Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point results in M_ν-metric spaces with an application

Mohammad Asim[1], Izhar Uddin[2*] and Mohammad Imdad[1]

*Correspondence:
izharuddin1@jmi.ac.in
[2]Department of Mathematics, Jamia Millia Islamia, New Delhi, India
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the concept of M_ν-metric as a generalization of M-metric and ν-generalized metric and also prove an analogue of Banach contraction principle in an M_ν-metric space. Also, we adopt an example to highlight the utility of our main result which extends and improves the corresponding relevant results of the existing literature. Finally, we use our main result to examine the existence and uniqueness of solution for a Fredholm integral equation.

MSC: 47H10; 54H25

Keywords: M_ν-metric space; Fixed point; Integral equations

1 Introduction

In metric fixed point theory, the classical Banach contraction principle [11] remains a vital instrument which ensures the existence and uniqueness of fixed points of contraction maps in the setting of complete metric spaces. However, many researchers generalized and extended the Banach contraction principle in numerous ways by improving contraction conditions, using auxiliary mappings, and enlarging the class of metric spaces for this kind of results. One may recall the existing notions, namely of partial metric space [16], partial symmetric space [9], partial JS-metric space [7], metric like space [1], b-metric space [14], rectangular metric space [8, 12], cone metric space [15], M-metric space [5], M_b-metric space [18], rectangular M-metric space [25], and several others. Very recently, Asim et al. [10] introduced the class of rectangular M_{rb}-metric spaces to enlarge the classes of M_b-metric spaces and rectangular M-metric spaces wherein the newly refined ideas are utilized to prove some fixed point results.

In 2000, Branciari [12] enlarged the class of metric spaces by introducing an interesting class of ν-generalized metric spaces wherein the triangular inequality is replaced by a more general inequality, often called polygonal inequality (namely, involving $x, u_1, u_2, \dots, u_\nu, y$ points instead of three). In [12], Branciari proved a generalization of Banach contraction principle whose proof was erroneous (see [28, 29]). However, one is required to be careful while proving results involving ν-generalized metric spaces because such spaces need not have a compatible topology (see [30]).

In 2014, Asadi et al. [5] extended the partial metric spaces (see [17]) by introducing M-metric spaces and utilized the same to prove fixed point results, which were extended in many ways (see [2–4, 6, 13, 19–27]). Thereafter, Özgür [25] extended both the rectangular metric spaces and M-metric spaces by introducing rectangular M_r-metric spaces which were used to prove fixed point results.

Inspired by the concepts of M-metric spaces and ν-generalized metric spaces, we introduce the notion of an M_ν-metric space and utilize the same approach to prove an analogue of the Banach contraction principle in such a space. Also, we adopt an example to establish the genuineness of our main result. Finally, as an application of our main result, we prove a result establishing the existence and uniqueness of solution for a Fredholm integral equation.

2 Preliminaries

In this section, we begin with some notions and definitions which are needed in our subsequent discussions.

Notation 1 ([5]) The following notations will be utilized in our presentation:

(1) $m_{x,y} = \min\{m(x,x), m(y,y)\}$,

(2) $M_{x,y} = \max\{m(x,x), m(y,y)\}$.

(3) $m_{r_{x,y}} = \min\{m_r(x,x), m_r(y,y)\}$,

(4) $M_{r_{x,y}} = \max\{m_r(x,x), m_r(y,y)\}$.

In 2014, Asadi et al. [5] introduced the notion of an M-metric spaces as follows:

Definition 2.1 ([5]) Let X be a nonempty set. A mapping $m : X \times X \to \mathbb{R}_+$ is said to be an M-metric, if m satisfies the following (for all $x, y, z \in X$):

$(1m)$ $m(x,x) = m(x,y) = m(y,y)$ if and only if $x = y$,

$(2m)$ $m_{x,y} \leq m(x,y)$,

$(3m)$ $m(x,y) = m(y,x)$,

$(4m)$ $(m(x,y) - m_{x,y}) \leq (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$.

Then the pair (X, m) is said to be an M-metric space.

In 2018, Özgür [25] introduced the notion of rectangular M_r-metric spaces as follows:

Definition 2.2 ([25]) Let X be a nonempty set. A mapping $m_r : X \times X \to \mathbb{R}_+$ is said to be a rectangular M_r-metric, if m_r satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$):

$(1m_r)$ $m_r(x,x) = m_r(x,y) = m_r(y,y)$ if and only if $x = y$,

$(2m_r)$ $m_{r_{x,y}} \leq m_r(x,y)$,

$(3m_r)$ $m_r(x,y) = m_r(y,x)$,

$(4m_r)$ $(m_r(x,y) - m_{r_{x,y}}) \leq (m_r(x,u) - m_{r_{x,u}}) + (m_r(u,v) - m_{r_{u,v}}) + (m_r(v,y) - m_{r_{v,y}})$.

Then the pair (X, m_r) is said to be a rectangular M_r-metric space.

In 2000, Branciari [12] introduced the notion of rectangular metric spaces as follows:

Definition 2.3 ([12]) Let X be a nonempty set. A mapping $r : X \times X \to \mathbb{R}^+$ is said to be a rectangular metric on X, if r satisfies the following (for all $x, y \in X$ and all distinct $u, v \in X \setminus \{x, y\}$):

$(1r)$ $r(x,y) = 0$ if and only if $x = y$,

(2r) $r(x,y) = r(y,x)$,
(3r) $r(x,y) \leq r(x,u) + r(u,v) + r(v,y)$.
Then the pair (X,r) is said to be a rectangular metric space.

In 2000, Branciari [12] introduced the following very interesting metric.

Definition 2.4 ([12]) Let X be a nonempty set. A mapping $r_\nu : X \times X \to \mathbb{R}^+$ is said to be a ν-generalized metric on X, if r_ν satisfies the following (for all distinct $x, u_1, u_2, \ldots, u_\nu, y \in X$):
(1r_ν) $r_\nu(x,y) = 0$ if and only if $x = y$,
(2r_ν) $r_\nu(x,y) = r_\nu(y,x)$,
(3r_ν) $r_\nu(x,y) \leq r_\nu(x,u_1) + r_\nu(u_1,u_2) + \cdots + r_\nu(u_\nu,y)$.
Then the pair (X,r_ν) is said to be a ν-generalized metric space.

Remark 2.1 Observe that when $\nu = 1$ the ν-generalized metric space coincides with a metric space whereas for $\nu = 2$ the same space coincides with a rectangular metric space.

3 Main results

In this section, we introduce the notion of an M_ν-metric space (for any fixed $\nu \in \mathbb{N}$) and utilize it to prove a fixed point theorem besides deriving some lemmas, propositions, and corollaries. Some natural examples are also furnished. The following notations will be utilized in the sequel.

Notation 2
(1) $m_{\nu_{x,y}} = \min\{m_\nu(x,x), m_\nu(y,y)\}$,
(2) $M_{\nu_{x,y}} = \max\{m_\nu(x,x), m_\nu(y,y)\}$.

Definition 3.1 Let X be a nonempty set. A mapping $m_\nu : X \times X \to \mathbb{R}_+$ is said to be an M_ν-metric, if m_ν satisfies the following (for all $x, u_1, u_2, \ldots, u_\nu, y \in X$):
(1m_ν) $m_\nu(x,x) = m_\nu(x,y) = m_\nu(y,y)$ if and only if $x = y$,
(2m_ν) $m_{\nu_{x,y}} \leq m_\nu(x,y)$,
(3m_ν) $m_\nu(x,y) = m_\nu(y,x)$,
(4m_ν) $(m_\nu(x,y) - m_{\nu_{x,y}}) \leq (m_\nu(x,u_1) - m_{\nu_{x,u_1}}) + (m_\nu(u_1,u_2) - m_{\nu_{u_1,u_2}}) + \cdots + (m_\nu(u_\nu,y) - m_{\nu_{u_\nu,y}})$ such that $x, u_1, u_2, \ldots, u_\nu, y$ are distinct.
Then the pair (X,m_ν) is said to be an M_ν-metric space.

