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An intermixed iteration for constrained convex minimization problem and split feasibility problem

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Abstract

In this paper, we first introduce the two-step intermixed iteration for finding the common solution of a constrained convex minimization problem, and also we prove a strong convergence theorem for the intermixed algorithm. By using our main theorem, we prove a strong convergence theorem for the split feasibility problem. Finally, we apply our main theorem for the numerical example.

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Keywords: Constrained convex minimization problem; Split feasibility problem; Variational inequality

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of a real Hilbert space H.

We denote the fixed point set of a mapping T by $F(T)$. Fixed point theory can be applied to variational inequality problems, equilibrium problems, split feasibility problems, optimization problems, etc. These problems are encountered in various fields such as engineering, physics, game theory, and economics.

A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

In mathematics, conventional optimization problems arise in the process of making a trading system more effective and are usually stated in terms of minimization problems. In this paper, we give a new iteration for solving two constrained convex minimization problems.

Convex constrained minimization problem is popular and very important to various branches in physics, engineering and economics, e.g., to find the minimum travel distance or to find the lowest cost. Consider the constrained convex minimization problem as follows:

$$\text{minimize}\ \{f(x) : x \in C\}, \tag{1}$$

where $f : C \to \mathbb{R}$ is a real-valued convex function. If f is (Fréchet) differentiable, then the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C\big(x_n - \lambda \nabla f(x_n)\big), \quad \forall n \geq 0, \tag{2}$$

or more generally,

$$x_{n+1} = P_C\big(x_n - \lambda_n \nabla f(x_n)\big), \quad \forall n \geq 0, \tag{3}$$

where both in (2) and (3) the initial guess x_0 is taken from C arbitrarily, and the parameters, λ or λ_n, are positive real numbers satisfying certain conditions. The convergence of the algorithms (2) and (3) depends on the behavior of the gradient ∇f. In fact, it is known that if ∇f is α-strongly monotone and L-Lipschitz with constants $\alpha, L \geq 0$, then the operator

$$T := P_C(I - \lambda \nabla f) \tag{4}$$

is a contraction; hence, the sequence $\{x_n\}$ defined by the algorithm (2) converges in norm to the unique minimizer of (1). However, if the gradient ∇f fails to be strongly monotone, the operator T defined by (4) could fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (2) may fail to converge strongly [1]. If ∇f is Lipschitz, then the algorithms (2) and (3) can still converge in the weak topology under certain conditions [2–4].

The variational inequality problem is to find a point $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{5}$$

We denote the set of solutions of the variational inequality by $\mathrm{VI}(C, A)$. Many models of variational inequalities are used in practice, including a mathematical theory, some interesting connections to numerous disciplines and a wide range of important applications in engineering, physics, optimization, minimax problems, game theory, and economics; for more details, see [5, 6].

Su and Xu [3] introduced the relation of a solution to the minimization problem (1) and solutions of the variational inequality (5) as stated in the following Lemma 1, and this lemma helps to prove the theorem about the minimization problem more effectively; for more details, see [7–9].

Lemma 1 (Optimality condition, [3]) *A necessary condition for a point $x^* \in C$ to be a solution of the minimization problem (1) is that x^* solves the variational inequality*

$$\big\langle \nabla f\big(x^*\big), x - x^* \big\rangle \geq 0, \quad \forall x \in C. \tag{6}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\big(x^* - \lambda \nabla f\big(x^*\big)\big),$$

for every constant $\lambda > 0$. If, in addition, f is convex, then the optimality condition (6) is also sufficient.

By U_f we denote the set of solutions of (1).

In 2011, Ceng et al. [10] introduced the following iterative scheme that generates a sequence $\{x_n\}$ in an explicit way:

$$x_{n+1} = P_C\big[s_n\gamma Vx_n + (I - s_n\mu F)T_n x_n\big], \quad \forall n \ge 0,$$

where $s_n = \frac{2-\lambda_n L}{4}$ and $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n)T_n$ for each $n \ge 0$. He proved that the sequence $\{x_n\}$ converges strongly to a minimizer $x^* \in S$ of (1).

In 2014, Ming and Lei [11] introduced an explicit composite iterative method for finding the common element of the set of solutions to an equilibrium problem and the solution set to a constrained convex minimization problem, as well as proved a strong convergence theorem, as follows:

Algorithm 1 Given $x_1 \in C$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated iteratively by

$$\begin{cases} \phi(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = \alpha_n\gamma Vu_n + (I - \alpha_n A)T_n u_n, & \forall n \in \mathbb{N}, \end{cases}$$

where T_n is a nonexpansive mapping from $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$, which is $\frac{2+\lambda_n L}{4}$-averaged with $s_n = \frac{2-\lambda_n L}{4}$, and ∇f is an L-Lipschitz mapping, for all $L \ge 0$, $V : C \to C$ is an l-Lipschitz mapping with constant $l \ge 0$, $A : C \to C$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma} \ge 0$ and $0 < \gamma < \frac{\bar{\gamma}}{l}$, $u_n = Q_{\beta_n}x_n$, $\{\lambda_n\} \subset (0, \frac{2}{L})$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,\infty)$ and $\{s_n\} \subset (0, \frac{1}{2})$.

In 2015, Yao et al. [12] introduced the intermixed algorithm for two strict pseudocontractions S and T as follows:

Algorithm 2 For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], & n \ge 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n], & n \ge 0, \end{cases} \tag{7}$$

where $S, T : C \to C$ are λ-strictly pseudocontractions, $f : C \to H$ is a ρ_1-contraction, and $g : C \to H$ is a ρ_2-contraction, $k \in (0, 1-\lambda)$ is a constant, and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Furthermore, under some control conditions, they proved that the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (7) converge independently to $P_{F(T)}f(y^*)$ and $P_{F(S)}g(x^*)$, respectively, where $x^* \in F(T) = \{z \in C : Tz = z\}$ and $y^* \in F(S) = \{z^* \in C : Tz^* = z^*\}$.

Motivated by Yao et al. [12] and Ming et al. [11], we introduce the new iterative method as follows:

Algorithm 3 Given $x_1, y_1 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{cases} x_{n+1} = (1 - \mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1 - \alpha_n)T_n^{\widetilde{f}_1} x_n), \\ y_{n+1} = (1 - \mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1 - \alpha_n)T_n^{\widetilde{f}_2} y_n), \end{cases} \tag{8}$$

where $f,g : H \to H$ are a_f- and a_g-contraction mappings with $a_f, a_g \in (0,1)$ and $a = \max\{a_f, a_g\}$, $\nabla\widetilde{f_i}$ is an $\frac{1}{L_i}$-inverse strongly monotone with $L_i \geq 0$, for all $i = 1,2$, $\{\mu_n\}$, $\{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f_i}) = s_n^i I + (1 - s_n^i)T_n^{\widetilde{f_i}}$, $\forall i = 1,2$ and $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$ and $0 < \overline{\theta} \leq \mu_n \leq \theta$ for all $n \in \mathbb{N}$ and for some $\overline{\theta}, \theta > 0$.

The purpose of this article is to combine the GPA and averaged mapping approach to design a two-step intermixed iteration for finding the common solution of a constrained convex minimization problem, and also prove a strong convergence theorem for the intermixed algorithm generated by (8). Applying our main result, we prove a strong convergence theorem for the split feasibility problem. Moreover, we utilize our main theorem in the numerical example.

