac{31|{}_3x_2|^3}{140(1 + |{}_3x_2|^3)} + \frac{3}{2}. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} &\bigl\| T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_2x_1, {}_3x_1, {}_3x_2) \bigr\|_p \\ &\quad = \sup\bigl\{|\nu| : \nu \in T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_2x_1, {}_3x_1, {}_3x_2)\bigr\} \\ &\quad \le \frac{4087}{1260} + \frac{1}{119}\bigl(|{}_0x_5| + 1\bigr) + \frac{1}{135}\bigl(|{}_3x_1| + 1\bigr), \end{aligned}$$

for all $t \in \bar{J}$ and ${}_ix_j \in \mathbb{R}$. It is obvious that T has convex and compact values and is of Carathéodory type. Put $g_{0j}(t) = 1$ here $j \in N_5$, $\phi_1({}_0x_1) = \frac{4}{5}$, $\phi_2({}_0x_2) = \frac{1}{2}$, $\phi_3({}_0x_3) = \frac{1}{5}$, $\phi_4({}_0x_4) = \frac{1}{2}$, $\phi_5({}_0x_5) = \frac{1}{119}(|{}_0x_5| + 1)$, and

$$g_{ij}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \psi_{ij}({}_ix_j) = \begin{bmatrix} \frac{8}{9} & 0 \\ \frac{1}{60} & 0 \\ \frac{1}{135}(|{}_ix_j| + 1) & \frac{31}{140} \end{bmatrix},$$

for each $t \in \bar{J}$ and ${}_ix_j \in \mathbb{R}$. Hence,

$$\begin{aligned} &\bigl\| T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_2x_1, {}_3x_1, {}_3x_2) \bigr\|_p \\ &\quad = \sup\bigl\{|\nu| : \nu \in T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_2x_1, {}_3x_1, {}_3x_2)\bigr\} \end{aligned}$$

$$\begin{aligned}&\leq \left(\sum_{j=1}^{5} g_{0j}(t)\phi_j\big(|_0x_j|\big)\right) + g_{11}(t)\psi_{11}\big(|_1x_1|\big) + g_{21}(t)\psi_{21}\big(|_2x_1|\big)\\&\quad + g_{31}(t)\psi_{31}\big(|_3x_1|\big) + g_{32}(t)\psi_{32}\big(|_3x_2|\big)\end{aligned}$$

for all $t \in \overline{J}$ and ${}_ix_j \in \mathbb{R}$. According to data values of problem (28), we have

$$\begin{aligned}\Lambda_2 &= \frac{19}{11\Gamma_q(\frac{16}{13})} + \frac{19\Gamma_q(\frac{9}{5})}{11\Gamma_q(\frac{77}{15})} + \frac{106\Gamma_q(\frac{4}{3})}{99\Gamma_q(\frac{11}{3})} + \frac{215\Gamma_q(\frac{13}{7})}{693\Gamma_q(\frac{67}{21})} + \frac{383\Gamma_q(\frac{13}{7})}{504\Gamma_q(6721)}\\&\quad + \frac{1}{\Gamma_q(\frac{13}{3})} + \frac{\Gamma_q(\frac{9}{5})}{\Gamma_q(\frac{77}{15})} + \frac{14\Gamma_q(\frac{4}{3})}{3\Gamma_q(\frac{11}{3})} + \frac{185\Gamma_q(\frac{7}{4})}{126\Gamma_q(\frac{67}{21})}\\&\quad + \frac{1}{\Gamma_q(\frac{10}{3})} + \frac{\frac{4}{3}}{\Gamma_q(\frac{11}{3})} + \frac{7\Gamma_q(\frac{13}{7})}{4\Gamma_q(\frac{67}{21})} + \frac{1}{\Gamma_q(\frac{7}{3})} + \frac{\Gamma_q(\frac{13}{7})}{\Gamma_q(\frac{67}{21})}.\end{aligned}$$