Notice that (X,m_ν) is an M-metric space if and only if (X,m_ν) is an M_1-metric space and a rectangular M_r-metric space if and only if (X,m_ν) is an M_2-metric space.

Now, we adopt an example in support of Definition 3.1 which is as follows:

Example 3.1 Let $X = \mathbb{R}$. Define $m_\nu : X \times X \to \mathbb{R}_+$ by

$$m_\nu(x,y) = \frac{|x| + |y|}{2}, \quad \text{for all } x, y \in X.$$

Here, one can easily check that conditions (1m_ν)-(3m_ν) are trivially satisfied. Now, we merely need to show that condition (4m_ν) holds. In doing so, we distinguish the following six cases:

Case 1. Firstly, assume that $|u_1| \leq |u_2| \leq \cdots \leq |u_\nu| \leq |x| \leq |y|$. Hence, $m_{v_{x,y}} = |x|$, $m_{v_{x,u_1}} = |u_1|$, $m_{v_{u_1,u_2}} = |u_1|, m_{v_{u_2,u_3}} = |u_2|, \ldots, m_{v_{u_\nu,y}} = |u_\nu|$. Then $(4m_\nu)$ can be written as

$$\begin{aligned}\frac{|x|+|y|}{2} - |x| &\leq \frac{|x|+|u_1|}{2} - |u_1| + \frac{|u_1|+|u_2|}{2} - |u_1| + \cdots + \frac{|u_\nu|+|y|}{2} - |u_\nu| \\ &= \frac{|x|+|y|}{2} - |u_1| + |u_1| + \cdots + |u_\nu| - |u_1| - \cdots - |u_\nu| \\ &= \frac{|x|+|y|}{2} - |u_1|,\end{aligned}$$

and since $|u_1| \leq |x|$, the above inequality is correct.

Case 2. Next, assume that $|u_1| \leq \cdots \leq |u_r| \leq |x| \leq |u_{r+1}| \leq \cdots \leq |u_\nu| \leq |y|$, for some $1 < r < \nu$. Then, $(4m_\nu)$ can be written as

$$\begin{aligned}\frac{|x|+|y|}{2} - |x| &\leq \frac{|x|+|u_1|}{2} - |u_1| + \frac{|u_1|+|u_2|}{2} - |u_1| + \cdots + \frac{|u_r|+|x|}{2} - |u_r| \\ &\quad + \frac{|x|+|u_{r+1}|}{2} - |x| + \cdots + \frac{|u_\nu|+|y|}{2} - |u_\nu| \\ &= \frac{|x|+|y|}{2} - |u_1| + |u_1| + \cdots + |u_\nu| - |u_1| - \cdots - |u_\nu| \\ &= \frac{|x|+|y|}{2} - |u_1|,\end{aligned}$$

and since $|u_\nu| \leq |x|$, then above inequality is correct.

Case 3. Now, assume that $|u_1| \leq \cdots \leq |u_r| \leq |x| \leq |y| \leq |u_{r+1}| \leq \cdots \leq |u_\nu|$, for some $1 < r < \nu$. Then, $(4m_\nu)$ can be written as

$$\begin{aligned}\frac{|x|+|y|}{2} - |x| &\leq \frac{|x|+|u_1|}{2} - |u_1| + \frac{|u_1|+|u_2|}{2} - |u_1| + \cdots + \frac{|u_r|+|x|}{2} - |u_r| \\ &\quad + \frac{|x|+|y|}{2} - |x| + \frac{|y|+|u_{r+1}|}{2} - |y| + \cdots + \frac{|u_\nu|+|y|}{2} - |y| \\ &= \frac{|x|+|y|}{2} - |u_1| + |u_1| + \cdots + |u_\nu| - |u_1| - \cdots - |y| \\ &= \frac{|x|+|y|}{2} - \bigl(|x| + |y| - |u_\nu|\bigr),\end{aligned}$$

and since $|x| + |y| - |u_\nu| \leq |x|$, then above inequality is correct.

Case 4. Now, assume that $|x| \leq |u_1| \leq |u_2| \leq \cdots \leq |u_\nu| \leq |y|$. Then, $(4m_\nu)$ can be written as

$$\begin{aligned}\frac{|x|+|y|}{2} - |x| &\leq \frac{|x|+|u_1|}{2} - |x| + \frac{|u_1|+|u_2|}{2} - |u_1| + \cdots \frac{|u_\nu|+|y|}{2} - |u_\nu| \\ &= \frac{|x|+|y|}{2} - |x| + |u_1| + \cdots + |u_\nu| - |u_1| - \cdots - |u_\nu| \\ &= \frac{|x|+|y|}{2} - |x|.\end{aligned}$$

Case 5. Now, assume that $|x| \leq |u_1| \leq \cdots \leq |u_r| \leq |y| \leq |u_{r+1}| \leq \cdots \leq |u_\nu|$, for some $1 < r < \nu$. Then, $(4m_\nu)$ can be written as:

$$\frac{|x|+|y|}{2} - |x| \leq \frac{|x|+|u_1|}{2} - |x| + \frac{|u_1|+|u_2|}{2} - |u_1| + \cdots + \frac{|u_r|+|y|}{2} - |u_r|$$

$$\begin{aligned} &+\frac{|y|+|u_{r+1}|}{2}-|y|+\cdots+\frac{|u_\nu|+|y|}{2}-|u_\nu| \\ &=\frac{|x|+|y|}{2}-|x|+|u_1|+\cdots+|u_\nu|-|u_1|-\cdots-|u_\nu| \\ &=\frac{|x|+|y|}{2}-|x|. \end{aligned}$$

Case 6. Finally, assume that $|x| \leq |y| \leq |u_1| \leq |u_2| \leq \cdots \leq |u_\nu|$. Then, $(4m_\nu)$ can be written as:

$$\begin{aligned} \frac{|x|+|y|}{2}-|x| &\leq \frac{|x|+|u_1|}{2}-|x|+\frac{|u_1|+|u_2|}{2}-|u_1|+\cdots+\frac{|u_\nu|+|y|}{2}-|y| \\ &=\frac{|x|+|y|}{2}-|x|+|u_1|+\cdots+|u_\nu|-|u_1|-\cdots-|y| \\ &=\frac{|x|+|y|}{2}-\bigl(|x|+|y|-|u_\nu|\bigr), \end{aligned}$$

and since $|x|+|y|-|u_\nu| \leq |x|$, then above inequality is correct.

Now, we furnish two examples by which one can obtain a ν-generalized metric space from an M_ν-metric space.

Example 3.2 Let (X, m_ν) be an M_ν-metric space. Define a function $m_\nu^* : X \times X \to \mathbb{R}_+$ by (for all $x, y \in X$)

$$m_\nu^*(x,y) = m_\nu(x,y) - 2m_{\nu_{x,y}} + M_{\nu_{x,y}}. \tag{3.1}$$

Then m_ν^* is a ν-generalized metric and the pair (X, m_ν^*) is ν-generalized metric space.