2 Preliminaries

Throughout this article, we always assume that C is a nonempty, closed, and convex subset of a real Hilbert space H. We use "$\rightharpoonup$" for weak convergence and "$\to$" for strong convergence. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Assume that C is a nonempty closed and convex subset of H. A mapping $V : C \to C$ is said to be an l-Lipschitz if there exists a constant $l \geq 0$ such that

$$\|Vx - Vy\| \leq l\|x - y\|, \quad \forall x, y \in C.$$

If $l \in [0,1)$, then V is called a contraction. Obviously, if $l = 1$, V is a nonexpansive mapping.

Definition 1 A mapping $T : H \to H$ is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive.

Definition 2 (Positive operator) An operator A is called *positive* if it is self-adjoint and $\langle Ax, x\rangle \geq 0$ for all $x \in H$.

An operator A on H is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

Lemma 2 ([13]) *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \quad \Longleftrightarrow \quad \langle u - z, v - u \rangle \geq 0, \quad \forall v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C.

Lemma 3 ([14]) *Let H be a real Hilbert space. Then the following results hold:*

(i) *For all $x, y \in H$ and $\alpha \in [0,1]$,*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, *for each $x, y \in H$.*

Lemma 4 ([4]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Definition 3 A mapping $T : H \to H$ is said to be an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1-\alpha)I + \alpha S, \tag{9}$$

where α is a number in $(0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when (9) holds, we say that T is α-averaged.

Clearly, a firmly nonexpansive mapping is a $\frac{1}{2}$-averaged mapping.

Proposition 1 *For given operators $S, T, V : H \to H$:*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composition of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composition $T_1 \circ T_2 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged, and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composition $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Lemma 5 ([11]) *For given $x \in H$ and let $P_C : H \to C$ be a metric projection. Then*

(a) *$z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$.*

(b) *$z = P_C x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$.*

(c) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, $\forall x, y \in H$.

Consequently, P_C is nonexpansive and monotone.

Lemma 6 ([15]) *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$, the inequality*

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

holds for every $v \in H$ with $v \neq u$.

Definition 4 A nonlinear operator T whose domain $D(T) \subseteq H$ and range $R(T) \subseteq H$ is said to be:

(a) monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T);$$

(b) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in D(T);$$

(c) ν-inverse strongly monotone (for short, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

Proposition 2 *Let T be an operator from H to itself. Then*

(a) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism;*
(b) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism;*
(c) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Lemma 7 ([16]) *Assume $A : H \to H$ is a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < t \leq \|A\|^{-1}$. Then $\|I - tA\| \leq 1 - t\overline{\gamma}$.*

3 Main results

Let $V : C \to C$ be l-Lipschitz with coefficient $l \geq 0$, and $A : C \to C$ a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and $0 < \gamma < \frac{\overline{\gamma}}{l}$. Let $f : C \to \mathbb{R}$ be a real-valued convex function and assume that ∇f is an L-Lipschitz mapping with $L \geq 0$. From Xu [1], we have that $P_C(I - \lambda\nabla f)$ is $\frac{2+\lambda L}{4}$-averaged for $0 < \lambda < \frac{2}{L}$ and for each $n \in \mathbb{N}$, that is, we can write

$$P_C(I - \lambda_n \nabla f) = (1 - s_n)I + s_n T_n^f,$$

where T_n^f is nonexpansive and $s_n = \frac{2+\lambda_n L}{4}$.

Theorem 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2$, $\widetilde{f_i} : C \to \mathbb{R}$ be a real-valued convex function and assume that $\nabla\widetilde{f_i}$ is an $\frac{1}{L_i}$-inverse strongly monotone with $L_i > 0$ and $U_{\widetilde{f_i}} \neq \emptyset$. Let $f, g : H \to H$ be a_f- and a_g-contraction mappings, respectively, with $a_f, a_g \in (0, 1)$ and $a = \max\{a_f, a_g\}$. Let the sequences $\{x_n\}$, $\{y_n\}$*

be generated by $x_1, y_1 \in C$ *and*

$$\begin{cases} x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n), \\ y_{n+1} = (1-\mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f}_2}y_n), \end{cases} \tag{10}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f}_i) = s_n^i I + (1-s_n^i)T_n^{\widetilde{f}_i}$, $s_n^i = \frac{2-\lambda_n^i L_i}{4}$ *and* $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$ *for all* $i = 1,2$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < \overline{\theta} \le \mu_n \le \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly as* $s_n^i \to 0$ ($\Longleftrightarrow \lambda_n^i \to \frac{2}{L_i}$) $\forall i = 1,2$, *to* $x^* = P_{\mathcal{U}_{\widetilde{f}_1}} f(y^*)$ *and* $y^* = P_{\mathcal{U}_{\widetilde{f}_2}} g(x^*)$, *respectively.*

Proof First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded. Assume that $\widetilde{x} \in \mathcal{U}_{\widetilde{f}_1}$ and $\widetilde{y} \in \mathcal{U}_{\widetilde{f}_2}$. Then we have

$$\begin{aligned} \|x_{n+1} - \widetilde{x}\| &= \big\|(1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n\big) - \widetilde{x}\big\| \\ &= \big\|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n\big(P_C\big(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n\big) - \widetilde{x}\big)\big\| \\ &\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\| \\ &\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\big(\alpha_n\big\|f(y_n) - \widetilde{x}\big\| + (1-\alpha_n)\big\|T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|\big) \\ &\le (1-\mu_n)\|x_n - \widetilde{x}\| + \mu_n\big(\alpha_n\big\|f(y_n) - \widetilde{x}\big\| + (1-\alpha_n)\|x_n - \widetilde{x}\|\big) \\ &= (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n\big\|f(y_n) - \widetilde{x}\big\| \\ &\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n\big(\big\|f(y_n) - f(\widetilde{y})\big\| + \big\|f(\widetilde{y}) - \widetilde{x}\big\|\big) \\ &\le (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\| + \alpha_n\mu_n a\|y_n - \widetilde{y}\| + \alpha_n\mu_n\big\|f(\widetilde{y}) - \widetilde{x}\big\|. \end{aligned} \tag{11}$$

Similarly, we get

$$\|y_{n+1} - \widetilde{y}\| \le (1-\alpha_n\mu_n)\|y_n - \widetilde{y}\| + \alpha_n\mu_n a\|x_n - \widetilde{x}\| + \alpha_n\mu_n\big\|g(\widetilde{x}) - \widetilde{y}\big\|. \tag{12}$$

Combining (11) and (12), we have

$$\begin{aligned} \|x_{n+1} - \widetilde{x}\| + \|y_{n+1} - \widetilde{y}\| \le{}& \big(1 - \alpha_n\mu_n(1-a)\big)\big(\|x_n - \widetilde{x}\| + \|y_n - \widetilde{y}\|\big) \\ &+ \alpha_n\mu_n\big(\big\|f(\widetilde{y}) - \widetilde{x}\big\| + \big\|g(\widetilde{x}) - \widetilde{y}\big\|\big). \end{aligned}$$