Table 4 shows the some numerical values of Λ_2 from Eq. (17), for five examples of $q \in \frac{1}{8}, \frac{1}{5}, \frac{1}{2}, \frac{3}{4}, \frac{8}{9}$ which yield 12.891036, 11.28628, 7.188979, 5.412507, 4.758008, respectively, which that have been shown by underlined rows. On the other hand,

$$\begin{aligned}A(\Delta) &= \left(\sum_{j=1}^{4} \|g_{0j}\|_1\phi_j(\Delta)\right) + \|g_{05}\|_1\phi_5\Bigg({}_0c_1\gamma_1^0 + \Delta\gamma_1^0\Bigg[\left(\sum_{j=1}^{4} {}_1\eta_j\right)\\&\quad + {}_1\eta_5\frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6\frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7\frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg]\Bigg)\\&\quad + \|g_{06}\|_1\phi_6\Bigg({}_0c_2\gamma_2^0 + \Delta\gamma_2^0\Bigg[\left(\sum_{j=1}^{4} {}_2\eta_j\right)\\&\quad + {}_2\eta_5\frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2\eta_6\frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7\frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg)\\&\quad + \sum_{j=1}^{k_1} \|g_{1j}\|_1\psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\Delta\right) + \sum_{j=1}^{k_2} \|g_{2j}\|_1\psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}\Delta\right)\\&\quad + \sum_{j=1}^{k_3} \|g_{3j}\|_1\psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Delta\right).\end{aligned}$$

With the right choice for Δ from Eq. (16), the conditions of Theorem 11 hold and so problem (28) has at least one solution.

Example 2 Consider the fractional differential inclusion

$$\begin{aligned}{}^cD_q^{\frac{16}{3}} u(t) \in T\big(&t, u(t), u'(t), u''(t), u'''(t), \varphi_1 u(t),\\&{}^cD_q^{\frac{1}{19}} u(t), {}^cD_q^{\frac{2}{15}} u(t), {}^cD_q^{\frac{20}{17}} u(t), {}^cD_q^{\frac{11}{5}} u(t)\big),\end{aligned} \tag{29}$$

for $t \in [0,1]$, with boundary value conditions $u^{(4)}(0) = u^{(5)}(0) = 0$, $u(0) - 3u(1) = 0$ and

$${}^cD_q^{\frac{3}{20}} u(0) = -{}^cD_q^{\frac{3}{20}} u(1), \qquad {}^cD_q^{\frac{15}{14}} u(0) = -{}^cD_q^{\frac{15}{14}} u(1), \qquad {}^cD_q^{\frac{19}{9}} u(0) = -{}^cD_q^{\frac{19}{9}} u(1),$$

Table 4 Some numerical results of Λ_2 in Example 1 for $q \in \{\frac{1}{8}, \frac{1}{5}, \frac{1}{2}, \frac{3}{4}, \frac{8}{9}\}$ which is given by Algorithm 5

n	$\frac{1}{8}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{8}{9}$
1	12.877066	11.226759	6.254225	2.677980	0.916173
2	12.889290	11.274367	6.713328	3.270871	1.205723
3	12.890818	11.283897	6.949071	3.754125	1.490385
4	12.891009	11.285804	7.068503	4.138890	1.765248
5	12.891033	11.286185	7.128610	4.440300	2.027099
6	<u>12.891036</u>	11.286261	7.158762	4.673700	2.273948
7	12.891036	11.286277	7.173862	4.852932	2.504695
8	12.891036	<u>11.286280</u>	7.181419	4.989733	2.718900
9	12.891036	11.286280	7.185198	5.093679	2.916602
⋮	⋮	⋮	⋮	⋮	⋮
21	12.891036	11.286280	7.188978	5.402246	4.265943
22	12.891036	11.286280	<u>7.188979</u>	5.404810	4.319178
23	12.891036	11.286280	7.188979	5.406734	4.366798
⋮	⋮	⋮	⋮	⋮	⋮
54	12.891036	11.28628	7.188979	5.412506	4.747670
55	12.891036	11.28628	7.188979	<u>5.412507</u>	4.748823
56	12.891036	11.28628	7.188979	5.412507	4.749849
⋮	⋮	⋮	⋮	⋮	⋮
96	12.891036	11.28628	7.188979	5.412507	4.757986
97	12.891036	11.28628	7.188979	5.412507	4.757994
98	12.891036	11.28628	7.188979	5.412507	4.758001
99	12.891036	11.28628	7.188979	5.412507	<u>4.758008</u>
100	12.891036	11.28628	7.188979	5.412507	4.758013