Proof To verify condition $(1r_\nu)$, for any $x, y \in X$, we have

$$\begin{aligned} &m_\nu^*(x,y) = 0 \\ &\quad\Longleftrightarrow\quad m_\nu(x,y) - 2m_{\nu_{x,y}} + M_{\nu_{x,y}} = 0 \\ &\quad\Longleftrightarrow\quad m_\nu(x,y) = 2m_{\nu_{x,y}} - M_{\nu_{x,y}}. \end{aligned}$$

Also,

$$\begin{aligned} &m_{\nu_{x,y}} \leq 2m_{\nu_{x,y}} - M_{\nu_{x,y}} \\ &\quad\Longleftrightarrow\quad M_{\nu_{x,y}} \leq m_{\nu_{x,y}} \\ &\quad\Longleftrightarrow\quad m_\nu(x,y) = m_{\nu_{x,y}} = M_{\nu_{x,y}} \\ &\quad\Longleftrightarrow\quad x = y. \end{aligned}$$

Now, for condition $(2r_\nu)$, for any $x, y \in X$, we have

$$\begin{aligned} m_\nu^*(x,y) &= m_\nu(x,y) - 2m_{\nu_{x,y}} + M_{\nu_{x,y}} \\ &= m_\nu(y,x) - 2m_{\nu_{y,x}} + M_{\nu_{y,x}} \\ &= m_\nu^*(y,x). \end{aligned}$$

Finally, we show that condition $(3r_\nu)$ holds. Observe that for all distinct $x, u_1, u_2, \ldots, u_\nu, y \in X$, we have

$$\begin{aligned} m_\nu^*(x,y) &= m_\nu(x,y) - 2m_{\nu_{x,y}} + M_{\nu_{x,y}} \\ &= \bigl(m_\nu(x,y) - m_{\nu_{x,y}}\bigr) + (M_{\nu_{x,y}} - m_{\nu_{x,y}}) \\ &\leq \bigl[\bigl(m_\nu(x,u_1) - m_{\nu_{x,u_1}}\bigr) + \bigl(m_\nu(u_1,u_2) - m_{\nu_{u_1,u_2}}\bigr) \\ &\quad + \cdots + \bigl(m_\nu(u_\nu,y) - m_{\nu_{u_\nu,y}}\bigr)\bigr] + \bigl[(M_{\nu_{x,u_1}} - m_{\nu_{x,u_1}}) \\ &\quad + (M_{\nu_{u_1,u_2}} - m_{\nu_{u_1,u_2}}) + \cdots + (M_{\nu_{u_\nu,y}} - m_{\nu_{u_\nu,y}})\bigr] \\ &= m_\nu^*(x,u_1) + m_\nu^*(u_1,u_2) + \cdots + m_\nu^*(u_\nu,y). \end{aligned}$$

Thus, (X, m_ν^*) is a ν-generalized metric space. □

Example 3.3 Let (X, m_ν) be an M_ν-metric space. Define a function $m_\nu^{**} : X \times X \to \mathbb{R}_+$ by (for all $x, y \in X$)

$$m_\nu^{**}(x,y) = m_\nu(x,y) - m_{\nu_{x,y}}. \tag{3.2}$$

Then m_ν^{**} is a ν-generalized metric and the pair (X, m_ν^{**}) is ν-generalized metric space.

Proof By similar arguments as in Example 3.2, one can easily show that m_ν^{**} is a ν-generalized metric. □

With a view to discuss topology corresponding to new M_ν-metric, let (X, m_ν) be an M_ν-metric space. Then, for all $x \in X$ and $\epsilon > 0$, the open ball with center x and radius ϵ is defined by

$$B_{m_\nu}(x,\epsilon) = \bigl\{y \in X : m_\nu(x,y) < m_{\nu_{x,y}} + \epsilon\bigr\}.$$

Observe that $x \in B_{m_\nu}(x,\epsilon)$ for each $\epsilon > 0$. Indeed, we have

$$m_\nu(x,x) - m_{\nu_{x,x}} = m_\nu(x,x) - m_\nu(x,x) = 0 < \epsilon.$$

Similarly, for all $x \in X$ and $\epsilon > 0$, the closed ball with center x and radius ϵ is defined by

$$B_{m_\nu}[x,\epsilon] = \bigl\{y \in X : m_\nu(x,y) \leq m_{\nu_{x,y}} + \epsilon\bigr\}.$$

Lemma 3.1 *Let (X, m_ν) be an M_ν-metric space. Then the collection of all open balls on X,*

$$\mathcal{U}_{m_\nu} = \bigl\{B_{m_\nu}(x,\epsilon) : x \in X, \epsilon > 0\bigr\},$$

forms a basis on X.

Proof Let $u_0 \in B_{m_\nu}(x,\epsilon)$. Then by the definition of $B_{m_\nu}(x,\epsilon)$, we have

$$m_\nu(x,u_0) < m_{\nu_{x,u_0}} + \epsilon.$$

Let $\delta = \epsilon + m_{\nu_{x,u_0}} - m_\nu(x,u_0) > 0$. We claim that

$$B_{m_\nu}(u_0,\delta) \subseteq B_{m_\nu}(x,\epsilon).$$

Let $u_1 \in B_{m_\nu}(u_0,\delta)$. Then by the definition, we have

$$m_\nu(u_1,u_0) < m_{\nu_{u_1,u_0}} + \delta.$$

Again let $\delta_1 = \delta + m_{\nu_{u_1,u_0}} - m_\nu(u_1,u_0)$. Inductively, let $u_\nu \in B_{m_\nu}(u_{\nu-1},\delta_{\nu-1})$, for any finite $\nu \geq 2$. Then

$$m_\nu(u_\nu,u_{\nu-1}) < m_{\nu_{u_\nu,u_{\nu-1}}} + \delta_{\nu-1}.$$

Let us choose $\delta_\nu > 0$ such that

$$\delta_\nu = \delta_{\nu-1} + m_{\nu_{u_\nu,u_{\nu-1}}} - m_\nu(u_\nu,u_{\nu-1}).$$

Now, from condition $(4m_\nu)$, we have

$$\begin{aligned}
\bigl(m_\nu(x,u_\nu) - m_{\nu_{x,u_\nu}}\bigr) &\leq \bigl(m_\nu(x,u_0) - m_{\nu_{x,u_0}}\bigr) + \bigl(m_\nu(u_0,u_1) - m_{\nu_{u_0,u_1}}\bigr) \\
&\quad + \cdots + \bigl(m_\nu(u_{\nu-1},u_\nu) - m_{\nu_{u_{\nu-1},u_\nu}}\bigr) \\
&< (\epsilon - \delta) + (\delta - \delta_1) + (\delta_1 - \delta_2) \\
&\quad + \cdots + (\delta_{\nu-2} - \delta_{\nu-1}) + (\delta_{\nu-1} - \delta_\nu) \\
&< (\epsilon - \delta_\nu).
\end{aligned}$$

Hence, $B_{m_\nu}(u_0,\delta) \subseteq B_{m_\nu}(x,\epsilon)$. Therefore, $\mathcal{U}_{m_\nu}$ forms a basis on X. □

Definition 3.2 Let (X,m_ν) be an M_ν-metric space and τ_{m_ν} a topology generated by the open balls $B_{m_\nu}(x,\epsilon)$. Then the pair (X,τ_{m_ν}) is called an M_ν-space.

Proposition 3.1 *An M_ν-space is a T_0-space.*

Proof Let (X,τ_{m_ν}) be an M_ν-metric space and $x,y \in X$ are two distinct points. Then from condition $(2m_\nu)$, we have

$$m_{\nu_{x,y}} \leq m_\nu(x,y) \quad \Rightarrow \quad \min\bigl\{m_\nu(x,x), m_\nu(y,y)\bigr\} \leq m_\nu(x,y),$$

that is,

$$m_\nu(x,x) \leq m_\nu(x,y) \quad \text{or} \quad m_\nu(y,y) \leq m_\nu(x,y).$$

Firstly, assume that $m_\nu(x,x) = m_\nu(y,y)$. Then we have

$$m_{\nu_{x,y}} = m_\nu(x,x) = m_\nu(y,y) < m_\nu(x,y),$$

yielding

$$m_\nu(x,y) - m_{\nu_{x,y}} = m_\nu(x,y) - m_\nu(x,x) > 0.$$

If we choose $\epsilon > 0$ such that $m_\nu(x,y) - m_\nu(x,x) = \epsilon$, then $m_\nu(x,y) = m_{\nu_{x,y}} + \epsilon$, so that $y \notin B_{m_\nu}(x,\epsilon)$.