By induction, we can derive that

$$\|x_n - \widetilde{x}\| + \|y_n - \widetilde{y}\| \le \max\big\{\|x_1 - \widetilde{x}\| + \|y_1 - \widetilde{y}\|, \big\|f(\widetilde{y}) - \widetilde{x}\big\| + \big\|g(\widetilde{x}) - \widetilde{y}\big\|\big\},$$

for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{y_n\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$. Observe that

$$\begin{aligned} &\big\|T_n^{\widetilde{f}_1}x_n - T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\| \\ &\quad\le \big\|T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}x_{n-1}\big\| + \big\|T_n^{\widetilde{f}_1}x_{n-1} - T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|x_n - x_{n-1}\| + \left\| \left(\frac{4P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) - (2 - \lambda_n^1 L_1)}{2 + \lambda_n^1 L_1} \right) x_{n-1} \right. \\
&\quad \left. - \left(\frac{4P_C(I - \lambda_{n-1}^1 \nabla\widetilde{f}_1) - (2 - \lambda_{n-1}^1 L_1)}{2 + \lambda_{n-1}^1 L_1} \right) x_{n-1} \right\| \\
&\leq \|x_n - x_{n-1}\| + \left\| \left(\frac{4P_C(I - \lambda_n^1 \nabla\widetilde{f}_1)}{2 + \lambda_n^1 L_1} \right) x_{n-1} - \left(\frac{4P_C(I - \lambda_{n-1}^1 \nabla\widetilde{f}_1)}{2 + \lambda_{n-1}^1 L_1} \right) x_{n-1} \right\| \\
&\quad + \left\| \left(\frac{2 - \lambda_{n-1}^1 L_1}{2 + \lambda_{n-1}^1 L_1} \right) x_{n-1} - \left(\frac{2 - \lambda_n^1 L_1}{2 + \lambda_n^1 L_1} \right) x_{n-1} \right\| \\
&= \|x_n - x_{n-1}\| \\
&\quad + \left\| \frac{4(2 + \lambda_{n-1}^1 L_1) P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) x_{n-1} - 4(2 + \lambda_n^1 L_1) P_C(I - \lambda_{n-1}^1 \nabla\widetilde{f}_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&\quad + \left\| \frac{(2 - \lambda_{n-1}^1 L_1)(2 + \lambda_n^1 L_1) x_{n-1} - (2 - \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&= \|x_n - x_{n-1}\| \\
&\quad + \left\| \frac{4(2 + \lambda_{n-1}^1 L_1) P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) x_{n-1} - 4(2 + \lambda_n^1 L_1) P_C(I - \lambda_{n-1}^1 \nabla\widetilde{f}_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&\quad + \left(\frac{4L_1 |\lambda_n^1 - \lambda_{n-1}^1|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right) \|x_{n-1}\| \\
&= \|x_n - x_{n-1}\| \\
&\quad + \left\| \frac{4L_1(\lambda_{n-1}^1 - \lambda_n^1) P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) x_{n-1}}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right. \\
&\quad \left. + \frac{4(2 + \lambda_n^1 L_1)(P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) x_{n-1} - P_C(I - \lambda_{n-1}^1 \nabla\widetilde{f}_1) x_{n-1})}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right\| \\
&\quad + \left(\frac{4L_1 |\lambda_n^1 - \lambda_{n-1}^1|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right) \|x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| \\
&\quad + \frac{4L_1 |\lambda_{n-1}^1 - \lambda_n^1| \|P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) x_{n-1}\|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} + \frac{4\|\lambda_{n-1}^1 \nabla\widetilde{f}_1 x_{n-1} - \lambda_n^1 \nabla\widetilde{f}_1 x_{n-1}\|}{2 + \lambda_{n-1}^1 L_1} \\
&\quad + \left(\frac{4L_1 |\lambda_n^1 - \lambda_{n-1}^1|}{(2 + \lambda_n^1 L_1)(2 + \lambda_{n-1}^1 L_1)} \right) \|x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + L_1 \bigl|\lambda_{n-1}^1 - \lambda_n^1\bigr| \bigl\|P_C\bigl(I - \lambda_n^1 \nabla\widetilde{f}_1\bigr) x_{n-1}\bigr\| \\
&\quad + 4\bigl|\lambda_{n-1}^1 - \lambda_n^1\bigr| \|\nabla\widetilde{f}_1 x_{n-1}\| + L_1 \bigl|\lambda_n^1 - \lambda_{n-1}^1\bigr| \|x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| \\
&\quad + \bigl|\lambda_{n-1}^1 - \lambda_n^1\bigr| \bigl(L_1 \bigl\|P_C\bigl(I - \lambda_n^1 \nabla\widetilde{f}_1\bigr) x_{n-1}\bigr\| + 4\|\nabla\widetilde{f}_1 x_{n-1}\| + L_1 \|x_{n-1}\|\bigr) \\
&\leq \|x_n - x_{n-1}\| + M_1 \bigl|\lambda_{n-1}^1 - \lambda_n^1\bigr|,
\end{aligned} \tag{13}
$$

for some $M_1 > 0$ such that $M_1 \geq L_1 \|P_C(I - \lambda_n^1 \nabla\widetilde{f}_1) x_{n-1}\| + 4\|\nabla\widetilde{f}_1 x_{n-1}\| + L_1 \|x_{n-1}\|$, $\forall n \geq 1$.

From the definition of x_n and (13), we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad= \Big\|(1-\mu_n)x_n+\mu_n P_C\big(\alpha_n f(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n\big) \\
&\qquad-\big((1-\mu_{n-1})x_{n-1}+\mu_{n-1}P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\big)\Big\| \\
&\quad\le (1-\mu_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+\mu_n\Big\|P_C\big(\alpha_n f(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n\big) \\
&\qquad-P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\quad\le (1-\mu_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+\mu_n\Big\|\alpha_n f(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\quad\le (1-\mu_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+\mu_n\Big(\big\|\alpha_n f(y_n)-\alpha_{n-1}f(y_{n-1})\big\|+\big\|(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|\Big) \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\quad\le (1-\mu_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+\mu_n\Big(\alpha_n\big\|f(y_n)-f(y_{n-1})\big\|+|\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\qquad+(1-\alpha_n)\big\|T_n^{\widetilde{f_1}}x_n-T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|+|\alpha_{n-1}-\alpha_n|\big\|T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|\Big) \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\quad\le (1-\mu_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+\mu_n\Big(\alpha_n\big\|f(y_n)-f(y_{n-1})\big\|+|\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\qquad+(1-\alpha_n)\big(\|x_n-x_{n-1}\|+M_1\big|\lambda_{n-1}^1-\lambda_n^1\big|\big)+|\alpha_{n-1}-\alpha_n|\big\|T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|\Big) \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\quad\le (1-\mu_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+\mu_n\bigg(\alpha_n\big\|f(y_n)-f(y_{n-1})\big\|+|\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\qquad+(1-\alpha_n)\|x_n-x_{n-1}\|+(1-\alpha_n)\frac{4M_1}{L_1}\big|s_n^1-s_{n-1}^1\big| \\
&\qquad+|\alpha_{n-1}-\alpha_n|\big\|T_{n-1}^{\widetilde{f_1}}x_{n-1}\big\|\bigg) \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\| \\
&\quad\le (1-\mu_n\alpha_n)\|x_n-x_{n-1}\|+|\mu_{n-1}-\mu_n|\|x_{n-1}\| \\
&\qquad+|\mu_n-\mu_{n-1}|\Big\|P_C\big(\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})T_{n-1}^{\widetilde{f_1}}x_{n-1}\big)\Big\|
\end{aligned}
$$

$$+ \mu_n\Big(\alpha_n a\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| + (1-\alpha_n)\frac{4M_1}{L_1}\big|s_n^1 - s_{n-1}^1\big| + |\alpha_{n-1} - \alpha_n|\big\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\|\Big). \tag{14}$$

Using the same method as derived in (14), we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\le (1-\mu_n\alpha_n)\|y_n - y_{n-1}\| + |\mu_{n-1} - \mu_n|\|y_{n-1}\| \\
&\quad + |\mu_n - \mu_{n-1}|\big\|P_C\big(\alpha_{n-1}g(x_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_2}y_{n-1}\big)\big\| \\
&\quad + \mu_n\Big(\alpha_n a\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|g(x_{n-1})\big\| \\
&\quad + (1-\alpha_n)\frac{4M_2}{L_2}\big|s_n^2 - s_{n-1}^2\big| + |\alpha_{n-1} - \alpha_n|\big\|T_{n-1}^{\widetilde{f}_2}y_{n-1}\big\|\Big),
\end{aligned} \tag{15}$$

for some $M_2 > 0$ such that $M_2 \ge L_2\|P_C(I - \lambda_n^2\nabla\widetilde{f}_2)y_{n-1}\| + 4\|\nabla\widetilde{f}_2 y_{n-1}\| + L_2\|y_{n-1}\|$, $\forall n \ge 1$.