where

$$\begin{aligned}\varphi_1 u(t) = {}& \int_0^t \frac{(s-t)^2 e^{-(s-t)^3}}{1350} \left[\frac{t^3 \cos^2 t}{e^{\pi t}(1+t^2)} + \frac{|u(s) + u'(s) + {}^cD_q^{\frac{1}{12}} u(s)|}{9416\pi (1 + |u(s) + u'(s) + {}^cD_q^{\frac{1}{12}} u(s)|)} \right. \\ & + \frac{t^3 \sin^2 t \cos s}{759(36\sqrt{\pi} + e^{3s})} \arctan\left(\frac{3}{2} + \frac{|u''(s) + {}^cD_q^{\frac{7}{4}} u(s)|}{1 + |u''(s) + {}^cD_q^{\frac{7}{4}} u(s)|} \right) \\ & \left. + \frac{e^{st}}{8190(1+e^{st})} \cos u'''(s) + \frac{s^2 e^{-\pi s^3} |{}^cD_q^{\frac{41}{20}} u(s)|}{(1200 + \arcsin(\frac{1}{3}) e^{3t^2})(1 + |{}^cD_q^{\frac{41}{20}} u(s)|)} \right] ds.\end{aligned}$$

Put $\alpha = \frac{16}{3}$,

$$\beta_{ij} = \begin{bmatrix} \frac{1}{19} & \frac{2}{15} \\ \frac{20}{17} & 0 \\ \frac{11}{5} & 0 \end{bmatrix}, \qquad p_j = \begin{bmatrix} \frac{3}{20} \\ \frac{15}{14} \\ \frac{19}{9} \end{bmatrix}, \qquad \gamma_{1j} = \begin{bmatrix} \frac{1}{12} \\ \frac{7}{4} \\ \frac{41}{20} \end{bmatrix},$$

$a_1 = 1$, $a_2 = -3$ ($a_1 + a_2 \neq 0$), $\gamma_1^0 = \frac{e-1}{4050}$,

$${}_i c_j = \begin{bmatrix} \frac{1}{9416\pi} & \frac{1}{9416\pi} & \frac{1}{759(36\sqrt{\pi}+1)} & \frac{1}{8190} & \frac{1}{9416\pi} & \frac{1}{759(36\sqrt{\pi}+1)} & \frac{1}{1200+\arcsin(\frac{1}{3})} \end{bmatrix}.$$

Algorithm 5 The proposed method for calculating Λ_2

```
Input: q, n, α, p, δ, a
L1 = zeros(n, 1);
for m = 1 to n do
    L1(m, 1) = (abs(a(1)) + 2 * abs(a(2)))./(abs(a(1) + a(2)) * qGamma(q, α, m));
    L1(m, 2) = ((abs(a(1)) + 2 * abs(a(2))) * qGamma(q, 2 − p(1), m))./(abs(a(1) + a(2))
                    *qGamma(q, α − p(1), m));
    L1(m, 3) = ((abs(a(2)) * p(1) + abs(a(1) + a(2)) * (4 − p(1))) * qGamma(q, 3 − p(2), m))
                    ./(2 * abs(a(1) + a(2)) * (2 − p(1)) * qGamma(q, α − p(2), m));
    L1(m, 4) = (abs(a(2)) * (6 * (p(2) − p(1)) + (2 − p(1)) * (3 − p(1)) * p(2)) * qGamma(q, 4 − p(3), m))
                    ./(6 * abs(a(1) + a(2)) * (2 − p(1)) * (3 − p(1)) * (3 − p(2)) * qGamma(q, α − p(3), m));
    L1(m, 5) = (6 * (p(2) − p(1)) + (2 − p(1)) * (3 − p(1)) * (6 − p(2)) * qGamma(q, 4 − p(3), m))
                    ./(6 * (2 − p(1)) * (3 − p(1)) * (3 − p(2)) * qGamma(q, α − p(3), m));
end for
s = zeros(n, 1);
for m = 1 to n do
    for j = 1 to 5 do
        s(m, 1) = s(m, 1) + L1(m, j);
    end for
end for
s = δ^(α−1) * s;
for m = 1 to n do
    L1(m, 6) = 1./qGamma(q, α − 1, m) + qGamma(q, 2 − p(1), m)
                    ./qGamma(q, α − p(1), m) + ((3 − p(1)) * qGamma(q, 3 − p(2), m))./((2 − p(1))
                    *qGamma(q, α − p(2), m)) + ((2 * (p(2) − p(1)) + (2 − p(1)) * (3 − p(1)) * (5 − p(2)))
                    *qGamma(q, 4 − p(3), m))./(2 * (2 − p(1)) * (3 − p(1)) * (3 − p(2)) * qGamma(q, α − p(3), m));
end for
for m = 1 to n do
    s(m, 1) = s(m, 1) + δ^(α−2) * L1(m, 6);
end for
for m = 1 to n do
    L1(m, 7) = 1./qGamma(q, alpha − 2, m) + qGamma(q, 3 − p(2), m)./qGamma(q, α − p(2), m)
                    +((4 − p(2)) * qGamma(q, 4 − p(3), m))./((3 − p(2)) * qGamma(q, α − p(3), m));
end for
for m = 1 to n do
    s(m, 1) = s(m, 1) + δ^(α−3) * L1(m, 7);
end for
for m = 1 to n do
    L1(m, 8) = 1./qGamma(q, alpha − 3, m) + qGamma(q, 4 − p(3), m)./qGamma(q, α − p(3), m);
end for
for m = 1 to n do
    s(m, 1) = s(m, 1) + δ^(α−4) * L1(m, 8);
end for
Λ2 ← s;
Output: Λ2
```