Next, assume that $m_\nu(x,x) < m_\nu(y,y)$. Then

$$m_{\nu_{x,y}} = m_\nu(x,x) < m_\nu(x,y),$$

implying

$$m_\nu(x,y) - m_{\nu_{x,y}} = m_\nu(x,y) - m_\nu(x,x) > 0.$$

Again, if we choose $\epsilon > 0$ such that $m_\nu(x,y) - m_\nu(x,x) = \epsilon$, then

$$m_\nu(x,y) = m_{\nu_{x,y}} + \epsilon$$

so that $y \notin B_{m_\nu}(x,\epsilon)$.

Similarly, for $m_\nu(x,x) > m_\nu(y,y)$, one can easily show that $x \in B_{m_\nu}(x,\epsilon)$ and $y \notin B_{m_\nu}(x,\epsilon)$.

Therefore, for any two distinct points in $x, y \in X$, there is a ball containing one and not containing the other point. Hence, (X, m_ν) is a T_0-space. □

In an M_ν-metric space, the concepts of basic topological notions, namely of m_ν-Cauchy sequence, m_ν-convergent sequence, and m_ν-complete M_ν-metric space can be easily adopted as follows.

Definition 3.3 A sequence $\{x_n\}$ in (X, m_ν) is said to be m_ν-convergent to $x \in X$ if and only if

$$\lim_{n\to\infty}\bigl(m_\nu(x_n,x) - m_{\nu_{x_n,x}}\bigr) = 0.$$

Definition 3.4 A sequence $\{x_n\}$ in (X, m_ν) is said to be m_ν-convergent uniquely to $x \in X$ if and only if $\lim_{n\to\infty}(m_\nu(x_n,x) - m_{\nu_{x_n,x}}) = 0$ holds and $\lim_{n\to\infty}(m_\nu(x_n,y) - m_{\nu_{x_n,y}}) = 0$ does not hold for $y \in X \setminus \{x\}$.

Definition 3.5 A sequence $\{x_n\}$ in (X, m_ν) is said to be m_ν-Cauchy if and only if

$$\lim_{n,m\to\infty}\bigl(m_\nu(x_n,x_m) - m_{\nu_{x_n,x_m}}\bigr) \quad \text{and} \quad \lim_{n,m\to\infty}(M_{\nu_{x_n,x_m}} - m_{\nu_{x_n,x_m}})$$

exist and are finite.

Definition 3.6 An M_ν-metric space (X, m_ν) is said to be m_ν-complete if every m_ν-Cauchy sequence in X is m_ν-convergent to a point $x \in X$ such that

$$\lim_{n\to\infty}\bigl(m_\nu(x_n,x) - m_{\nu_{x_n,x}}\bigr) = 0 \quad \text{and} \quad \lim_{n\to\infty}(M_{\nu_{x_n,x}} - m_{\nu_{x_n,x}}) = 0.$$

Definition 3.7 A self-mapping f on (X, m_ν) is said to be sequentially m_ν-continuous if and only if the fact that $\{x_n\}$ m_ν-converges to x implies that $\{fx_n\}$ m_ν-converges to fx.

Definition 3.8 A sequence $\{x_n\}$ in (X, m_ν) is said to be m_ν-κ-Cauchy if and only if

$$\lim_{n\to\infty}\left(m_\nu(x_n, x_{n+1+j\kappa}) - m_{\nu_{x_n,x_{n+1+j\kappa}}}\right) \quad \text{and} \quad \lim_{n\to\infty}\left(M_{\nu_{x_n,x_{n+1+j\kappa}}} - m_{\nu_{x_n,x_{n+1+j\kappa}}}\right)$$

exist and are finite with $\kappa \in \mathbb{N}$ and $j \in \mathbb{N}_0$.

Definition 3.9 An M_ν-metric space (X, m_ν) is said to be an m_ν-κ-complete if every m_ν-κ-Cauchy in X is m_ν-convergent to a point $x \in X$ such that

$$\lim_{n\to\infty}\left(m_\nu(x_n, x) - m_{\nu_{x_n,x}}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(M_{\nu_{x_n,x}} - m_{\nu_{x_n,x}}\right) = 0.$$

Remark 3.1 We prefer to write "m_ν-Cauchy" instead of "m_ν-1-Cauchy" and "m_ν-complete" instead of "m_ν-1-complete".

Lemma 3.2 *Let (X, m_ν) be an M_ν-metric space. Then, we have*

(i) *A sequence $\{x_n\}$ in (X, m_ν) is m_ν-Cauchy in (X, m_ν) if and only if $\{x_n\}$ in (X, m_ν) is m_ν-Cauchy in (X, m_ν^*) (resp. (X, m_ν^{**})).*

(ii) *(X, m_ν) is m_ν-complete if and only if (X, m_ν^*) (resp. (X, m_ν^{**})) is m_ν-complete. Moreover,*

$$\lim_{n\to\infty} m_\nu^*(x_n, x) = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty}\left(m_\nu(x_n, x) - m_{\nu_{x_n,x}}\right) = 0 = \lim_{n\to\infty}\left(M_{\nu_{x_n,x}} - m_{\nu_{x_n,x}}\right).$$

Proof By using Examples 3.2 and 3.3, one can easily prove this lemma. □

Proposition 3.2 *Let (X, m_ν) be an M_ν-metric space and $\kappa, \lambda \in \mathbb{N}$ such that κ divides λ. Then*

(i) *Every m_ν-κ-Cauchy sequence is m_ν-λ-Cauchy.*

(ii) *If X is m_ν-κ-complete, then X is m_ν-λ-complete.*

Proof (i) Let $\{x_n\}$ be an m_ν-κ-Cauchy sequence in X. By the definition of an m_ν-κ-Cauchy sequence, we have that

$$\lim_{n\to\infty}\left(m_\nu(x_n, x_{n+1+j\kappa}) - m_{\nu_{x_n,x_{n+1+j\kappa}}}\right) \quad \text{and} \quad \lim_{n\to\infty}\left(M_{\nu_{x_n,x_{n+1+j\kappa}}} - m_{\nu_{x_n,x_{n+1+j\kappa}}}\right)$$

exist and are finite with $j \in \mathbb{N}_0$. Since, $\kappa, \lambda \in \mathbb{N}$ are such that κ divides λ, one can find a $j \in \mathbb{N}_0$ such that $\lambda = j\kappa$. Thus,

$$\lim_{n\to\infty}\left(m_\nu(x_n, x_{n+1+j\lambda}) - m_{\nu_{x_n,x_{n+1+j\lambda}}}\right) \quad \text{and} \quad \lim_{n\to\infty}\left(M_{\nu_{x_n,x_{n+1+j\lambda}}} - m_{\nu_{x_n,x_{n+1+j\lambda}}}\right)$$

exist and are finite with $j \in \mathbb{N}_0$. Therefore, $\{x_n\}$ is an m_ν-λ-Cauchy.