From (14) and (15), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
&\le \big(1 - (1-a)\mu_n\alpha_n\big)\big(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|\big) \\
&\quad + |\mu_{n-1} - \mu_n|\big(\|x_{n-1}\| + \|y_{n-1}\| \\
&\quad + \big\|P_C\big(\alpha_{n-1}f(y_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_1}x_{n-1}\big)\big\| \\
&\quad + \big\|P_C\big(\alpha_{n-1}g(x_{n-1}) + (1-\alpha_{n-1})T_{n-1}^{\widetilde{f}_2}y_{n-1}\big)\big\|\big) \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|f(y_{n-1})\big\| + \big\|g(x_{n-1})\big\| + \big\|T_{n-1}^{\widetilde{f}_1}x_{n-1}\big\| + \big\|T_{n-1}^{\widetilde{f}_2}y_{n-1}\big\|\big) \\
&\quad + (1-\alpha_n)\Big(\frac{4M_1}{L_1}\big|s_n^1 - s_{n-1}^1\big| + \frac{4M_2}{L_2}\big|s_n^2 - s_{n-1}^2\big|\Big).
\end{aligned}$$

Applying Lemma 4 and condition (iii), we can conclude that

$$\|x_{n+1} - x_n\| \to 0 \quad \text{and} \quad \|y_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{16}$$

Next, we show that $\|x_n - W_n\| \to 0$ where $W_n = \alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n$ and $\|y_n - V_n\| \to 0$ where $V_n = \alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f}_2}y_n$. Let $\widetilde{x} \in U_{\widetilde{f}_1}$ and $\widetilde{y} \in U_{\widetilde{f}_2}$. Then we derive that

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\|^2 &= \big\|(1-\mu_n)x_n + \mu_n P_C W_n - \widetilde{x}\big\|^2 \\
&= \big\|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n(P_C W_n - \widetilde{x})\big\|^2 \\
&= (1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|P_C W_n - \widetilde{x}\|^2 \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2 \\
&\le (1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\big\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1}x_n - \widetilde{x}\big\|^2 \\
&\quad - (1-\mu_n)\mu_n\|x_n - P_C W_n\|^2
\end{aligned}$$

$$\begin{aligned}
&= (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\big\|\alpha_n\big(f(y_n)-T_n^{\widetilde{f_1}}x_n\big) + T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\|^2 \\
&\quad - (1-\mu_n)\mu_n\|x_n-P_CW_n\|^2 \\
&\le (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\big(\big\|T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\|^2 \\
&\quad + 2\alpha_n\big\langle f(y_n)-T_n^{\widetilde{f_1}}x_n, \alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\rangle\big) \\
&\quad - (1-\mu_n)\mu_n\|x_n-P_CW_n\|^2 \\
&\le (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\big(\big\|T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\|^2 \\
&\quad + 2\alpha_n\big\|f(y_n)-T_n^{\widetilde{f_1}}x_n\big\|\big\|\alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\|\big) \\
&\quad - (1-\mu_n)\mu_n\|x_n-P_CW_n\|^2 \\
&\le (1-\mu_n)\|x_n-\widetilde{x}\|^2 + \mu_n\|x_n-\widetilde{x}\|^2 \\
&\quad + 2\alpha_n\mu_n\big\|f(y_n)-T_n^{\widetilde{f_1}}x_n\big\|\big\|\alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\| \\
&\quad - (1-\mu_n)\mu_n\|x_n-P_CW_n\|^2 \\
&= \|x_n-\widetilde{x}\|^2 \\
&\quad + 2\alpha_n\mu_n\big\|f(y_n)-T_n^{\widetilde{f_1}}x_n\big\|\big\|\alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\| \\
&\quad - (1-\mu_n)\mu_n\|x_n-P_CW_n\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\mu_n)\mu_n\|x_n-P_CW_n\|^2 \\
&\quad\le \|x_n-\widetilde{x}\|^2 - \|x_{n+1}-\widetilde{x}\|^2 \\
&\qquad + 2\alpha_n\mu_n\big\|f(y_n)-T_n^{\widetilde{f_1}}x_n\big\|\big\|\alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\| \\
&\quad\le \|x_n-x_{n+1}\|\big(\|x_n-\widetilde{x}\| + \|x_{n+1}-\widetilde{x}\|\big) \\
&\qquad + 2\alpha_n\mu_n\big\|f(y_n)-T_n^{\widetilde{f_1}}x_n\big\|\big\|\alpha_nf(y_n)+(1-\alpha_n)T_n^{\widetilde{f_1}}x_n-\widetilde{x}\big\|.
\end{aligned}$$

By (16), as well as conditions (i) and (ii), we get

$$\|P_CW_n-x_n\| \to 0 \quad \text{as } n\to\infty. \tag{17}$$

From definition of x_n and applying the same method as (17), we have

$$\|P_CV_n-y_n\| \to 0 \quad \text{as } n\to\infty. \tag{18}$$

Considering

$$\begin{aligned}
\|P_CW_n-\widetilde{x}\|^2 &= \|P_CW_n-P_C\widetilde{x}\|^2 \\
&\le \langle W_n-\widetilde{x}, P_CW_n-\widetilde{x}\rangle \\
&= \frac{1}{2}\big(\|W_n-\widetilde{x}\|^2 + \|P_CW_n-\widetilde{x}\|^2 - \|W_n-P_CW_n\|^2\big)
\end{aligned}$$

implies that

$$\|P_C W_n - \widetilde{x}\| \leq \|W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2. \tag{19}$$

Observe that

$$\begin{aligned}
\|W_n - \widetilde{x}\|^2 &= \left\|\alpha_n\big(f(y_n) - \widetilde{x}\big) + (1-\alpha_n)\big(T_n^{\widetilde{f}_1} x_n - \widetilde{x}\big)\right\|^2 \\
&\leq \alpha_n \left\|f(y_n) - \widetilde{x}\right\|^2 + (1-\alpha_n)\left\|T_n^{\widetilde{f}_1} x_n - \widetilde{x}\right\|^2 \\
&\leq \alpha_n \left\|f(y_n) - \widetilde{x}\right\|^2 + (1-\alpha_n)\|x_n - \widetilde{x}\|^2.
\end{aligned} \tag{20}$$