Note that the variables p, a in Algorithm 5 are matrices.

Define the multifunction $T : [0,1] \times \mathbb{R}^9 \to P(\mathbb{R})$ by

$$
\begin{aligned}
&T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_1x_2, {}_2x_1, {}_3x_1) \\
&\quad = \bigl[Q_1\bigl(t, {}_ix_j, {}_ix_j'\bigr), Q_2\bigl(t, {}_ix_j, {}_ix_j'\bigr)\bigr] \cup \bigl[Q_3\bigl(t, {}_ix_j, {}_ix_j'\bigr), Q_4\bigl(t, {}_ix_j, {}_ix_j'\bigr)\bigr] \\
&\qquad \cup \bigl[Q_5\bigl(t, {}_ix_j, {}_ix_j'\bigr), Q_6\bigl(t, {}_ix_j, {}_ix_j'\bigr)\bigr],
\end{aligned}
$$

for all $t \in [0,1]$ and ${}_ix_j \in \mathbb{R}$, where

$$
\begin{aligned}
Q_1\bigl(t, {}_ix_j, {}_ix_j'\bigr) = {} & -\frac{e^{-\pi t}}{1+t^2} \\
& - \arctan\left(1 + \frac{e^t |{}_0x_1 + {}_0x_2 + {}_0x_3 + {}_0x_4|}{9600(\frac{1}{3} + e^t)(1 + |{}_0x_1 + {}_0x_2 + {}_0x_3 + {}_0x_4|)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&Q_2\big(t, {}_i x_j, {}_i x_j'\big) = \cos^2 t,\\
&Q_3\big(t, {}_i x_j, {}_i x_j'\big) = 7 + \frac{\sin \pi t}{\sqrt{2 + t^3}},\\
&Q_4\big(t, {}_i x_j, {}_i x_j'\big) = 20\big(t^3 + 1\big) + \cos\left(t^3 + \frac{e^t|{}_1x_1 + {}_1x_2|}{8719\pi(1 + |{}_1x_1 + {}_1x_2|)}\right)\\
&\qquad + \frac{|{}_2x_1 + {}_3x_1|}{5170(9+t)^4(1 + |{}_2x_1 + {}_3x_1|)},\\
&Q_5\big(t, {}_i x_j, {}_i x_j'\big) = \frac{3}{2},\\
&Q_6\big(t, {}_i x_j, {}_i x_j'\big) = \frac{|{}_0x_5|}{7491(t+25)^5(1 + |{}_0x_5|)} + t^2 + \frac{5}{2}.
\end{aligned}
$$