(ii) Let (X, m_ν) be m_ν-κ-complete. Then every m_ν-κ-Cauchy sequence is also m_ν-λ-Cauchy which is m_ν-convergent to some point in X. Hence, (X, m_ν) is m_ν-λ-complete. □

Now, we prove the following lemma which is used in our subsequent discussion:

Lemma 3.3 *Let (X, m_ν) be an M_ν-metric space. Let $\{x_n\}$ be a sequence in X such that all x_ns are distinct and $\sum_{n=1}^{\infty}(m_\nu(x_n, x_{n+1}) - m_{\nu_{x_n,x_{n+1}}}) < \infty$. Then $\{x_n\}$ is m_ν-ν-Cauchy.*

Proof Fix $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty}(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i,x_{i+1}}}) < \epsilon$. Fix $n \in \mathbb{N}$ with $n \geq N$. We will show that

$$\bigl(m_\nu(x_n, x_{n+1+j\nu}) - m_{\nu_{x_n,x_{n+1+j\nu}}}\bigr) \leq \sum_{i=n}^{n+j\nu}\bigl(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i,x_{i+1}}}\bigr). \tag{3.3}$$

For $j = 0$, (3.3) trivially holds. Now, from $(4m_\nu)$, we have (for some $j \in \mathbb{N}$)

$$\begin{aligned}\bigl(m_\nu(x_n, x_{n+1+(j+1)\nu}) - m_{\nu_{x_n,x_{n+1+(j+1)\nu}}}\bigr) &\leq \bigl(m_\nu(x_n, x_{n+1+j\nu}) - m_{\nu_{x_n,x_{n+1+j\nu}}}\bigr) \\ &\quad + \sum_{i=n+1+j\nu}^{n+(j+1)\nu}\bigl(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i,x_{i+1}}}\bigr) \\ &\leq \sum_{i=n}^{n+(j+1)\nu}\bigl(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i,x_{i+1}}}\bigr).\end{aligned}$$

Then (3.5) holds for $k = k + 1$. Thus, by mathematical induction, (3.3) holds for any $j \in \mathbb{N}_0$. Hence,

$$\begin{aligned}\bigl(m_\nu(x_n, x_{n+1+j\nu}) - m_{\nu_{x_n,x_{n+1+j\nu}}}\bigr) &\leq \sum_{i=n}^{n+j\nu}\bigl(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i,x_{i+1}}}\bigr) \\ &\leq \sum_{i=N}^{\infty}\bigl(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i,x_{i+1}}}\bigr) < \epsilon.\end{aligned}$$

Also, by condition $(2m_\nu)$ and recalling our notation, we have

$$(M_{\nu_{x_n,x_{n+1+j\nu}}} - m_{\nu_{x_n,x_{n+1+j\nu}}}) \leq \bigl(m_\nu(x_n, x_{n+1+j\nu}) - m_{\nu_{x_n,x_{n+1+j\nu}}}\bigr) < \epsilon.$$

Therefore, $\{x_n\}$ is an m_ν-ν-Cauchy sequence. □

Proposition 3.3 *Let (X, m_ν) be an M_ν-metric space where ν is odd. Let $\{x_n\}$ be an m_ν-ν-Cauchy sequence such that all x_n are distinct. Then $\{x_n\}$ is an m_ν-Cauchy sequence.*

Proof We first note that if $\nu = 1$, then from Remark 3.1 the conclusion clearly holds. Now, we assume that $\nu \geq 3$. Fix $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that

$$\bigl(m_\nu(x_n, x_{n+1+j\nu}) - m_{\nu_{x_n,x_{n+1+j\nu}}}\bigr) < \epsilon, \quad \text{for } n \geq N \text{ and } j \in \mathbb{N}_0. \tag{3.4}$$

Next, we fix $j \in \mathbb{N}_0$ with $n \geq N$. Now, we first show that

$$\bigl(m_\nu(x_n, x_{n+1+j\nu+2k}) - m_{\nu_{x_n,x_{n+1+j\nu+2k}}}\bigr) < (k\nu + 1)\epsilon, \quad \text{for } k = 0, 1, \ldots, (\nu - 1)/2. \tag{3.5}$$

If $k = 0$, then (3.5) trivially holds by (3.4). So, let $0 < k \leq (\nu - 1)/2$. Now using $(4m_\nu)$, we have

$$\begin{aligned}
&\big(m_\nu(x_n, x_{n+1+j\nu+2(k+1)}) - m_{\nu_{x_n, x_{n+1+j\nu+2(k+1)}}}\big) \\
&\quad\leq \big(m_\nu(x_n, x_{n+1+j\nu+2k}) - m_{\nu_{x_n, x_{n+1+j\nu+2k}}}\big) \\
&\qquad + \big(m_\nu(x_{n+1+j\nu+2k}, x_{n+2+(j+1)\nu+2k}) - m_{\nu_{x_{n+1+j\nu+2k}, x_{n+2+(j+1)\nu+2k}}}\big) \\
&\qquad + \sum_{i=n+1+j\nu+2(k+1)}^{n+1+(j+1)\nu+2k} \big(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i, x_{i+1}}}\big) \\
&\quad\leq (k\nu + 1)\epsilon + \epsilon + (\nu - 1)\epsilon \\
&\quad= \big((k+1)\nu + 1\big)\epsilon.
\end{aligned}$$

Then, (3.5) holds for $k = k + 1$. Thus, by mathematical induction, (3.5) holds for every k, which implies

$$\big(m_\nu(x_n, x_{n+1+j\nu+2k}) - m_{\nu_{x_n, x_{n+1+j\nu+2k}}}\big) < \left(\frac{\nu^2}{2} - \frac{\nu}{2} + 1\right),$$

for $j \in \mathbb{N}_0$, $k = 0, 1, \ldots, (\nu - 1)/2$ with $n \geq N$. Therefore, we have

$$\begin{aligned}
&\big(m_\nu(x_n, x_{n+1+j\nu+2k+1}) - m_{\nu_{x_n, x_{n+1+j\nu+2k+1}}}\big) \\
&\quad\leq \big(m_\nu(x_n, x_{n+1+j\nu+2k}) - m_{\nu_{x_n, x_{n+1+j\nu+2k}}}\big) \\
&\qquad + \big(m_\nu(x_{n+1+j\nu+2k}, x_{n+2+j\nu+2k+\nu-1}) - m_{\nu_{x_{n+1+j\nu+2k}, x_{n+2(j\nu+2k+\nu-1}}}\big) \\
&\qquad + \sum_{i=n+2+j\nu+2k}^{n+(j+1)\nu+2k} \big(m_\nu(x_i, x_{i+1}) - m_{\nu_{x_i, x_{i+1}}}\big) \\
&\quad\leq 2\left(\frac{\nu^2}{2} - \frac{\nu}{2} + 1\right)\epsilon + (\nu - 1)\epsilon \\
&\quad= \big(\nu^2 + 1\big)\epsilon
\end{aligned}$$

for $j \in \mathbb{N}_0$, $k = 0, 1, \ldots, (\nu - 3)/2$ with $n \geq N$. Also, by condition $(2m_\nu)$ and our notation, we have

$$(M_{\nu_{x_n, x_{n+1+j\nu+2k}}} - m_{\nu_{x_n, x_{n+1+j\nu+2k}}}) \leq \big(m_\nu(x_n, x_{n+1+j\nu+2k}) - \nu_{x_n, x_{n+1+j\nu+2k}}\big) < \epsilon.$$

Therefore, $\{x_n\}$ is an m_ν-Cauchy sequence. □

Proposition 3.4 *Let (X, m_ν) be an M_ν-metric space where ν is even. Let $\{x_n\}$ be an m_ν-ν-Cauchy sequence such that all x_n are distinct. Then $\{x_n\}$ is an m_ν-2-Cauchy sequence.*

Proof Fix $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that

$$\big(m_\nu(x_n, x_{n+1+j\nu}) - m_{\nu_{x_n, x_{n+1+j\nu}}}\big) < \epsilon, \quad \text{for } n \geq N \text{ and } j \in \mathbb{N}_0. \tag{3.6}$$

Next, we fix $j \in \mathbb{N}_0$ with $n \geq N$. Then by mathematical induction as in the proof of Proposition 3.3, one can show

$$\big(m_\nu(x_n, x_{n+1+j\nu+2k}) - m_{\nu_{x_n,x_{n+1+j\nu+2k}}}\big) < (k\nu + 1)\epsilon, \quad \text{for } k = 0, 1, \ldots, \nu/2 - 1. \tag{3.7}$$

Therefore, we have

$$\big(m_\nu(x_n, x_{n+1+2j}) - m_{\nu_{x_n,x_{n+1+2j}}}\big) < \left(\frac{\nu^2}{2} - \nu + 1\right)\epsilon, \quad \text{for any } j \in \mathbb{N}_0.$$

Furthermore, by using condition $(2m_\nu)$ and our notation, we obtain

$$(M_{\nu_{x_n,x_{n+1+2j}}} - m_{\nu_{x_n,x_{n+1+2j}}}) \leq \big(m_\nu(x_n, x_{n+1+2j}) - m_{\nu_{x_n,x_{n+1+2j}}}\big) < \left(\frac{\nu^2}{2} - \nu + 1\right)\epsilon$$

for any $j \in \mathbb{N}_0$. So that $\{x_n\}$ is m_ν-2-Cauchy. □

Remark 3.2 Observe that every m_ν-Cauchy sequence is m_ν-2-Cauchy but the converse is not true in general. For converse part, we prove the following lemma.