From (19) and (20), we obtain

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\|^2 &= \left\|(1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n\big) - \widetilde{x}\right\|^2 \\
&= \left\|(1-\mu_n)(x_n - \widetilde{x}) + \mu_n(P_C W_n - \widetilde{x})\right\|^2 \\
&\leq (1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\|P_C W_n - \widetilde{x}\|^2 \\
&\leq (1-\mu_n)\|x_n - \widetilde{x}\|^2 + \mu_n\big(\|W_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2\big) \\
&\leq (1-\mu_n)\|x_n - \widetilde{x}\|^2 \\
&\quad + \mu_n\big(\alpha_n\left\|f(y_n) - \widetilde{x}\right\|^2 + (1-\alpha_n)\|x_n - \widetilde{x}\|^2 - \|W_n - P_C W_n\|^2\big),
\end{aligned}$$

implying that

$$\begin{aligned}
\mu_n\|W_n - P_C W_n\|^2 &\leq (1-\alpha_n\mu_n)\|x_n - \widetilde{x}\|^2 - \|x_{n+1} - \widetilde{x}\|^2 + \alpha_n\mu_n\left\|f(y_n) - \widetilde{x}\right\|^2 \\
&\leq \|x_n - \widetilde{x}\|^2 - \|x_{n+1} - \widetilde{x}\|^2 + \alpha_n\mu_n\left\|f(y_n) - \widetilde{x}\right\|^2 \\
&\leq \|x_n - x_{n+1}\|\big(\|x_n - \widetilde{x}\| + \|x_{n+1} - \widetilde{x}\|\big) + \alpha_n\mu_n\left\|f(y_n) - \widetilde{x}\right\|^2.
\end{aligned}$$

From $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and condition (i), we have

$$\|W_n - P_C W_n\| \to 0 \quad \text{as } n \to \infty. \tag{21}$$

From definition of V_n and applying the same argument as (21), we also obtain

$$\|V_n - P_C V_n\| \to 0 \quad \text{as } n \to \infty. \tag{22}$$

Since

$$\begin{aligned}
\|x_n - W_n\| &= \|x_n - P_C W_n + P_C W_n - W_n\| \\
&\leq \|x_n - P_C W_n\| + \|P_C W_n - W_n\|.
\end{aligned}$$

From (17) and (21), we have

$$\|x_n - W_n\| \to 0 \quad \text{as } n \to \infty. \tag{23}$$

From definition of y_n and applying the same method as in (23), we also have

$$\|y_n - V_n\| \to 0 \quad \text{as } n \to \infty. \tag{24}$$

Next, we show that $\|W_n - P_C(I - \frac{2}{L_1}\nabla\widetilde{f}_1)W_n\| \to 0$ as $n \to \infty$ and $\|V_n - P_C(I - \frac{2}{L_2}\nabla\widetilde{f}_2)V_n\| \to 0$ as $n \to \infty$. Observe that

$$W_n - x_n = \alpha_n\bigl(f(y_n) - x_n\bigr) + (1-\alpha_n)\bigl(T_n^{\widetilde{f}_1}x_n - x_n\bigr),$$

which yields

$$(1-\alpha_n)\bigl\|T_n^{\widetilde{f}_1}x_n - x_n\bigr\| \le \|W_n - x_n\| + \alpha_n\bigl\|f(y_n) - x_n\bigr\|.$$

From (23) and condition (i), we have

$$\bigl\|T_n^{\widetilde{f}_1}x_n - x_n\bigr\| \to 0 \quad \text{as } n \to \infty. \tag{25}$$

Since

$$\begin{aligned}\bigl\|W_n - T_n^{\widetilde{f}_1}W_n\bigr\| &= \bigl\|W_n - x_n + x_n - T_n^{\widetilde{f}_1}x_n + T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}W_n\bigr\| \\ &\le \|W_n - x_n\| + \bigl\|x_n - T_n^{\widetilde{f}_1}x_n\bigr\| + \bigl\|T_n^{\widetilde{f}_1}x_n - T_n^{\widetilde{f}_1}W_n\bigr\| \\ &\le \|W_n - x_n\| + \bigl\|x_n - T_n^{\widetilde{f}_1}x_n\bigr\| + \|x_n - W_n\| \\ &= 2\|x_n - W_n\| + \bigl\|T_n^{\widetilde{f}_1}x_n - x_n\bigr\|.\end{aligned}$$

From (23) and (25), we get

$$\bigl\|T_n^{\widetilde{f}_1}W_n - W_n\bigr\| \to 0 \quad \text{as } n \to \infty. \tag{26}$$

Observe that

$$\begin{aligned}\bigl\|P_C\bigl(I - \lambda_n^1\nabla\widetilde{f}_1\bigr)W_n - W_n\bigr\| &= \bigl\|s_n^1W_n + \bigl(1 - s_n^1\bigr)T_n^{\widetilde{f}_1}W_n - W_n\bigr\| \\ &= \bigl(1 - s_n^1\bigr)\bigl\|T_n^{\widetilde{f}_1}W_n - W_n\bigr\| \\ &\le \bigl\|T_n^{\widetilde{f}_1}W_n - W_n\bigr\|,\end{aligned} \tag{27}$$

where $s_n^1 = \frac{2-\lambda_n^1L_1}{4} \in (0, \frac{1}{2})$.

From (27), we have

$$\begin{aligned}&\left\|P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n - W_n\right\| \\ &\quad\le \left\|P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n - P_C\bigl(I - \lambda_n^1\nabla\widetilde{f}_1\bigr)W_n\right\| + \bigl\|P_C\bigl(I - \lambda_n^1\nabla\widetilde{f}_1\bigr)W_n - W_n\bigr\| \\ &\quad\le \left\|\left(I - \frac{2}{L_1}\nabla\widetilde{f}_1\right)W_n - \bigl(I - \lambda_n^1\nabla\widetilde{f}_1\bigr)W_n\right\| + \bigl\|P_C\bigl(I - \lambda_n^1\nabla\widetilde{f}_1\bigr)W_n - W_n\bigr\| \\ &\quad\le \left(\frac{2}{L_1} - \lambda_n^1\right)\bigl\|\nabla\widetilde{f}_1(W_n)\bigr\| + \bigl\|T_n^{\widetilde{f}_1}W_n - W_n\bigr\|.\end{aligned}$$

From the boundedness of $\{W_n\}$, $s_n^1 \to 0$ ($\Longleftrightarrow \lambda_n^1 \to \frac{2}{L_1}$) and (26), we conclude that

$$\lim_{n\to\infty} \left\| W_n - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f_1}\right)W_n \right\| = 0. \tag{28}$$

Applying the same method as for (28), we also have

$$\lim_{n\to\infty} \left\| V_n - P_C\left(I - \frac{2}{L_2}\nabla\widetilde{f_2}\right)V_n \right\| = 0. \tag{29}$$

Next, we show that $\limsup_{n\to\infty}\langle f(y^*) - x^*, W_n - x^*\rangle \leq 0$, where $x^* = P_{U_{\widetilde{f_1}}} f(y^*)$ and $\limsup_{n\to\infty}\langle g(x^*) - y^*, V_n - y^*\rangle \leq 0$, where $y^* = P_{U_{\widetilde{f_2}}} g(x^*)$.