It is clear that T has compact values. From the above assumptions, we have

$$
\begin{aligned}
&d_H\big(T(t, {}_0x_1, {}_0x_2, {}_0x_3, {}_0x_4, {}_0x_5, {}_1x_1, {}_1x_2, {}_2x_1, {}_3x_1),\\
&\quad T\big(t, {}_0x_1', {}_0x_2', {}_0x_3', {}_0x_4', {}_0x_5', {}_1x_1', {}_1x_2', {}_2x_1', {}_3x_1'\big)\big)\\
&\quad \le \frac{e^t}{9600(\frac{1}{3} + e^t)} \sum_{j=1}^{4} \big|{}_0x_j - {}_0x_j'\big| + \frac{1}{7491(t+25)^5}\big|{}_0x_5 - {}_0x_5'\big|\\
&\qquad + \frac{e^t}{8719\pi}\big(\big|{}_1x_1 - {}_1x_1'\big| + \big|{}_1x_2 - {}_1x_2'\big|\big)\\
&\qquad + \frac{1}{5170(9+t)^4}\big(\big|{}_2x_1 - {}_2x_1'\big| + \big|{}_3x_1 - {}_3x_1'\big|\big),
\end{aligned}
$$

for all $t \in [0,1]$ and ${}_ix_j \in \mathbb{R}$. According to data values of problem (29), we have

$$
\begin{aligned}
\Lambda_2 &= \frac{3}{\Gamma_q(\frac{19}{3})} + \frac{3\Gamma_q(\frac{27}{16})}{\Gamma_q(\frac{289}{48})} + \frac{101\Gamma_q(\frac{18}{11})}{81\Gamma_q(\frac{194}{33})} + \frac{39{,}275\Gamma_q(\frac{16}{9})}{125{,}388\Gamma_q(\frac{-275}{144})} + \frac{-13{,}817\Gamma_q(\frac{16}{9})}{37{,}152\Gamma_q(\frac{37}{9})}\\
&\quad + \frac{1}{\Gamma_q(\frac{16}{3})} + \frac{\Gamma_q(\frac{27}{16})}{\Gamma_q(\frac{299}{16})} + \frac{43\Gamma_q(\frac{18}{11})}{27\Gamma_q(\frac{194}{33})} + \frac{430\Gamma_q(\frac{16}{9})}{387\Gamma_q(\frac{37}{9})}\\
&\quad + \frac{1}{\Gamma_q(\frac{13}{3})} + \frac{\Gamma_q(\frac{18}{11})}{\Gamma_q(\frac{164}{33})} + \frac{29\Gamma_q(\frac{16}{9})}{18\Gamma_q(\frac{37}{9})} + \frac{1}{\Gamma_q(\frac{10}{3})} + \frac{\Gamma_q(\frac{16}{9})}{\Gamma_q(\frac{37}{9})}.
\end{aligned}
$$

Table 5 shows the some numerical values of Λ_2 from Eq. (17), for five examples of $q = \frac{1}{8}, \frac{1}{5}, \frac{1}{2}, \frac{3}{4}, \frac{8}{9}$ that yield 15.400781, 11.914180, 4.521162, 2.316636, 1.704692, respectively, which have been shown by the underlined rows. Also

$$
\begin{aligned}
A(\Delta) &= \left(\sum_{j=1}^{4} \|g_{0j}\|_1 \phi_j(D)\right) + \|g_{05}\|_1 \phi_5 \left({}_0c_1 \gamma_1^0 + \Delta \gamma_1^0 \left[\left(\sum_{j=1}^{4} {}_1\eta_j\right)\right.\right.\\
&\quad \left.\left. + {}_1\eta_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1\eta_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1\eta_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\right]\right)\\
&\quad + \|g_{06}\|_1 \phi_6 \left({}_0c_2 \gamma_2^0 + \Delta \gamma_2^0 \left[\left(\sum_{j=1}^{4} {}_2\eta_j\right)\right.\right.
\end{aligned}
$$

Table 5 Some numerical results of $\varLambda_2$ in Example 2 for $q \in \{\frac{1}{8}, \frac{1}{5}, \frac{1}{2}, \frac{3}{4}, \frac{8}{9}\}$ which is given by Algorithm 5