Lemma 3.4 *Let (X, m_ν) be an M_ν-metric space. Let $\{x_n\}$ be an m_ν-2-Cauchy sequence in X such that all x_ns are distinct and*

$$\lim_{n\to\infty} \big(m_\nu(x_n, x_{n+2}) - m_{\nu_{x_n,x_{n+2}}}\big) = 0.$$

Then $\{x_n\}$ is an m_ν-Cauchy sequence.

Proof Since $\{x_n\}$ is an m_ν-2-Cauchy, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\big(m_\nu(x_n, x_{n+1+2j}) - m_{\nu_{x_n,x_{n+1+2j}}}\big) < \epsilon \quad \text{and} \quad \big(m_\nu(x_n, x_{n+2}) - m_{\nu_{x_n,x_{n+2}}}\big) < \epsilon$$

for any $j \in \mathbb{N}_0$ with $n \geq N$. Fix $j \in \mathbb{N}_0$ and for $n \geq N$. Then for $\nu = 1$, we have

$$\begin{aligned}\big(m_\nu(x_n, x_{n+2+2j}) - m_{\nu_{x_n,x_{n+2+2j}}}\big) &\leq \big(m_\nu(x_n, x_{n+1+2j}) - m_{\nu_{x_n,x_{n+1+2j}}}\big)\\ &\quad + \big(m_\nu(x_{n+1+2j}, x_{n+2+2j}) - m_{\nu_{x_{n+1+2j},x_{n+2+2j}}}\big)\\ &< \epsilon.\end{aligned}$$

Now, we take the case $\nu \geq 2$, and then have

$$\begin{aligned}\big(m_\nu(x_n, x_{n+2+2j}) - m_{\nu_{x_n,x_{n+2+2j}}}\big) &\leq \big(m_\nu(x_n, x_{n+1+2j}) - m_{\nu_{x_n,x_{n+1+2j}}}\big)\\ &\quad + \big(m_\nu(x_{n+1+2j}, x_{n+2j+2\nu}) - m_{\nu_{x_{n+1+2j},x_{n+2j+2\nu}}}\big)\\ &\quad + \sum_{i=n+2+2j}^{n+2(\nu-1)+2j} \big(m_\nu(x_i, x_{i+2}) - m_{\nu_{x_i,x_{i+2}}}\big)\\ &< (\nu + 1)\epsilon.\end{aligned}$$

By using condition $(2m_\nu)$ and our notation, we obtain

$$(M_{\nu_{x_n,x_{n+2+2j}}} - m_{\nu_{x_n,x_{n+2+2j}}}) \leq \left(m_\nu(x_n, x_{n+2+2j}) - m_{\nu_{x_n,x_{n+2+2j}}}\right) < (\nu + 1)\epsilon.$$

Therefore, $\{x_n\}$ is an m_ν-Cauchy sequence. □

Next, we present the following lemma required in the sequel.

Lemma 3.5 *Let (X, m_ν) be an M_ν-metric space and $f : X \to X$ a self-mapping on X such that*

$$m_\nu(fx, fy) \leq \lambda m_\nu(x, y) \tag{3.8}$$

for some $\lambda \in [0, 1)$. Consider the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n$. If $x_n \to x$ as $n \to \infty$, then $fx_n \to fx$ as $n \to \infty$.

Proof Assume that $m_\nu(fx_n, fx) = 0$, then $m_{\nu_{fx_n,fx}} \leq m_\nu(fx_n, fx) = 0$, so that $m_\nu(fx_n, fx) - m_{\nu_{fx_n,fx}} \to 0$ as $n \to \infty$ and $fx_n \to fx$ as $n \to \infty$.

On the other hand, assume that $m_\nu(fx_n, fx) > 0$. By (3.8) we have $m_\nu(fx_n, fx) \leq \lambda m_\nu(x_n, x)$. Here, we distinguish two cases as follows:

Firstly, assume that $m_\nu(x, x) \leq m_\nu(x_n, x_n)$. Then, by using (3.8), we have

$$m_\nu(x_n, x_n) = m_\nu(fx_{n-1}, fx_{n-1}) \leq \lambda m_\nu(x_{n-1}, x_{n-1}) \leq \cdots \leq \lambda^{n-1} m_\nu(x_0, x_0).$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} m_\nu(x_n, x_n) = 0 \quad \Longrightarrow \quad m_\nu(x, x) = 0.$$

Since $m_\nu(fx, fx) < m_\nu(x, x) = 0$, we obtain that $m_\nu(fx, fx) = \lambda m_\nu(x, x) = 0$ (for $\lambda \in [0, 1)$). Then, by the definition of m_ν-convergence of a sequence x_n, which converges to x, we have

$$\lim_{n\to\infty} \left(m_\nu(x_n, x) - m_{\nu_{x_n,x}}\right) = 0.$$

Since, $m_{\nu_{x_n,x}} = \min\{m_\nu(x_n, x_n), m_\nu(x, x)\}$ and hence $m_{\nu_{x_n,x}} \to 0$ as $n \to \infty$ so that $m_\nu(x_n, x) \to 0$, $n \to \infty$. Hence, we obtain $m_\nu(fx_n, fx) < m_\nu(x_n, x) \to 0$. Therefore, $m_\nu(fx_n, fx) - m_{\nu fx_n,fx} \to 0$ so that $fx_n \to fx$.

Secondly, assume that $m_\nu(x, x) \geq m_\nu(x_n, x_n)$. Similarly, one can show that

$$\lim_{n\to\infty} m_\nu(x_n, x_n) = 0 \quad \Longrightarrow \quad m_{\nu x_n,x} \to 0.$$

Hence, $m_\nu(x_n, x) \to 0$. Since $m_\nu(fx_n, fx) < m_\nu(x_n, x) \to 0$, we have $m_\nu(fx_n, fx) - m_{\nu fx_n,fx} \to 0$ so that $fx_n \to fx$. This finishes the proof. □

Now, we are equipped to prove our main result as follows:

Theorem 3.1 *Let (X, m_ν) be an M_ν-metric space and $f : X \to X$. Assume that the following conditions are satisfied*:

(i) *there exists* $\lambda \in [0,1)$ *such that* (*for all* $x, y \in X$)

$$m_\nu(fx, fy) \le \lambda m_\nu(x, y) \tag{3.9}$$

(ii) (X, m_ν) *is* m_ν*-complete.*

Then f *has a unique fixed point* $x \in X$ *such that* $m_\nu(x,x) = 0$.