Indeed, take a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ such that

$$\limsup_{n\to\infty}\langle f(y^*) - x^*, W_n - x^*\rangle = \limsup_{k\to\infty}\langle f(y^*) - x^*, W_{n_k} - x^*\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$. From (23), we obtain $W_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$. Assume that $\widehat{x} \neq P_C(I - \frac{2}{L_1}\nabla\widetilde{f_1})\widehat{x}$. By nonexpansiveness of $P_C(I - \frac{2}{L_1}\nabla\widetilde{f_1})$, (28) and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty}\|W_{n_k} - \widehat{x}\| &< \liminf_{k\to\infty}\left\| W_{n_k} - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f_1}\right)\widehat{x}\right\| \\
&\leq \liminf_{k\to\infty}\left(\left\| W_{n_k} - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f_1}\right)W_{n_k}\right\| \right. \\
&\quad \left. + \left\| P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f_1}\right)W_{n_k} - P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f_1}\right)\widehat{x}\right\| \right) \\
&\leq \liminf_{k\to\infty}\|W_{n_k} - \widehat{x}\|.
\end{aligned}$$

This is a contradiction, thus we have

$$\widehat{x} \in F\left(P_C\left(I - \frac{2}{L_1}\nabla\widetilde{f_1}\right)\right) = U_{\widetilde{f_1}}. \tag{30}$$

Since $W_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$, due to (30) and Lemma 2, we can derive that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(y^*) - x^*, W_n - x^*\rangle &= \limsup_{k\to\infty}\langle f(y^*) - x^*, W_{n_k} - x^*\rangle \\
&= \langle f(y^*) - x^*, \widehat{x} - x^*\rangle \\
&\leq 0.
\end{aligned} \tag{31}$$

Similarly, take a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$\limsup_{n\to\infty}\langle g(x^*) - y^*, V_n - y^*\rangle = \limsup_{k\to\infty}\langle g(x^*) - y^*, V_{n_k} - y^*\rangle.$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$. From (24), we obtain $V_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$. Following the same method as for (31), we easily obtain that

$$\limsup_{n\to\infty}\langle g(x^*) - y^*, V_n - y^*\rangle \leq 0. \tag{32}$$

Finally, we show that $\{x_n\}$ converges strongly to x^*, where $x^* = P_{U_{\widetilde{f}_1}} f(y^*)$ and $\{y_n\}$ converges strongly to y^*, where $y^* = P_{U_{\widetilde{f}_2}} g(x^*)$.

Let $W_n = \alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n$ and $V_n = \alpha_n g(x_n) + (1-\alpha_n)T_n^{\widetilde{f}_2} y_n$. From the definition of x_n, we get

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|(1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n) - x^*\|^2 \\
&\quad = \|(1-\mu_n)(x_n - x^*) + \mu_n(P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n) - x^*)\|^2 \\
&\quad = (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n) - x^*\|^2 \\
&\quad \leq (1-\mu_n)\|x_n - x^*\|^2 + \mu_n\|\alpha_n f(y_n) + (1-\alpha_n)T_n^{\widetilde{f}_1} x_n - x^*\|^2 \\
&\quad = (1-\mu_n)\|x_n - x^*\|^2 \\
&\qquad + \mu_n\|\alpha_n(f(y_n) - x^*) + (1-\alpha_n)(T_n^{\widetilde{f}_1} x_n - x^*)\|^2 \\
&\quad \leq (1-\mu_n)\|x_n - x^*\|^2 \\
&\qquad + \mu_n((1-\alpha_n)\|T_n^{\widetilde{f}_1} x_n - x^*\|^2 + 2\alpha_n\langle f(y_n) - x^*, W_n - x^*\rangle) \\
&\quad \leq (1-\mu_n)\|x_n - x^*\|^2 \\
&\qquad + \mu_n(1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\mu_n\langle f(y_n) - x^*, W_n - x^*\rangle \\
&\quad = (1-\alpha_n\mu_n)\|x_n - x^*\|^2 + 2\alpha_n\mu_n\langle f(y_n) - x^*, W_n - x^*\rangle \\
&\quad = (1-\alpha_n\mu_n)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n\mu_n(\langle f(y_n) - f(y^*), W_n - x^*\rangle + \langle f(y^*) - x^*, W_n - x^*\rangle) \\
&\quad \leq (1-\alpha_n\mu_n)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n\mu_n(\|f(y_n) - f(y^*)\|\|W_n - x^*\| + \langle f(y^*) - x^*, W_n - x^*\rangle) \\
&\quad \leq (1-\alpha_n\mu_n)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n\mu_n\|f(y_n) - f(y^*)\|(\|W_n - x_{n+1}\| + \|x_{n+1} - x^*\|) \\
&\qquad + 2\alpha_n\mu_n\langle f(y^*) - x^*, W_n - x^*\rangle \\
&\quad \leq (1-\alpha_n\mu_n)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n\mu_n a\|y_n - y^*\|\|W_n - x_{n+1}\| + 2\alpha_n\mu_n a\|y_n - y^*\|\|x_{n+1} - x^*\| \\
&\qquad + 2\alpha_n\mu_n\langle f(y^*) - x^*, W_n - x^*\rangle \\
&\quad \leq (1-\alpha_n\mu_n)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n\mu_n a\|y_n - y^*\|\|W_n - x_{n+1}\| + \alpha_n\mu_n a(\|y_n - y^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\qquad + 2\alpha_n\mu_n\langle f(y^*) - x^*, W_n - x^*\rangle,
\end{aligned}$$

which yields

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\leq \frac{1-\alpha_n\mu_n}{1-\alpha_n\mu_n a}\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\| \\
&\quad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle \\
&=\left(1-\frac{\alpha_n\mu_n-\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\right)\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\| \\
&\quad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle \\
&=\left(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\right)\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|\|W_n-x_{n+1}\| \\
&\quad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle f(y^*)-x^*,W_n-x^*\rangle.
\end{aligned} \tag{33}$$

Similarly, as derived above, we also have

$$\begin{aligned}
&\|y_{n+1}-y^*\|^2 \\
&\leq\left(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\right)\|y_n-y^*\|^2+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|x_n-x^*\|\|V_n-y_{n+1}\| \\
&\quad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\|x_n-x^*\|^2+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\langle g(x^*)-y^*,V_n-y^*\rangle.
\end{aligned} \tag{34}$$

From (33) and (34), we deduce that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 \\
&\leq\left(1-\frac{\alpha_n\mu_n(1-a)}{1-\alpha_n\mu_n a}\right)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\
&\quad+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\left(\|y_n-y^*\|\|W_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|\right) \\
&\quad+\frac{\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\
&\quad+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\left(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle\right) \\
&=\left(1-\frac{\alpha_n\mu_n(1-2a)}{1-\alpha_n\mu_n a}\right)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\
&\quad+\frac{2\alpha_n\mu_n a}{1-\alpha_n\mu_n a}\left(\|y_n-y^*\|\|W_n-x_{n+1}\|+\|x_n-x^*\|\|V_n-y_{n+1}\|\right) \\
&\quad+\frac{2\alpha_n\mu_n}{1-\alpha_n\mu_n a}\left(\langle f(y^*)-x^*,W_n-x^*\rangle+\langle g(x^*)-y^*,V_n-y^*\rangle\right).
\end{aligned} \tag{35}$$

By (16), (23), (24), (31), (32), condition (i) and Lemma 4, we have $\lim_{n\to\infty}(\|x_n-x^*\|+\|y_n-y^*\|)=0$. It implies that the sequences $\{x_n\}$, $\{y_n\}$ converge to $x^*=P_{U_{\tilde{f}_1}}f(y^*)$, $y^*=P_{U_{\tilde{f}_2}}g(x^*)$, respectively. This completes the proof. □

Corollary 1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\widetilde{f}: C \to \mathbb{R}$ be a real-valued convex function and assume that $\nabla\widetilde{f}$ is $\frac{1}{L}$-inverse strongly monotone with $L > 0$ and $U_{\widetilde{f}} \neq \emptyset$. Let $f : H \to H$ be an a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C\big(\alpha_n f(x_n) + (1-\alpha_n)T_n^{\widetilde{f}} x_n\big), \tag{36}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n^i \nabla\widetilde{f}) = s_n I + (1-s_n)T_n^{\widetilde{f}}$ and $s_n = \frac{2-\lambda_n L}{4}$, $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \le \mu_n \le \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ converges strongly, as $s_n \to 0$ ($\Longleftrightarrow \lambda_n \to \frac{2}{L}$), to $x^* = P_{U_{\widetilde{f}}} f(x^*)$.