n	$\frac{1}{8}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{8}{9}$
1	15.383428	11.848048	3.825547	0.882238	0.148607
2	15.398612	11.900941	4.164294	1.159149	0.222799
3	15.400510	11.911531	4.340431	1.400094	0.304966
4	15.400747	11.913650	4.430219	1.600908	0.392048
5	15.400777	11.914074	4.475546	1.763441	0.481449
6	<u>15.400781</u>	11.914158	4.498318	1.892313	0.571048
7	15.400781	11.914175	4.509731	1.993005	0.659175
8	15.400781	11.914179	4.515444	2.070845	0.744560
9	15.400781	11.914179	4.518303	2.130552	0.826279
10	15.400781	<u>11.914180</u>	4.519733	2.176087	0.903699
11	15.400781	11.914180	4.520447	2.210667	0.976425
⋮	⋮	⋮	⋮	⋮	⋮
20	15.400781	11.91418	4.521161	2.308563	1.421107
21	15.400781	11.91418	<u>4.521162</u>	2.310579	1.450892
22	15.400781	11.91418	4.521162	2.312093	1.477722
⋮	⋮	⋮	⋮	⋮	⋮
50	15.400781	11.91418	4.521162	2.316635	1.695907
51	15.400781	11.91418	4.521162	<u>2.316636</u>	1.696882
52	15.400781	11.91418	4.521162	2.316636	1.697749
⋮	⋮	⋮	⋮	⋮	⋮
117	15.400781	11.91418	4.521162	2.316637	1.704691
118	15.400781	11.91418	4.521162	2.316637	<u>1.704692</u>
119	15.400781	11.91418	4.521162	2.316637	1.704692
120	15.400781	11.91418	4.521162	2.316637	1.704692

$$+ {}_2\eta_5 \frac{\delta^{1-\gamma_{21}}}{\Gamma_q(2-\gamma_{21})} + {}_2\eta_6 \frac{\delta^{2-\gamma_{22}}}{\Gamma_q(3-\gamma_{22})} + {}_2\eta_7 \frac{\delta^{3-\gamma_{23}}}{\Gamma_q(4-\gamma_{23})}\Bigg]\Bigg)$$
$$+ \sum_{j=1}^{k_1} \|g_{1j}\|_1 \psi_{1j}\left(\frac{\delta^{1-\beta_{1j}}}{\Gamma_q(2-\beta_{1j})}\Delta\right) + \sum_{j=1}^{k_2} \|g_{2j}\|_1 \psi_{2j}\left(\frac{\delta^{2-\beta_{2j}}}{\Gamma_q(3-\beta_{2j})}\Delta\right)$$
$$+ \sum_{j=1}^{k_3} \|g_{3j}\|_1 \psi_{3j}\left(\frac{\delta^{3-\beta_{3j}}}{\Gamma_q(4-\beta_{3j})}\Delta\right).$$

Put

$$m_{0j}(t) = \begin{bmatrix} \frac{e^t}{9600(\frac{1}{3}+e^t)} & \frac{e^t}{9600(\frac{1}{3}+e^t)} & \frac{e^t}{9600(\frac{1}{3}+e^t)} & \frac{e^t}{9600(\frac{1}{3}+e^t)} & \frac{1}{7491(t+25)^5} \end{bmatrix},$$
$$m_{ij}(t) = \begin{bmatrix} \frac{e^t}{8719\pi} & \frac{e^t}{8719\pi} \\ \frac{1}{5170(9+t)^4} & 0 \\ \frac{1}{5170(9+t)^4} & 0 \end{bmatrix}.$$

With the right choice Δ, we get

$$\varLambda_2' = \varLambda_2\Bigg[\left(\sum_{j=1}^{4}\|m_{0j}(t)\|_1\right) + \|m_{05}(t)\|_1 \gamma_1^0\Bigg(\left(\sum_{j=1}^{4} {}_1c_j\right)$$
$$+ {}_1c_5 \frac{\delta^{1-\gamma_{11}}}{\Gamma_q(2-\gamma_{11})} + {}_1c_6 \frac{\delta^{2-\gamma_{12}}}{\Gamma_q(3-\gamma_{12})} + {}_1c_7 \frac{\delta^{3-\gamma_{13}}}{\Gamma_q(4-\gamma_{13})}\Bigg)$$

$$+ m_{11}(t)\|_1 \frac{1}{\Gamma_q(2-\beta_{11})} + m_{11}(t)\|_1 \frac{1}{\Gamma_q(2-\beta_{12})} + \left\|m_{21}(t)\right\|_1 \frac{1}{\Gamma_q(3-\beta_{21})} + \left\|m_{31}(t)\right\|_1 \frac{1}{\Gamma_q(4-\beta_{31})}\Bigg] < 1.$$

Now, by using Theorem 13, the inclusion problem (29) has at least one solution.

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