Proof Let $x_0 \in X$. Construct an iterative sequence $\{x_n\}$ by:

$$x_1 = fx_0, \qquad x_2 = f^2x_0, \qquad x_3 = f^3x_0, \qquad \ldots, \qquad x_n = f^nx_0, \qquad \ldots.$$

Now, we assert that $\lim_{n\to\infty} m_\nu(x_n, x_{n+1}) = 0$. On setting $x = x_n$ and $y = x_{n+1}$ in (3.9), we have

$$\begin{aligned} m_\nu(x_n, x_{n+1}) &= m_\nu(fx_{n-1}, fx_n) \\ &\le \lambda m_\nu(x_{n-1}, x_n) \\ &\le \lambda^n m_\nu(x_0, x_1), \end{aligned}$$

which, letting $n \to \infty$, gives rise to

$$\lim_{n\to\infty} m_\nu(x_n, x_{n+1}) = 0.$$

Now, by taking $x = x_n$ and $y = x_{n+2}$ in (3.9), we obtain

$$\begin{aligned} m_\nu(x_n, x_{n+2}) &= m_\nu(fx_{n-1}, fx_{n+1}) \\ &\le \lambda m_\nu(x_{n-1}, x_{n+1}) \\ &\le \lambda^{n-1} m_\nu(x_0, x_2), \end{aligned}$$

and, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} m_\nu(x_n, x_{n+2}) = 0.$$

Similarly, from condition (3.9), we get

$$m_\nu(x_n, x_n) = m_\nu(fx_{n-1}, fx_{n-1}) \le \lambda m_\nu(x_{n-1}, x_{n-1}) \le \cdots \le \lambda^{n-1} m_\nu(x_0, x_0).$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} m_\nu(x_n, x_n) = 0. \tag{3.10}$$

Also, we have

$$\sum_{n=1}^{\infty} m_\nu(x_n, x_{n+1}) \le \sum_{n=1}^{\infty} \lambda^n m_\nu(x_0, x_1) < \infty \tag{3.11}$$

and

$$\sum_{n=1}^{\infty} m_\nu(x_n,x_n) \le \sum_{n=1}^{\infty} \lambda^n m_\nu(x_0,x_0) < \infty. \tag{3.12}$$

Therefore, from equations (3.11), (3.12) and by recalling our notation, we obtain

$$\sum_{n=1}^{\infty} \left(m_\nu(x_n,x_{n+1}) - m_{\nu_{x_n,x_{n+1}}} \right) < \infty.$$

Firstly, we show that $x_n \neq x_m$ for any $n \neq m$. Let on the contrary $x_n = x_m$ for some $n > m$, then we have $x_{n+1} = fx_n = fx_m = x_{m+1}$. Then from (3.9), we get

$$m_\nu(x_m,x_{m+1}) = m_\nu(x_n,x_{n+1}) < m_\nu(x_{n-1},x_n) < \cdots < m_\nu(x_m,x_{m+1}),$$

a contradiction. Thus, in what follows, we can assume that $x_n \neq x_m$ for all $n \neq m$.

Now, we assert that $\{x_n\}$ is an m_ν-Cauchy sequence in (X, m_ν). By Lemma 3.3, $\{x_n\}$ is an m_ν-ν-Cauchy sequence. By Propositions 3.3 and 3.4, $\{x_n\}$ is an m_ν-Cauchy sequence. Thus, we have

$$\lim_{n,m\to\infty} \left(m_\nu(x_n,x_m) - m_{\nu_{x_n,x_m}} \right) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} (M_{\nu_{x_n,x_m}-m_{\nu_{x_n,x_m}}}) = 0.$$

Since X is m_ν-complete, there exists $x \in X$ such that $x_n \to x$. Now, we show that $fx = x$. By Lemma 3.5 we have

$$\begin{aligned} \lim_{n\to\infty} \left(m_\nu(x_n,x) - m_{\nu_{x_n,x}} \right) = 0 \\ &= \lim_{n\to\infty} \left(m_\nu(x_{n+1},x) - m_{\nu_{x_{n+1},x}} \right) \\ &= \lim_{n\to\infty} \left(m_\nu(fx_n,x) - m_{\nu_{fx_n,x}} \right) \\ &= \lim_{n\to\infty} \left(m_\nu(fx,x) - m_{\nu_{fx,x}} \right) \end{aligned}$$

so that $m_\nu(fx,x) = m_{\nu_{x,fx}}$. Since $m_{\nu_{x,fx}} = \min\{m_\nu(x,x), m_\nu(fx,fx)\}$, therefore $m_{\nu_{x,fx}} = m_\nu(x,x)$ or $m_{\nu_{x,fx}} = m_\nu(fx,fx)$ which amounts to saying that $fx = x$.

Now, we show the uniqueness of the fixed point x. Suppose on the contrary that f has two fixed points $x, y \in X$, that is, $fx = x$ and $fy = y$. Thus

$$m_\nu(x,y) = m_\nu(fx,fy) \le \lambda m_\nu(x,y) < m_\nu(x,y),$$

which implies that $m_\nu(x,y) = 0$ and hence, $x = y$. Finally, we show that if x is a fixed point, then $m_\nu(x,x) = 0$. Assume that x is a fixed point of f. Observe that

$$m_\nu(x,x) = m_\nu(fx,fx) \le \lambda m_\nu(x,x) < m_\nu(x,x),$$

yielding $m_\nu(x,x) = 0$. This completes the proof. □

Now, we present an example which demonstrates the utility of our newly proved result:

Example 3.4 Consider $X = [0,1]$ and an M_ν-metric $m_\nu : X \times X \to \mathbb{R}_+$ defined by

$$m_\nu(x,y) = \frac{x+y}{2}, \quad \text{for all } x,y \in X.$$

Then (X, m_ν) is an m_ν-complete M_ν-metric space. Define a self-mapping f on X by

$$fx = \frac{3x}{5}, \quad \text{for all } x \in X.$$

Observe that, for all $x, y \in X$, we obtain

$$\begin{aligned} m_\nu(fx,fy) &= \frac{fx+fy}{2} = \frac{\frac{3x}{5}+\frac{3y}{5}}{2} \\ &\leq \frac{3}{5}\left(\frac{x+y}{2}\right) = \frac{3}{5} m_\nu(x,y). \end{aligned}$$

Thus, all conditions of Theorem 3.1 are satisfied and f has a unique fixed point (namely $x = 0$).

Observe that, by putting m_ν in (3.1) (or alternately in (3.2)) with $\nu = 1$, one can deduce a metric and henceforth the classical Banach contraction principle.

The following corollary is due to Asadi et al. [5].

Corollary 3.1 *Let (X, m) be an M-metric space and $f : X \to X$. Assume that the following conditions are satisfied*:

(i) *there exists $\lambda \in [0,1)$ such that (for all $x,y \in X$)*

$$m(fx,fy) \leq \lambda m(x,y)$$

(ii) *(X, m) is m-complete.*

Then f has a unique fixed point x such that $m(x,x) = 0$.

Proof By choosing $\nu = 1$ in Theorem 3.1, the above result is immediate. □

The following corollary is due to Özgür et al. [25].

Corollary 3.2 *Let (X, m_r) be a rectangular M_r-metric space and $f : X \to X$. Assume that the following conditions are satisfied*:

(i) *there exists $\lambda \in [0,1)$ such that (for all $x,y \in X$)*

$$m_r(fx,fy) \leq \lambda m_r(x,y)$$

(ii) *(X, m_r) is m_r-complete.*

Then f has a unique fixed point x such that $m_r(x,x) = 0$.

Proof The above result is immediate from Theorem 3.1 by choosing $\nu = 2$. □

Corollary 3.3 *Theorem 3.1 remains a genuinely sharpened version of Theorem 2.1 due to A. Branciari [12].*

4 An application to an integral equation

In this section, we endeavor to apply Theorem 3.1 to investigate the existence and uniqueness of solution of the Fredholm integral equation.

Let $X = C([0,1],\mathbb{R})$ be the set of continuous real-valued functions defined on $[0,1]$. Now, we consider the following Fredholm type integral equation:

$$x(t) = \int_0^1 G\bigl(t,s,x(t)\bigr)\,ds, \quad \text{for } t,s \in [0,1], \tag{4.1}$$

where $G \in C([0,1],\mathbb{R})$. Define $m_\nu : X \times X \to \mathbb{R}^+$ as in Example 3.1, that is,

$$m_\nu\bigl(x(t),y(t)\bigr) = \sup_{t\in[a,b]}\left(\frac{|x(t)| + |y(t)|}{2}\right), \quad \text{for all } x,y \in X.$$

Then (X, m_ν) is an m_ν-complete M_ν-metric space.