Proof If we put $f \equiv g$, $x_n = y_n$, in Theorem 1, we obtain the desired conclusion. □

4 Application

Let H_1, H_2 be two real Hilbert spaces. Let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively.

In 1994, Censor and Elfving [17] introduced the *split feasibility problem* (SFP), which is to find a point x such that

$$x \in C \quad \text{and} \quad Dx \in Q,$$

where $D : H_1 \to H_2$ is a bounded linear operator.

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f : \mathcal{H}_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem

$$\min_{x\in C} f(x) := \min_{x\in C} \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{37}$$

is ill-posed.

Before proving Theorem 2, we need the following:

Proposition 3 ([18]) *Given $x^* \in \mathcal{H}_1$, the following statements are equivalent*:

(i) *x^* solves the SFP*;

(ii) $P_C(I - \lambda\nabla f)x^* = P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$;

(iii) *x^* solves the variational inequality problem of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^* \rangle \ge 0, \quad \forall x \in C, \tag{38}$$

where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.

Theorem 2 *Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and let $A_i : H_1 \to H_2$ be bounded linear operators for all $i = 1,2$ with L_i being the spectral radius of $A_i^* A_i$ for all $i = 1,2$ with $\Gamma_i \neq \emptyset$. Let $f, g : H \to H$ be a_f- and a_g-contraction* *mappings with* $a_f, a_g \in (0,1)$ *and* $a = \max\{a_f, a_g\}$. *Let the sequences* $\{x_n\}$, $\{y_n\}$ *be generated by* $x_1, y_1 \in C$ *and*

$$\begin{cases} x_{n+1} = (1-\mu_n)x_n + \mu_n P_C(\alpha_n f(y_n) + (1-\alpha_n)T_n^{a_1}x_n), \\ y_{n+1} = (1-\mu_n)y_n + \mu_n P_C(\alpha_n g(x_n) + (1-\alpha_n)T_n^{a_2}y_n), \end{cases} \tag{39}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I-\lambda_n^i(A_i^*(I-P_Q)A_i)) = s_n^i I + (1-s_n^i)T_n^{a_i}$, $\forall i = 1,2$ *and* $s_n^i = \frac{2-\lambda_n^i L_i}{4}$, $\{\lambda_n^i\} \subset (0, \frac{2}{L_i})$. *Assume that the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \overline{\theta} \le \mu_n \le \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly, as* $s_n^i \to 0$ ($\Longleftrightarrow \lambda_n^i \to \frac{2}{L_i}$) $\forall i = 1,2$, *to* $x^* = P_{\Gamma_1} f(y^*)$ *with* $\Gamma_1 = \{x \in C; A_1 x \in Q\}$ *and* $y^* = P_{\Gamma_2} g(x^*)$ *with* $\Gamma_2 = \{\bar{x} \in C; A_2\bar{x} \in Q\}$, *respectively*.

Proof Letting $x, y \in C$ and $\nabla f_i = A_i^*(I-P_Q)A_i$ for all $i = 1,2$, we have

$$\begin{aligned} \|\nabla f_i(x) - \nabla f_i(y)\|^2 &= \|A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y\|^2 \\ &\le L_i \|(I-P_Q)A_i x - (I-P_Q)A_i y\|^2. \end{aligned} \tag{40}$$

From the property of P_C, we have

$$\begin{aligned} &\|(I-P_Q)A_i x - (I-P_Q)A_i y\|^2 \\ &\quad = \langle (I-P_Q)A_i x - (I-P_Q)A_i y, (I-P_Q)A_i x - (I-P_Q)A_i y\rangle \\ &\quad = \langle (I-P_Q)A_i x - (I-P_Q)A_i y, A_i x - A_i y\rangle \\ &\qquad - \langle (I-P_Q)A_i x - (I-P_Q)A_i y, P_Q A_i x - P_Q A_i y\rangle \\ &\quad = \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle \\ &\qquad - \langle (I-P_Q)A_i x - (I-P_Q)A_i y, P_Q A_i x - P_Q A_i y\rangle \\ &\quad = \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle \\ &\qquad - \langle (I-P_Q)A_i x, P_Q A_i x - P_Q A_i y\rangle \\ &\qquad + \langle (I-P_Q)A_i y, P_Q A_i x - P_Q A_i y\rangle \\ &\quad \le \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle. \end{aligned} \tag{41}$$

Substituting (41) into (40), we have

$$\begin{aligned} \|\nabla f_i(x) - \nabla f_i(y)\|^2 &\le L_i \langle A_i^*(I-P_Q)A_i x - A_i^*(I-P_Q)A_i y, x - y\rangle \\ &= L_i \langle \nabla f_i(x) - \nabla f_i(y), x - y\rangle. \end{aligned}$$

It follows that

$$\langle \nabla f_i(x) - \nabla f_i(y), x - y\rangle \ge \frac{1}{L_i}\|\nabla f_i(x) - \nabla f_i(y)\|^2.$$

Then ∇f_i is $\frac{1}{L_i}$-inverse strongly monotone, for all $i = 1,2$.

From Proposition 3 and Theorem 1, we can conclude that Theorem 2 is true. □

Corollary 2 *Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be bounded linear operator with L being the spectral radius of A^*A with $\Gamma \neq \emptyset$. Let $f : H \to H$ be an a-contraction mapping with $a \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = (1-\mu_n)x_n + \mu_n P_C\bigl(\alpha_n f(x_n) + (1-\alpha_n)T_n^{a_1}x_n\bigr), \tag{42}$$

where $\{\mu_n\}, \{\alpha_n\} \subseteq [0,1]$, $P_C(I - \lambda_n(A^*(I-P_Q)A)) = s_nI + (1-s_n)T_n^{a_1}$ and $s_n = \frac{2-\lambda_n L}{4}$, $\{\lambda_n\} \subset (0, \frac{2}{L})$. Assume that the following conditions hold:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \overline{\theta} \leq \mu_n \leq \theta$ *for all* $n \in \mathbb{N}$ *and for some* $\overline{\theta}, \theta > 0$,
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.
Then $\{x_n\}$ converges strongly, as $s_n \to 0$ ($\Longleftrightarrow \lambda_n \to \frac{2}{L}$), to $x^* = P_\Gamma f(x^*)$ with $\Gamma = \{x \in C; Ax \in Q\}$.

Proof If we put $f \equiv g$, $x_n = y_n$ in Theorem 2, then the conclusion follows. □

5 Numerical examples

Example 1 Let $C = [-10,10] \times [-10,10]$ and let $\langle \cdot,\cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an inner product defined by $\langle \mathbf{x},\mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2$, for all $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$. For every $i = 1,2$, let $\widetilde{f}_i : C \to \mathbb{R}$ be defined by $\widetilde{f}_1(x_1,x_2) = 2x_1^2 + x_2$ and $\widetilde{f}_2(x_1,x_2) = (x_1-1) + x_2^2$, $\forall x_1,x_2 \in \mathbb{R}$. Let $f,g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $f(x_1,x_2) = (\frac{x_1}{3}, \frac{x_2}{3})$ and $g(x_1,x_2) = (\frac{x_1}{4}, \frac{x_2}{4})$, be $\frac{1}{2}$- and $\frac{1}{3}$-contraction mappings and $a = \max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by $x_1, y_1 \in C$. Putting $\alpha_n = \frac{1}{4n}$ and $\mu_n = \frac{3n+1}{7n}$, we can rewrite (10) as follows:

$$\begin{cases} x_{n+1} = (\frac{4n-1}{7n})x_n + (\frac{3n+1}{7n})P_C(\frac{1}{4n}f(y_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}_1}x_n), \\ y_{n+1} = (\frac{4n-1}{7n})y_n + (\frac{3n+1}{7n})P_C(\frac{1}{4}g(x_n) + (\frac{4n-1}{4n})T_n^{\widetilde{f}_2}y_n), \end{cases} \tag{43}$$

where $P_C(x_1,x_2) = (\max\{\min\{x_1,10\},-10\}, \max\{\min\{x_2,10\},-10\})$ and also $P_C(I - \lambda_n^i \nabla\widetilde{f}_i) = s_n^i I + (1-s_n^i)T_n^{\widetilde{f}_i}$ and $s_n^i = \frac{2-\lambda_n^i(16)}{4}$, where $\lambda_n^i = \frac{n^2}{8n^2+1}$ $\forall i = 1,2$.