Now, we are equipped to state and prove our result as follows:

Theorem 4.1 *Assume that (for all $x,y \in C([0,1],\mathbb{R})$)*

$$\bigl|G\bigl(t,s,x(t)\bigr) + G\bigl(t,s,y(t)\bigr)\bigr| \le \lambda\bigl|x(t) + y(t)\bigr|, \quad \textit{for all } t,s \in [0,1], \tag{4.2}$$

where $\lambda \in [0,1)$. Then the integral equation (4.1) has a unique solution.

Proof Define $f : X \to X$ by

$$fx(t) = \int_0^1 G\bigl(t,s,x(t)\bigr)\,ds, \quad \text{for all } t,s \in [0,1].$$

Observe that existence of a fixed point of the operator f is equivalent to the existence of a solution of the integral equation (4.1). Now, for all $x,y \in X$, we have

$$\begin{aligned}
m_\nu(fx,fy) &= \left|\frac{fx(t) + fy(t)}{2}\right| = \left|\int_0^1 \left(\frac{G(t,s,x(t)) + G(t,s,y(t))}{2}\right) ds\right| \\
&\le \int_0^1 \left|\frac{G(t,s,x(t)) + G(t,s,y(t))}{2}\right| ds \\
&\le \lambda \int_0^1 \left|\frac{x(t) + y(t)}{2}\right| ds \\
&\le \lambda \int_0^1 \left(\frac{|x(t)| + |y(t)|}{2}\right) ds \\
&\le \lambda \sup_{t\in[a,b]}\left(\frac{|x(t)| + |y(t)|}{2}\right) \int_0^1 ds \\
&\le \lambda m_\nu(x,y).
\end{aligned}$$

Thus, condition (3.9) is satisfied. Therefore, all conditions of Theorem 3.1 are satisfied. Hence, operator f has a unique fixed point, which means that the Fredholm integral equation (4.1) has a unique solution. This completes the proof. □

Acknowledgements
The authors are thankful to learned referees for their critical and important suggestions.

Funding
The second author is thankful to University Grants Commission, India for the financial assistance in form of BSR-Startup research grant.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed to this work equally. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Aligarh Muslim University, Aligarh, India. [2]Department of Mathematics, Jamia Millia Islamia, New Delhi, India.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 2 May 2019 Accepted: 15 October 2019 Published online: 06 November 2019

References

1. Amini-Harandi, A.: Metric-like spaces, partial metric spaces and fixed points. Fixed Point Theory Appl. **2012**, Article ID 204 (2012)
2. Asadi, M.: Fixed point theorems for Meir–Keler mapping type in *M*-metric space with applications. Fixed Point Theory Appl. **2015**, 210 (2015)
3. Asadi, M.: On Ekeland's variational principle in *M*-metric spaces. J. Nonlinear Convex Anal. **17**(6), 1151–1158 (2016)
4. Asadi, M., Azhini, M., Karapınar, E., Monfared, H.: Simulation functions over *M*-metric spaces. East Asian Math. J. **33**(5), 559–570 (2017)
5. Asadi, M., Karapinar, E., Salimi, P.: New extension of *p*-metric spaces with some fixed-point results on *M*-metric spaces. J. Inequal. Appl. **2014**, 18 (2014)
6. Asadi, M., Moeini, B., Mukheimer, A., Aydi, H.: Complex valued *M*-metric spaces and related fixed point results via complex *C*-class function. J. Inequal. Spec. Funct. **10**(1), 101–110 (2019)
7. Asim, M., Imdad, M.: Partial JS-metric spaces and fixed fixed point results. Indian J. Math. **61**(2), 175–186 (2019)
8. Asim, M., Imdad, M., Radenovic, S.: Fixed point results in extended rectangular *b*-metric spaces with an application. UPB Sci. Bull., Ser. A **81**(2), 11–20 (2019)
9. Asim, M., Khan, A.R., Imdad, M.: Fixed point results in partial symmetric spaces with an application. Axioms **8**, 13 (2019). https://doi.org/10.3390/axioms8010013
10. Asim, M., Khan, A.R., Imdad, M.: Rectangular M_b-metric spaces and fixed point results. J. Math. Anal. **10**(1), 10–18 (2019)
11. Banach, S.: Sur les operations dans les ensembles abstraits et leur application aux equations integrals. Fundam. Math. **3**, 133–181 (1922)
12. Branciari, A.: A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces. Publ. Math. **57**, 31–37 (2000)
13. Budhia, L., Kir, M., Gopal, D., Kiziltunc, H.: New fixed point results in rectangular metric space and application to fractional calculus. Tbil. Math. J. **10**(1), 91–104 (2017)
14. Czerwik, S.: Contraction mappings in *b*-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**(1), 5–11 (1993)
15. Long-Guang, H., Xian, Z.: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468–1476 (2007)
16. Matthews, S.G.: Partial metric topology. Ann. N.Y. Acad. Sci. **728**, 183–197 (1994)
17. Matthews, S.G.: Partial metric topology. Ann. N.Y. Acad. Sci. **728**, 183–197 (1994)
18. Mlaiki, N., Zarrad, A., Souayah, N., Mukheimer, A., Abdeljawed, T.: Fixed point theorems in M_b-metric spaces. J. Math. Anal. **7**, 1–9 (2016)
19. Moeini, B., Asadi, M., Aydi, H.: C^*-Algebra-valued *M*-metric spaces and some related fixed point results. Ital. J. Pure Appl. Math. **41**, 708–723 (2019)
20. Monfared, H., Asadi, M., Azhini, M.: Coupled fixed point theorems for generalized contractions in ordered *M*-metric spaces. Results Fixed Point Theory Appl. **2018**, Article ID 2018004 (2018)
21. Monfared, H., Asadi, M., Azhini, M., O'Regan, D.: $F(\psi;\varphi)$-Contractions for α-admissible mappings on *M*-metric spaces. Fixed Point Theory Appl. **2018**, 22 (2018)
22. Monfared, H., Azhini, M., Asadi, M.: Fixed point results on *M*-metric spaces. J. Math. Anal. **7**(5), 85–101 (2016)
23. Monfared, H., Azhini, M., Asadi, M.: *C*-Class and $F(\psi;\varphi)$-contractions on *M*-metric spaces. Int. J. Nonlinear Anal. Appl. **8**(1), 209–224 (2017)
24. Monfared, H., Azhini, M., Asadi, M.: A generalized contraction principle with control function on *M*-metric spaces. Nonlinear Funct. Anal. Appl. **22**(2), 395–402 (2017)
25. Özgür, N.Y., Mlaiki, N., Tas, N., Souayah, N.: A new generalization of metric spaces: rectangular *M*-metric spaces. Math. Sci. **12**, 223–233 (2018)

26. Padcharoen, A., Gopal, D., Chaipunya, P., Kumam, P.: Fixed point and periodic point results for α-type F-contractions in modular metric spaces. Fixed Point Theory Appl. **2016**, 39 (2016)
27. Patle, P.R., Patel, D.P., Aydi, H., Gopal, D., Mlaiki, M.: Nadler and Kannan type set valued mappings in M-metric spaces and an application. Mathematics **7**(4), 373 (2019)
28. Samet, B.: Discussion on 'A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces' by A. Branciari. Publ. Math. **76**(4), 493–494 (2010)
29. Sarma, I.R., Rao, J.M., Rao, S.S.: Contractions over generalized metric spaces. J. Nonlinear Sci. Appl. **2**(3), 180–182 (2009)
30. Suzuki, T.: Generalized metric spaces do not have the compatible topology. Abstr. Appl. Anal. **2014**, Article ID 458098 (2014)