Then, since $\widetilde{f}_1(x_1,x_2) = 2x_1^2 + x_2$ and $\widetilde{f}_2(x_1,x_2) = (x_1-1) + x_2^2$, we have

$$\nabla\widetilde{f}_1(x_1,x_2) = (4x_1,1) \quad \text{and} \quad \nabla\widetilde{f}_2(x_1,x_2) = (1,2x_2).$$

It is obvious that $\nabla\widetilde{f}_i$ is a $\frac{1}{16}$-inverse strongly monotone, $\forall i = 1,2$.

Consider $(0,-10), (-10,0) \in [-10,10] \times [-10,10]$ for which

$$\begin{aligned} P_C\bigl(I - \lambda_n^1 \nabla\widetilde{f}_1\bigr)(0,-10) &= P_{[-10,10]\times[-10,10]}\left(I - \frac{1}{16}\nabla\widetilde{f}_1\right)(0,-10) \\ &= P_{[-10,10]\times[-10,10]}\left(0, \frac{-161}{16}\right) \\ &= \left(P_{[-10,10]}(0), P_{[-10,10]}\left(\frac{-161}{16}\right)\right) \\ &= \left(\max\bigl\{\min\{0,10\},-10\bigr\}, \max\left\{\min\left\{\frac{-161}{16},10\right\},-10\right\}\right) \end{aligned}$$

$$= (0, -10),$$

thus $(0, -10) \in U_{\widetilde{f_1}}$.

Similarly,

$$\begin{aligned} P_C\big(I - \lambda_n^2 \nabla \widetilde{f_2}\big)(-10, 0) &= P_{[-10,10]\times[-10,10]}\left(I - \frac{1}{16}\nabla \widetilde{f_2}\right)(-10, 0) \\ &= P_{[-10,10]\times[-10,10]}\left(\frac{-161}{16}, 0\right) \\ &= \left(P_{[-10,10]}\left(\frac{-161}{16}\right), P_{[-10,10]}(0)\right) \\ &= \left(\max\left\{\min\left\{\frac{-161}{16}, 10\right\}, -10\right\}, \max\{\min\{0, 10\}, -10\}\right) \\ &= (-10, 0), \end{aligned}$$

thus $(-10, 0) \in U_{\widetilde{f_2}}$.

It is clear that the sequences $\{\alpha_n\}$, $\{\mu_n\}$ satisfy all the conditions of Theorem 1, so we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(0, -10)$ and $(-10, 0)$, respectively. Table 1 shows the values of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$, and $n = N = 400$.

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$, and $n = N = 400$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−10.0000, 10.0000)	(10.0000, −10.0000)
2	(−6.0784, 8.0448)	(8.1639, −7.2059)
3	(−4.2192, 7.3380)	(7.5048, −5.8538)
4	(−3.0380, 6.9152)	(7.1109, −4.9143)
⋮	⋮	⋮
250	(−0.0050, −7.7785)	(−7.5696, −0.0076)
⋮	⋮	⋮
396	(−0.0043, −9.9937)	(−9.9937, −0.0065)
397	(−0.0042, −9.9937)	(−9.9937, −0.0064)
398	(−0.0042, −9.9937)	(−9.9937, −0.0064)
399	(−0.0042, −9.9937)	(−9.9937, −0.0064)
400	(−0.0042, −9.9937)	(−9.9937, −0.0064)

Figure 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, $y_n^1 = 10$, $y_n^2 = -10$, and $n = N = 400$

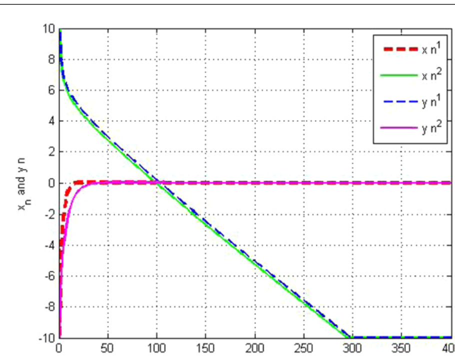
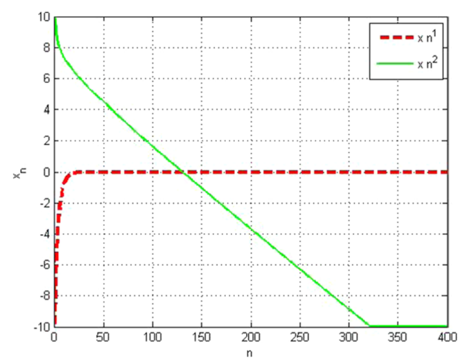


Table 2 The values of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, and $n = N = 400$

n	$x_n = (x_n^1, x_n^2)$
1	(−10.0000, 10.0000)
2	(−7.0308, 8.9972)
3	(−5.2235, 8.5685)
4	(−3.9531, 8.2876)
⋮	⋮
250	(0.0000,-6.2974)
⋮	⋮
396	(0.0000, −9.9958)
397	(0.0000, −9.9958)
398	(0.0000, −9.9958)
399	(0.0000, −9.9958)
400	(0.0000, −9.9958)

Figure 2 The convergence of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, and $n = N = 400$



Remark 1 If we choose $f \equiv g$ and $x_n = y_n$ in Example 1, we can rewrite (36) as follows:

$$x_{n+1} = \left(\frac{4n-1}{7n}\right)x_n + \left(\frac{3n+1}{7n}\right)P_C\left(\frac{1}{4n}f(x_n) + \left(\frac{4n-1}{4n}\right)T_n^{\tilde{f}}x_n\right),$$

where $P_C(x_1,x_2) = (\max\{\min\{x_1,10\},-10\}, \max\{\min\{x_2,10\},-10\})$ and also $P_C(I - \lambda_n\nabla\widetilde{f}) = s_nI + (1-s_n)T_n^{\tilde{f}}$ and $s_n = \frac{2-\lambda_n(16)}{4}$, where $\lambda_n = \frac{n^2}{8n^2+1}$. From Corollary 1, we can conclude that the sequence $\{x_n\}$ converges strongly to $(0,-10)$. Table 2 shows the values of $\{x_n\}$ with $x_n^1 = -10$, $x_n^2 = 10$, and $n = N = 400$.

Conclusion

1. Theorem 1 guarantees the convergence of $\{x_n\}$ and $\{y_n\}$ in Example 1.
2. Corollary 1 guarantees the convergence of $\{x_n\}$ in Remark 1.
3. By using the concepts of an intermixed algorithm and gradient-projection algorithm (GPA), we give a new iteration for solving two constrained convex minimization problems.

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Authors' contributions
The two authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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